Math 260 – Quiz # 11

Name: Solution Sr #: ____________

Consider the matrix $A = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & 0 \\ -1 & 1 & 2 \end{bmatrix}$.

(1) Write the characteristic polynomial of A.

(2) Find the eigen values of A.

(3) Find the eigen vectors of A which correspond to the eigen values in (2).

(4) Find bases and dimensions for the eigen spaces of A.

(5) Is A diagonalizable? Why?

(6) Find a matrix $\boldsymbol{P}$ that diagonalizes $\boldsymbol{A}$.

(7) Use part (6) to compute A^6.

(1) $$|A-\lambda I| = \begin{vmatrix} 3-\lambda & -1 & 0 \\ 0 & 2-\lambda & 0 \\ -1 & 1 & 2-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda)(2-\lambda)$$
$$= (3-\lambda)(2-\lambda)^2$$
$$= -\lambda^3 + 5\lambda^2 - 10\lambda + 12$$

The characteristic polynomial is $P(\lambda) = -\lambda^3 - 5\lambda^2 - 10\lambda + 12$

(2) From above, $|A-\lambda I| = (3-\lambda)(2-\lambda)^2 = 0$

$\therefore$ the eigen values of A are: $3, 2, 2$.

(3) <u>To find the eigenvectors:</u>

<u>For $\lambda = 3$</u>

$$(A-3I)X = 0 \Rightarrow \begin{bmatrix} 0 & -1 & 0 \\ 0 & -1 & 0 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow x_2 = 0, \quad x_1 = -x_3. \text{ Take } x_3 = -t \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ -t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

$\therefore E_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is an eigen vector corresponds to $\lambda = 3$

<u>For $\lambda = 2$</u>

$$(A-2I)\bar{X} = 0 \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

$$\Rightarrow x_1 = x_2 = t, \quad x_3 = s \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ s \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ we have two eigen vectors E_2, E_3 correspond to $\lambda = 2$, where $E_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $E_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(4) From part (3), we have two eigen spaces W_1, W_2, where W_1 is spanned by E_1 and W_2 is spanned by E_1, E_2.

i.e. $W_1 = \langle E_1 \rangle$ and $W_2 = \langle E_1, E_2 \rangle$.

Thus $\{E_1\}$ is a basis for the eigen space W_1, where

$E_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, and $\{E_2, E_3\}$ is a basis for the eigen space W_2, where

$E_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $E_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Also, $\dim W_1 = 1$, $\dim W_2 = 2$

(5) A is diagonalizable, since the eigen vectors E_1, E_2, E_3 are linearly independent.

(6) $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, so that

$P^{-1}AP = D$, where $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Find P^{-1} and check that $P^{-1}AP = D$

(7) <u>To find A^6:</u>

From part (6), we find that

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

Using the fact that $A = PDP^{-1}$, $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

we have $A^6 = PD^6P^{-1}$

$$\therefore A^6 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3^6 & 0 & 0 \\ 0 & 2^6 & 0 \\ 0 & 0 & 2^6 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 729 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & 64 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 729 & 64 & 0 \\ 0 & 64 & 0 \\ -729 & 0 & 64 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\therefore A^6 = \begin{bmatrix} 729 & -665 & 0 \\ 0 & 64 & 0 \\ -665 & 665 & 64 \end{bmatrix}$$