KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
Department of Mathematics and Statistics

MATH 550-(071)

Exam I

Time: 110 Minutes

Name: Solution Sec.# I.D. #

Show All Necessary Work

Question	Points
1	16
2	28
3	14
4	18
5	12
6	12
Total	100

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1. (a) Show that if V is an n-dimensional vector space, then any linearly independent subset S of V is a part of a basis for V.

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Let $S_0 = \{\alpha_1, \ldots, \alpha_k\}$ be linearly indep. subset of V. $(k \leq n)$

If $k = n$, then S_0 itself is a basis and the statement is true.

Assume that $k < n$. Then S_0 is not a spanning set for V; i.e. $\mathrm{Span}(S_0) \subsetneqq V$. Let $\alpha_{k+1} \in V \setminus \mathrm{Span}(S_0)$. Then $S_1 = S_0 \cup \{\alpha_{k+1}\}$ is linearly indep. If $\mathrm{Span}(S_1) = V$, then $n = k+1$ and we are done. If $\mathrm{Span}(S_1) \subsetneqq V$, then $\exists\, \alpha_{k+2} \in V \setminus \mathrm{Span}(S_1)$. Continue this process till we get $\alpha_{k+1}, \ldots, \alpha_n \ni S_0 \cup \{\alpha_{k+1}, \ldots, \alpha_n\}$ is linearly indep. and $V = \mathrm{Span}(S_0 \cup \{\alpha_1, \ldots, \alpha_n\})$. Hence $S_0 \cup \{\alpha_{k+1}, \ldots, \alpha_n\}$ is a basis for V.

8 (b) Let S be a linearly independent subset of a vector space V. Let β be a vector in V such that $\beta \notin span(S)$. Show that $S \cup \{\beta\}$ is linearly independent.

We show that each finite subset of $S \cup \{\beta\}$ is linearly indep.

Suppose that $c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n + b\beta = 0$, $c_i, b \in F$, $\alpha_i \in S$. $i = 1, \ldots, n$.

Then $b\beta = -(c_1\alpha_1 + \cdots + c_n\alpha_n)$. If $b \neq 0$, then $b^{-1} \in F$ and

$$\beta = -b^{-1}(c_1\alpha_1 + \cdots + c_n\alpha_n)$$
$$= (-b^{-1}c_1)\alpha_1 + \cdots + (-b^{-1}c_n)\alpha_n \in \mathrm{Span}(S)$$

which is a contradiction. Hence $b = 0$ and our assumption becomes $c_1\alpha_1 + \cdots + c_n\alpha_n = 0$. But S is linearly indep. $\Rightarrow c_1 = \cdots = c_n = 0$

$\therefore c_1 = c_2 = \cdots = c_n = b = 0$. Hence $S \cup \{\beta\}$ is linearly indep.

2. Let $T : R^3 \to R^2$ given by

$$T(x, y, z) = (2x + y, x + y - z)$$

Let S and B be the standard bases for R^3 and R^2 respectively, and let $S' = \{(2,0,0), (0,-1,0), (0,0,-2)\}$ and $B' = \{(1,1), (1,-1)\}$ be some ordered bases for R^3 and R^2 respectively.

2 (a) Show that T is a linear transformation.

Let $\alpha, \beta \in R^3$, $\alpha = (x_1, y_1, z_1)$, $\beta = (x_2, y_2, z_2)$. Then

$$T(\alpha+\beta) = T[(x_1+x_2, y_1+y_2, z_1+z_2)] = \left(2(x_1+x_2)+y_1+y_2, x_1+x_2+y_1+y_2-z_1-z_2\right)$$

$$= \left(2x_1+2x_2+y_1+y_2, x_1+y_1-z_1+x_2+y_2-z_2\right)$$

$$= (2x_1+y_1, x_1+y_1-z_1) + (2x_2+y_2, x_2+y_2-z_2) = T(\alpha)+T(\beta)$$

Also, if $c \in R$,

$$T(c\alpha) = T(cx_1, cy_1, cz_1) = (2cx_1+cy_1, cx_1+cy_1-cz_1) = c(2x_1+y_1, x_1+y_1-z_1)$$

$$= cT(\alpha).$$

6 (b) Is T singular, is it onto?

