A

Taibah University
Department of Mathematics

Math 241
Exam I
Thursday, November 2, 2017 Time: 90 minutes

Name: Solution I.D. #

Show All Necessary Work

Any Type of Calculator is NOT Allowed in the Exam

	Question #	Points
Part I	1 – 10 (10×2=20)	/20
Part II	1	/20
	2 (a:8 + b:5+c:7=20)	/20
	3 (2×10=20)	/20
	4	/20
Total		**/100**

Part I : M.C.Q

1- The given equation is linear:

a) $x^2 + y - z = -1$ b) $x + \cos y - z = 1$

c) $xy - y + z = 0$ d) $-x + y - 2z = -3$

2. The solution of the linear system :

$$\begin{cases} x - 3y = -1 \\ 3x + y = 7 \end{cases} \quad \text{is:}$$

a) $(x, y) = (-2,1)$ b) $(x, y) = (2,1)$

c) $(x, y) = (2,-1)$ d) $(x, y) = (-2,-1)$

3. If A is 2×3 matrix and B is 4×2 matrix and C is 3×4 matrix. The size of ACB is:

a) 2×4 b) 3×4 c) 2×2 d) 3×2

4. Let $A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$. The matrix A^2 is equal to:

a) $\begin{bmatrix} 1 & 4 \\ 1 & 4 \end{bmatrix}$ b) $\begin{bmatrix} -3 & 6 \\ 3 & -6 \end{bmatrix}$ c) $\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$ d) $\begin{bmatrix} 3 & -6 \\ -3 & 6 \end{bmatrix}$

5. Let $A = \begin{bmatrix} -2 & 1 \\ 4 & 0 \end{bmatrix}$. The transpose A^T is :

a) $\begin{bmatrix} 4 & 1 \\ 16 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 2 & -1 \\ -4 & 0 \end{bmatrix}$ c) $\begin{bmatrix} -2 & 4 \\ 1 & 0 \end{bmatrix}$ d) $\begin{bmatrix} 0 & 4 \\ 1 & -2 \end{bmatrix}$

6. Let $A = \begin{bmatrix} 3 & -1 & 0 \\ 1 & 5 & 6 \\ 0 & y & 2 \end{bmatrix}$ and $B = \begin{bmatrix} x & -1 & 0 \\ 1 & 5 & 6 \\ 0 & 1 & 2 \end{bmatrix}$. If $A = B+I$, then

a) $x=3, y=1$ b) $x=2, y=1$ c) $x=-2, y=-1$ d) $x=4, y=1$

7. The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 0 & 0 \end{bmatrix}$ is

a) *Elementary matrix* b) *Triangular* c) *Symmetric* d) *Not Diagonal*

In questions 8 and 9, assume that A, B and C are matrices of suitable sizes where addition and multiplication are defined. Then

8. a) $AB = BA$ b) $(A+B)^T = B^T + A^T$

c) $AB = AC \Rightarrow B = C$ d) $AC = I = CA$

9. a) $(AB)^{-1} = A^{-1}B^{-1}$ b) $(A+B)^2 = A^2 + 2AB + B^2$

c) $(AB)^2 = A^2B^2$ d) $I^{10} = I$

10. A non-homogeneous linear System $AX = B$

a) always inconsistent b) has only the trivial solution

c) always consistent d) has a unique solution if A is invertible

Part II

1. Use Gauss-Jordan-Elimination to solve the following system of linear equations:

$$
\begin{aligned}
3x_1 + 2x_2 - x_3 + x_4 &= 2 \\
x_1 - x_2 + x_3 + x_4 &= 4 \\
4x_1 + 2x_2 + x_3 + 2x_4 &= 8 \\
2x_1 - x_2 + x_3 + x_4 &= 0
\end{aligned}
$$

$$
\begin{bmatrix} 3 & 2 & -1 & 1 & 2 \\ 1 & -1 & 1 & 1 & 4 \\ 4 & 2 & 1 & 2 & 8 \\ 2 & -1 & 1 & 1 & 0 \end{bmatrix}
\xrightarrow{R_1 \leftrightarrow R_2}
\begin{bmatrix} 1 & -1 & 1 & 1 & 4 \\ 3 & 2 & -1 & 1 & 2 \\ 4 & 2 & 1 & 2 & 8 \\ 2 & -1 & 1 & 1 & 0 \end{bmatrix}
$$

