King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
(Term 091)
Math 260 - 2, 3, 5 & 6
Major Exam 1

Time: 1:30 hours

Name: Solution

ID #:

Serial #: Sec #:

No Calculator is Allowed in this Exam
Show All Necessary Work

Q1	/30
Q2	/40
Q3	/20
Q4	/20
Total	/110

Q1 Solve the following initial value problems:

(15) (a) $y + x^2\frac{dy}{dx} = 1, \qquad y(1) = 0$

$$\frac{dy}{dx} = \frac{1-y}{x^2}$$

[Separable]

Also linear: $\frac{dy}{dx} + \frac{1}{x^2}y = \frac{1}{x^2}$

$\rho(x) = e^{\int \frac{1}{x^2}dx} = e^{-\frac{1}{x}}$

$\frac{d}{dx}\left[y e^{-\frac{1}{x}}\right] = e^{-\frac{1}{x}}\frac{1}{x^2}$

$y e^{-\frac{1}{x}} = \int e^{-\frac{1}{x}}\frac{1}{x^2}dx$

$= e^{-\frac{1}{x}} + C$

$\boxed{y = 1 + Ce^{\frac{1}{x}}}$

$$\frac{dy}{1-y} = \frac{dx}{x^2}$$

$$-\ln|1-y| = -\frac{1}{x} + C_1$$

$$\ln|1-y| = \frac{1}{x} - C_1$$

$$|1-y| = e^{\frac{1}{x} - C_1}$$

$$1 - y = Ce^{\frac{1}{x}}, \quad C = \pm e^{-C_1}$$

$$\boxed{y = 1 - Ce^{\frac{1}{x}}}$$

$$y(1) = 0 \Rightarrow C = \frac{1}{e} \quad 2$$

$$\therefore \; y = 1 - \frac{1}{e}e^{\frac{1}{x}}$$

$$\boxed{y = 1 - e^{\frac{1-x}{x}}}$$

(15) (b) $x^2y' + 2xy = 5y^3, \qquad y(1) = 1$

Bernoulli: put $w = y^{1-3} = y^{-2} = \frac{1}{y^2}$

$\Rightarrow y = w^{-\frac{1}{2}}$

$\frac{dy}{dx} = -\frac{1}{2} w^{-3/2} \frac{dw}{dx}, \quad y^3 = w^{-3/2}$

Substitute in the given DE:

$-\frac{1}{2}x^2 w^{-3/2} \frac{dw}{dx} + 2x w^{-\frac{1}{2}} = 5 w^{-3/2}$

$\frac{dw}{dx} - \frac{4}{x} w = -\frac{10}{x^2} \quad$ (Linear)

$\rho(x) = e^{-\int \frac{4}{x} dx} = e^{-4\ln x} = \frac{1}{x^4}$

$\frac{d}{dx}\left[\frac{w}{x^4}\right] = -\frac{10}{x^6}$

$\frac{w}{x^4} = -10\int \frac{1}{x^6} dx = \frac{2}{x^5} + C$

$w = \frac{2}{x} + Cx^4$

$\frac{1}{y^2} = \frac{2}{x} + Cx^4 = \frac{2+Cx^5}{x}$

$\boxed{y^2 = \frac{x}{2+Cx^5}}$

$y(1) = 1 \Rightarrow 1 = \frac{1}{2+C} \Rightarrow C = -1$

$\therefore \boxed{y^2 = \frac{x}{2-x^5}}$

Q2 Solve the following differential equations:

14 **(a)** $(x^3 + y^3)dx + xy^2 dy = 0$

$$(*) \cdots \quad \frac{dy}{dx} = -\frac{x^3+y^3}{xy^2} = -\frac{1+\left(\frac{y}{x}\right)^3}{\left(\frac{y}{x}\right)^2} = f\left(\frac{y}{x}\right)$$

(homogeneous)