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Suppose $T(\alpha) = 0$. $\Rightarrow T(x,y,z) = 0 = (2x+y, x+y-z) = (0,0)$

$\Rightarrow \begin{cases} 2x+y=0 \\ x+y-z=0 \end{cases}$ This system has a non-trivial solution i.e. $\alpha \neq 0$. for example $\alpha = (1,-2,-1)$, $T(\alpha) = (0,0)$

Hence T is non singular.

Let $(x,y) \in R^2$. Check if $\exists\ (a,b,c) \in R^3 \ni T(a,b,c) = (x,y)$.

$\Rightarrow T(a,b,c) = (2a+b, a+b-c) = (x,y) \Rightarrow$ solving for a, b, c, we get $a = x-y$, $b = 2y-x$, $c = 0$ $\Rightarrow T(x-y, 2y-x, 0) = (x,y) \Rightarrow T$ is onto.

2 (c) What is the nullity of T?

Nullity of T + Rank $T = \dim R^3 = 3$

By (b) above, T is onto $\Rightarrow$ range $T = R^2 \Rightarrow$ rank $T = \dim R^2 = 2$

Nullity of $T = 3 - 2 = 1$

2 (d) What is the rank of T?

From (c), rank $T = 2$

(e) Find the transition matrix P from B' to B.

Recall, $[\alpha]_B = \underset{B \leftarrow B'}{P} [\alpha]_{B'}$, where $P_j = [\beta_j']_B$

$[\beta_1']_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$[\beta_2']_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\therefore P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$S = \{\alpha_1, \alpha_2, \alpha_3\}$
$= \{e_1, e_2, e_3\} = \{(1,0,0), \ldots$

$S' = \{\alpha_1', \alpha_2', \alpha_3'\}$
$= \{(2,0,0), (0,-1,0), (0,0,-2)\}$

$B = \{\beta_1, \beta_2\}$
$= \{e_1, e_2\} = \{(1,0), (0,1)\}$

$B' = \{\beta_1', \beta_2'\}$
$= \{(1,1), (1,-1)\}$

(f) Find the transition matrix Q from S' to S.

$[\alpha]_S = \underset{S \leftarrow S'}{Q} [\alpha]_{S'}$, where $Q_j = [\alpha_j']_S$

$[\alpha_1']_S = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

$[\alpha_2']_S = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

$[\alpha_3']_S = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$

$\therefore Q = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

$A = [T]_B$

$[T\alpha]_B = A(\alpha)_S$

4 (g) Find $[T]_B$; the matrix of T relative to the ordered bases S, B.

Note that the columns of this matrix are given by $[T(\alpha_j)]_B$.

$T(\alpha_1) = T(1,0,0) = (2,1) = 2e_1 + e_2 \Rightarrow [T\alpha_1]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$T(\alpha_2) = T(0,1,0) = (1,1) = e_1 + e_2 \Rightarrow [T\alpha_2]_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$T(\alpha_3) = T(0,0,1) = (0,-1) = 0e_1 - e_2 \Rightarrow [T\alpha_3]_B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\therefore$ the matrix of T relative to the ordered bases S, B is

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

$[T\alpha]_{B'} = A(\alpha)_{S'}$

4 (h) Find $[T]_{B'}$; the matrix of T relative to the ordered bases S', B'.

