Section 6.1 *Solution about ordinary points*

6.1.1 Review of power series

Here we will briefly review those results of powers series which we need to understand method of series solutions of differential equations. In case you have forgotten the stuff related to power series, my advice is to consult your Calculus notes to refresh your understanding of power series.

Learning Outcomes

After completing this sub-section, you will inshaAllah be able to

1. recall what is meant by power series
2. recall what is meant by interval & radius of convergence of a power series
3. recall how to find interval & radius of convergence of a power series
4. handle problems related to shifting the index of summation
5. understand what is meant by analytic points
6. recall how to write Taylor series of a function about analytic points

Power series in $x - x_0$

A power series about the point x_0 is series in the powers of $x - x_0$, of the form

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n = c_0 + c_1 (x - x_0) + c_2 (x - x_0)^2 + \cdots\cdots + c_n (x - x_0)^n + \cdots$$

Convergence of power series in $x - x_0$

Given a power series in $x - x_0$. Then either

1) it converges only for $x = x_0$.

OR

2) it converges for all values of x

OR

3) it converges for x in an open interval $(x_0 - R, x_0 + R)$ and diverges if $x < x_0 - R$, $x > x_0 + R$

- At the end points $x = x_0 - R$ or $x = x_0 + R$, the series may converge or diverge. (we need to investigate separately at the end points)

Interval of convergence
=
all values for which series converges

Radius of convergence
=
half of length of interval of convergence

Finding interval of convergence of power series in $x - x_0$

Main tool for finding interval of convergence

=

Ratio test

Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series and suppose that $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$.

1) If $L < 1$ then the series $\sum_{n=1}^{\infty} a_n$ absolutely converges (and hence converges)

2) If $L > 1$ or $L = \infty$ then the series $\sum_{n=1}^{\infty} a_n$ diverges

3) If $L = 1$, test fails and we cannot say anything about the convergence or divergence.

See Example 1 done in class

Differentiation and integration of power series (term by term)

Let $\boxed{f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n}$ with radius of convergence R.

Then in the radius of convergence

$$\boxed{f'(x) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(c_n (x - x_0)^n\right)}$$

and

$$\boxed{\int f(x)dx = \sum_{n=0}^{\infty} \int \left(c_n (x - x_0)^n\right) dx}.$$

Both have radius of convergence R.

Another fact

- Can add, subtract, multiply power series to get new power series.
- The sum, difference, product have same radius of convergence.

Shifting (or changing) index of a summation

Important tool

We will need to make changes of summation indices in calculating series solutions of differential equations

- Nothing new.
- An idea similar to change of variables in integration

See Examples 2, 3, 4 done in class

Analytic functions

- Not every function can be expressed as power series.
- Those which can be expressed as power series are given a special name.

A function $f(x)$ is said to be analytic at a point x_0 if it can be represented as a power series with a positive radius of convergence.

i.e. $$f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

Examples “Analytic”

- $f(x) = e^x$
- $f(x) = \sin x$
- $f(x) = ax^2 + bx + c$

Example “Non-Analytic”

- $f(x) = \dfrac{1}{x-1}$ at $x = 1$

- All polynomials are analytic
- If $P(x), Q(x)$ are polynomials with no common factors.

  Then $\dfrac{P(x)}{Q(x)}$ is analytic at points where $Q(x) \neq 0$

Taylor Series

Gives a way of finding power series of analytic functions

If $f(x)$ is analytic at x_0 then

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \\
&= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \cdots
\end{aligned}
$$

Important basic series

Series	Interval of convergence
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots\cdots$	$-1 < x < 1$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots$	$-\infty < x < \infty$
$\sin x = \sum_{k=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\cdots$	$-\infty < x < \infty$
$\cos x = \sum_{k=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\cdots$	$-\infty < x < \infty$
$\tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\cdots$	$-1 \le x \le 1$

End of 6.1.1
Do Qs. 1-14