Put $y = ux$

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

substitute in the given DE: i.e. in $(*)$

$$u + x\frac{du}{dx} = -\frac{x^3+u^3x^3}{u^2x^3} = -\frac{1+u^3}{u^2}$$

$$x\frac{du}{dx} = \frac{-1-u^3-u^3}{u^2} = \frac{-1-2u^3}{u^2}$$

$$\frac{x}{dx} = \frac{-1-2u^3}{u^2\,du}$$

$$\frac{dx}{x} = \frac{u^2}{-1-2u^3}\,du \qquad \text{(separable)}$$

$$\ln|x| = -\frac{1}{6}\ln|-1-2u^3| + \ln|c_1|$$

$$\ln|x| = \ln\left|\frac{c_1}{\sqrt[6]{-1-2u^3}}\right|$$

$$x = \frac{c_1}{\sqrt[6]{-1-2u^3}} \Rightarrow x^6 = \frac{c}{-1-2u^3}$$

$$-1-2u^3 = \frac{c_1}{x^6} \Rightarrow -1-2\frac{y^3}{x^3} = \frac{c_1}{x^6} \Rightarrow \frac{2y^3}{x^3} = \frac{c}{x^6} - 1 = \frac{c-x^6}{x^6}$$

$$\boxed{\therefore\ y = \sqrt[3]{\frac{c-x^6}{2x^3}}}$$

11 (b) $x^2y'' + 3xy' = 2$

put $\left.\begin{array}{l} u = y' \\ u' = y'' \end{array}\right\} \Rightarrow x^2 u' + 3xu = 2$ [Linear]

$$u' + \frac{3}{x}u = \frac{2}{x^2}$$

$$\rho(x) = e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$$

$$\frac{d}{dx}\left[ux^3\right] = 2x \Rightarrow ux^3 = \int 2x\,dx = x^2 + c_1$$

$$u = \frac{1}{x} + \frac{c_1}{x^3} \Rightarrow y' = \frac{1}{x} + \frac{c_1}{x^3} \Rightarrow y = \int\left(\frac{1}{x} + \frac{c_1}{x^3}\right)dx$$

$$\therefore \boxed{y = \ln|x| - \frac{c_1}{2x^2} + c_2}$$

15 (c) $(xe^{2y} + 1)dx + x^2e^{2y}dy = 0$

M N

$$\frac{\partial M}{\partial y} = 2xe^{2y} = \frac{\partial N}{\partial x} \Rightarrow \text{Exact.}$$

$\Rightarrow \exists f$ such that $\frac{\partial f}{\partial x} = M = xe^{2y}+1$ & $\frac{\partial f}{\partial y} = N = x^2e^{2y}$

$$\therefore f(x,y) = \int (xe^{2y}+1)dx = \frac{1}{2}x^2e^{2y} + x + g(y)$$

$$\frac{\partial f}{\partial y} = x^2e^{2y} + g'(y) \overset{!}{=} N = x^2e^{2y}$$

$$\Rightarrow g'(y) = 0 \Rightarrow g(y) = c_1$$

$$\therefore f(x,y) = \frac{1}{2}x^2e^{2y} + x + c_1$$

The solution is $\boxed{\frac{1}{2}x^2e^{2y} + x = C}$

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or method: put $u = e^{2y} \Rightarrow \frac{du}{dx} = 2e^{2y}\frac{dy}{dx}$ and substitute to get

$\frac{du}{dx} + \frac{2}{x}u = -\frac{2}{x^2}$ which is linear and you can solve

20 **Q3** A hot soup takes 10 minutes to cool down from 100° to 60° in a room with temperature 25°. How long will it take the soup to reach 40°?