Note that the columns of this matrix are given by $[T(\alpha'_j)]_{B'}$

$T(\alpha'_1) = T(2,0,0) = (4,2) = 3(1,1) + 1(1,-1) \Rightarrow [T(\alpha'_1)]_{B'} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$T(\alpha'_2) = T(0,-1,0) = (-1,-1) = -1(1,1) + 0(1,-1) \Rightarrow [T(\alpha'_2)]_{B'} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$T(\alpha'_3) = T(0,0,-2) = (0,2) = 1(1,1) + (-1)(1,-1) \Rightarrow [T(\alpha'_3)]_{B'} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\therefore$ the matrix of T relative to S', B' is

$$\begin{bmatrix} 3 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

2 (i) Verify that $[T]_{B'} = P^{-1}[T]_B Q$

From part (e), we have $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow P^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

$$P^{-1}[T]_B Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} = [T]_{B'}.$$

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3. 4 (a) Define what is meant by a linear functional on a vector space V and give an example for such function.

Let V be a vector space over the field F. A linear functional on V is a linear transformation $f: V \longrightarrow F$.
i.e. $f(c\alpha+\beta) = cf(\alpha)+f(\beta)$ $\forall \alpha, \beta \in V$, $c \in F$.
Example: Let F be a field and consider $V=F^n$. The func. $f_i: V \longrightarrow F$, given by $f_i(a_1,\ldots,a_n) = a_i$ is a linear functional

5 (b) State, without proof, the Dual basis theorem.

Let V be a finite dimensional vector space over the field F, and let $\mathcal{B}=\{\alpha_1,\ldots,\alpha_n\}$ be a basis for V. Consider $V^*=L(V,F)$. Then there is a unique basis $\mathcal{B}^*=\{f_1,\ldots,f_n\}$ for V^* $\ni$ $f_i(\alpha_j)=\delta_{ij}$. $\mathcal{B}^*$ is the dual basis for $\mathcal{B}$.

5 (c) Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis for a vector space V, and let $\{f_1, f_1, \ldots, f_n\}$ be its dual basis. Show that any vector $\alpha \in V$ can be written as $\alpha = \sum_{i=1}^{n} f_i(\alpha)\alpha_i$.

Since $\{\alpha_1,\ldots,\alpha_n\}$ is a basis for V, then we can write

$$\alpha = \sum_{i=1}^{n} c_i \alpha_i \quad , \quad c_i \in F. \qquad \text{-----} \quad (*)$$

So, $$f_j(\alpha) = f_j\left(\sum_{i=1}^{n} c_i\alpha_i\right) = \sum_{i=1}^{n} c_i f_j(\alpha_i) = \sum_{i=1}^{n} c_i \delta_{ij} = c_j.$$

Thus, $f_i(\alpha) = c_i$, and $(*)$ becomes

$$\alpha = \sum_{i=1}^{n} f_i(\alpha)\,\alpha_i .$$

4. Prove of disprove each one of the following statements:

5 (a) If T is a linear operator on a finite dimensional vector space V and B, B' are ordered Bases for V, then $\det([T]_B) = \det([T]_{B'})$

Let $A = [T]_\beta$, $C = [T]_{\beta'}$.

Then A and C are Similar.

$\Rightarrow \exists$ a non-singular matrix $P \ni C = P^{-1}AP$

$$\det(C) = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P)$$
$$= (\det P)^{-1}\det(A)\det(P) = \det(A)(\det(P))^{-1}(\det(P))$$
$$= \det(A)$$

i.e. $\det([T]_\beta) = \det([T]_{\beta'})$.

5 (b) There is exactly one linear transformation $T : R^2 \to R^2$ such that $T(1,5) = (1,0)$ and $T(5,1) = (0,1)$.

This statement is true if $(1,5)$, $(5,1)$ are linearly indep.

$c_1(1,5) + c_2(5,1) = (0,0) \Rightarrow c_1 = c_2 = 0$

$\Rightarrow$ $(1,5)$, $(5,1)$ are linearly independent ad hence they form a basis for R^2. Thus, there is exactly one linear transformation T such that $T(1,5) = (1,0)$
$T(5,1) = (0,1)$

8 (c) Consider **P**[X], the vector space of all polynomials over the field of real numbers.