$$
\xrightarrow[-2R_1+R_4]{-3R_1+R_2,\ -4R_1+R_3}
\begin{bmatrix} 1 & -1 & 1 & 1 & 4 \\ 0 & 5 & -4 & -2 & -10 \\ 0 & 6 & -3 & -2 & -8 \\ 0 & 1 & -1 & -1 & -8 \end{bmatrix}
\xrightarrow{R_2 \leftrightarrow R_4}
\begin{bmatrix} 1 & -1 & 1 & 1 & 4 \\ 0 & 1 & -1 & -1 & -8 \\ 0 & 6 & -3 & -2 & -8 \\ 0 & 5 & -4 & -2 & -10 \end{bmatrix}
$$

$$
\xrightarrow[-5R_2+R_4]{-6R_2+R_3}
\begin{bmatrix} 1 & -1 & 1 & 1 & 4 \\ 0 & 1 & -1 & -1 & -8 \\ 0 & 0 & 3 & 4 & 40 \\ 0 & 0 & 1 & 3 & 30 \end{bmatrix}
\xrightarrow{R_3 \leftrightarrow R_4}
\begin{bmatrix} 1 & -1 & 1 & 1 & 4 \\ 0 & 1 & -1 & -1 & -8 \\ 0 & 0 & 1 & 3 & 30 \\ 0 & 0 & 3 & 4 & 40 \end{bmatrix}
$$

$$
\xrightarrow[-3R_3+R_4]{R_2+R_1}
\begin{bmatrix} 1 & 0 & 0 & 0 & -4 \\ 0 & 1 & -1 & -1 & -8 \\ 0 & 0 & 1 & 3 & 30 \\ 0 & 0 & 0 & -5 & -50 \end{bmatrix}
\xrightarrow[-\frac{1}{5}R_4]{R_3+R_2}
\begin{bmatrix} 1 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 2 & 22 \\ 0 & 0 & 1 & 3 & 30 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix}
$$

$$
\xrightarrow[-3R_4+R_3]{-2R_4+R_2}
\begin{bmatrix} 1 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix}
$$

$\therefore$ The solution is $x_1 = -4,\quad x_2 = 2,\quad x_3 = 0,\quad x_4 = 10.$

2. Let $A = [2 \quad -1 \quad 4]$, $B = \begin{bmatrix} 7 & -4 \\ 0 & 3 \\ 5 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 0 & -4 \end{bmatrix}$. Find, if possible, the following:

a) AC^T

$C^T = \begin{bmatrix} 3 & 1 \\ 2 & 0 \\ 1 & -4 \end{bmatrix}$

$\underset{1\times 3}{A} \quad \underset{2\times 3}{C} \quad \underset{3\times 2}{C^T} \quad \underset{1\times 2}{AC^T}$

$AC^T = [2 \quad -1 \quad 4] \begin{bmatrix} 3 & 1 \\ 2 & 0 \\ 1 & -4 \end{bmatrix} = [2(3)+(-1)2+4(1) \quad 2(1)+(-1)(0)+4(-4)]$

$= [8 \quad -14]$

b) BA

$\underset{3\times 2}{B} \quad \underset{1\times 3}{A}$

not possible.

c) $tr(BC)$

$BC = \begin{bmatrix} 7 & -4 \\ 0 & 3 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 1 & 0 & -4 \end{bmatrix}$

$= \begin{bmatrix} 17 & 14 & 23 \\ 3 & 0 & -12 \\ 17 & 10 & -3 \end{bmatrix}$

$tr(BC) = 17+0-3 = 14$

3.

a) Prove that if A is an invertible symmetric matrix, then A^{-1} is symmetric.

Suppose that A is invertible, i.e. A^{-1} exists.

$$(A^{-1})^T = (A^T)^{-1}$$
$$= A^{-1}, \text{ since } A \text{ is symmetric}$$

$$\therefore (A^{-1})^T = A^{-1}$$

$\Rightarrow A^{-1}$ is symmetric.

b) Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ and consider the polynomial $p(x) = x^2 - 3x + 2$. Find $p(A)$.

$$A^2 = A.A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}$$

$$-3A = -3\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ -3 & -6 \end{bmatrix}$$

$$2I = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$P(A) = A^2 - 3A + 2I$$
$$= \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -3 & 0 \\ -3 & -6 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$\therefore P(A) = 0.$$

4. Find, if possible, the inverse A^{-1} of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\dashrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\dashrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\dashrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\dashrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 2 & -2 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\therefore\ A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{bmatrix}.$$