[$ln5 = 1.609$, $ln7 = 1.946$, $ln15 = 2.708$]

By Newton's law of Cooling:

$\frac{dT}{dt} \propto |T - T_m|$ from which we get

$(*) \rightarrow T(t) = T_m + Ce^{Kt}$ and we have:

$T(t) :=$ temperature of the soup at time t

$T_m := 25$; the room temperature

$T(0) = 100$

$T(10) = 60$

$t = ?$ at $T(t) = 40$

From $(*)$: $T(t) = 25 + Ce^{kt}$

$T(0) = 25 + C = 100 \Rightarrow C = 75$

$\therefore T(t) = 25 + 75e^{kt}$

$T(10) = 60 = 25 + 75e^{10K}$

$\Rightarrow 75e^{10K} = 35 \Rightarrow e^{10K} = \frac{35}{75} = \frac{7}{15} \Rightarrow 10K = \ln\frac{7}{15}$

$\therefore K = \frac{1}{10}\ln\frac{7}{15} = \frac{1}{10}[\ln 7 - \ln 15] = \frac{1}{10}[1.946 - 2.708] = \frac{1}{10}(-.76$

$= -.076$

So, we have $T(t) = 25 + 75e^{-.0762t}$

we need t when $T(t) = 40$:

$T(t) = 40 \Rightarrow 25 + 75e^{-.0762t} = 40$

$75e^{-.0762t} = 15 \Rightarrow e^{-.0762t} = \frac{1}{5}$

$\therefore -.0762t = \ln\frac{1}{5} = -\ln 5 = -1.609$

$t = \frac{1.609}{.0762} \simeq 21$

$\therefore$ The soup will reach 40° after 21 minutes.

Q4 Solve the following system of equations:

$$\begin{aligned} x+y-z+u+w &= 3 \\ 2x-y+2z-u-w &= -1 \\ x+2y+z-2u-2w &= 2 \end{aligned}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & 1 & 3 \\ 2 & -1 & 2 & -1 & -1 & -1 \\ 1 & 2 & 1 & -2 & -2 & 2 \end{array}\right] \xrightarrow[-R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & 1 & 3 \\ 0 & -3 & 4 & -3 & -3 & -7 \\ 0 & 1 & 2 & -3 & -3 & -1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -3 & -3 & -1 \\ 0 & -3 & 4 & -3 & -3 & -7 \end{array}\right] \xrightarrow{3R_2+R_3} \left[\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -3 & -3 & -1 \\ 0 & 0 & 10 & -12 & -12 & -10 \end{array}\right]$$

$$\xrightarrow{\frac{1}{10}R_3} \left[\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -3 & -3 & -1 \\ 0 & 0 & 1 & -\frac{6}{5} & -\frac{6}{5} & -1 \end{array}\right] \xrightarrow{-R_2+R_3} \left[\begin{array}{ccccc|c} 1 & 0 & -3 & 4 & 4 & 4 \\ 0 & 1 & 2 & -3 & -3 & -1 \\ 0 & 0 & 1 & -\frac{6}{5} & -\frac{6}{5} & -1 \end{array}\right]$$

$$\xrightarrow[-2R_3+R_2]{3R_3+R_1} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & \frac{2}{5} & \frac{2}{5} & 1 \\ 0 & 1 & 0 & -\frac{3}{5} & -\frac{3}{5} & 1 \\ 0 & 0 & 1 & -\frac{6}{5} & -\frac{6}{5} & -1 \end{array}\right] \leftarrow \text{in R.R.E.F.}$$

$\begin{cases} w, u \text{ are free variables} \\ x, y, z \text{ are leading } \sim \end{cases}$

$$\left.\begin{aligned} &w = t,\ u = s \\ &z = \tfrac{6}{5}t + \tfrac{6}{5}s - 1 \\ &y = \tfrac{3}{5}t + \tfrac{3}{5}s + 1 \\ &x = -\tfrac{2}{5}t - \tfrac{2}{5}s + 1 \end{aligned}\right\} \therefore \text{ the solution is: } \begin{cases} x = -\frac{2}{5}t - \frac{2}{5}s + 1 \\ y = \frac{3}{5}t + \frac{3}{5}s + 1 \\ z = \frac{6}{5}t + \frac{6}{5}s - 1 \\ u = s \\ w = t \qquad , s, t \in \mathbb{R} \end{cases}$$