Let $T : \mathrm{P}[\mathrm{X}] \to \mathrm{P}[\mathrm{X}]$ be a mapping defined by $T(p(x)) = \int_0^x p(t)\,dt$. Then T is a linear operator which is neither one-to-one nor onto.

- T is linear since if $p(x), q(x) \in P[X]$, then

$$T(p(x)+q(x)) = T[(p+q)(x)] = \int_0^x (p+q)(t)\,dt$$

$$= \int_0^x [p(t)+q(t)]\,dt$$

$$= \int_0^x p(t)\,dt + \int_0^x q(t)\,dt = T(p(x)) + T(q(x))$$

Similarly, we can show $T(cp(x)) = cT(p(x))$ for any $c \in \mathbb{R}$.

- Suppose that $T(p(x)) = T(q(x))$. Then

$$\int_0^x p(t)\,dt = \int_0^x q(t)\,dt$$

$$\Rightarrow \frac{d}{dx}\left[\int_0^x p(t)\,dt\right] = \frac{d}{dx}\left[\int_0^x q(t)\,dt\right]$$

$$\Rightarrow \quad p(x) = q(x)$$

$$\Rightarrow \quad T \text{ is } 1\text{-}1$$

- T is not onto since $\mathrm{Range}(T) \neq P[X]$. This is clear if we consider constant functions in $P[X]$.

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5. Let V and W be finite dimensional vector spaces over the field F.

Show that $\dim L(V, W) = (\dim V)(\dim W)$

See your notes

6.(a) Let $T : R^4 \to R^2$ be a linear transformation and let $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be an ordered basis for R^4, where

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$$\alpha_1 = (1,0,1,0), \alpha_2 = (0,1,-1,2), \alpha_3 = (0,2,2,1), \alpha_4 = (1,0,0,1).$$

Let $\alpha = (3,-5,-5,0)$ be a vector in R^4 such that $T(\alpha) = (4,7)$.

If $T(\alpha_1) = (1,2)$, $T(\alpha_3) = (0,0)$, $T(\alpha_4) = (2,0)$, find $T(\alpha_2)$.

$$\alpha = (3,-5,-5,0) = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4$$
$$= c_1(1,0,1,0) + c_2(0,1,-1,2) + c_3(0,2,2,1) + c_4(1,0,0,1)$$

$$\Rightarrow \left.\begin{aligned} c_1 \qquad\quad + c_4 &= 3 \\ c_2 + 2c_3 \qquad &= -5 \\ c_1 - c_2 + 2c_3 \qquad &= -5 \\ 2c_2 + c_3 + c_4 &= 0 \end{aligned}\right\} \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 2 & 0 & -5 \\ 1 & -1 & 2 & 0 & -5 \\ 0 & 2 & 1 & 1 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow \begin{aligned} c_1 &= 2 \\ c_2 &= 1 \\ c_3 &= -3 \\ c_4 &= 1 \end{aligned}$$

$$\therefore \alpha = 2\alpha_1 + \alpha_2 - 3\alpha_3 + \alpha_4$$

$$T(\alpha) = T(2\alpha_1 + \alpha_2 - 3\alpha_3 + \alpha_4) = 2T(\alpha_1) + T(\alpha_2) - 3T(\alpha_3) + T(\alpha_4)$$

$$T(\alpha_2) = T(\alpha) - 2T(\alpha_1) + 3T(\alpha_3) - T(\alpha_4)$$
$$= (4,7) - 2(1,2) + 3(0,0) - (2,0)$$
$$= (0,3)$$

(b) Let T be a linear operator on an n-dimensional vector space V such that the range and null space of T are identical. Show that n is even.

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Since range T = nullspace of T

$\Rightarrow$ rank(T) = nullity(T)

Now, nullity (T) + rank(T) = dim V = n

$\Rightarrow$ rank (T) + rank (T) = n

2 rank(T) = n

Hence n is even.