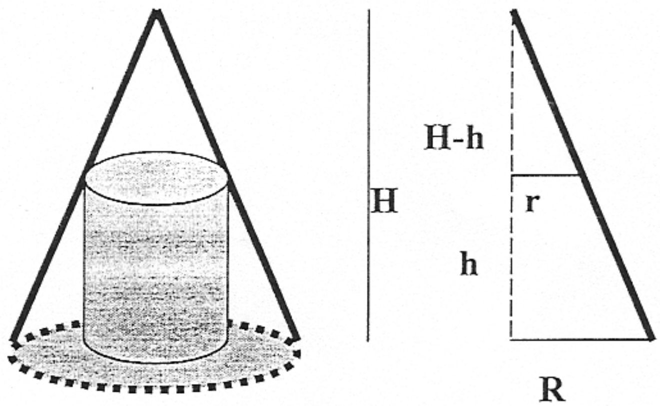
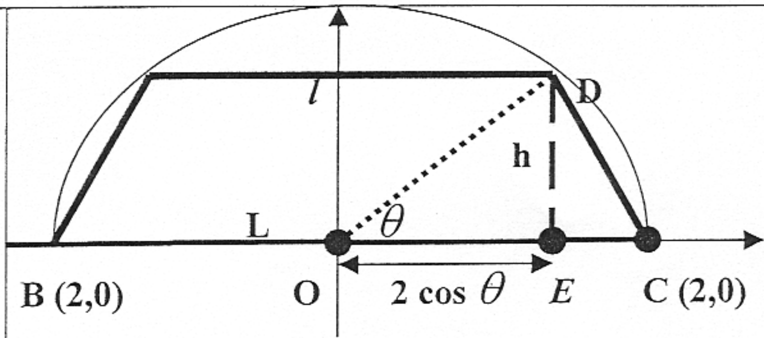
Q 26. Show that the right circular cylinder of greatest volume that can be inscribed in a right circular cone has volume that is 4/9 the volume of the cone.

Solution:

<table>
<tr><td>1</td><td>Draw the Diagram</td><td>
</td></tr>
<tr><td>2</td><td>Identify & Label the Unknowns</td><td>• <u>Cone</u>: Height = H; Radius = R
• <u>Cylinder</u>: Height = h; Radius = r
• Volume of Cylinder = V</td></tr>
<tr><td>3</td><td>Identify the Variable to be Optimized</td><td>V</td></tr>
<tr><td>4</td><td>Relation among the Variables</td><td>$V = \pi r^2 h$</td></tr>
<tr><td>5</td><td>Side Conditions (See Fig.)</td><td>$[(H-h)/h] = [r/R]$; $h = H(R-r)/R$ $0 \le r \le R$</td></tr>
<tr><td>6</td><td>Convert Main Equation To Equation of One Variable</td><td>$V = \pi r^2 h = \pi r^2 H(R-r)/R$
$= \pi H(r^2 R - r^3)/R$</td></tr>
<tr><td>7</td><td>Find C.N. of V</td><td>$dV/dr = \pi H r(2R-3r)/R = 0$
C.N : $r = 2R/3$</td></tr>
<tr><td>8</td><td>Use 2nd Deriv. To test C.N</td><td>$V'' = \pi H (2R-6r)/R$;
$V''(2R/3)<0$ [Maxim at C.N.]*</td></tr>
<tr><td>9</td><td>Test the C.N. and 0, R for Max/Mini Values</td><td><table>
<tr><td>r</td><td>h</td><td>V</td><td><u>V will be</u></td></tr>
<tr><td>0</td><td>H</td><td>0</td><td></td></tr>
<tr><td>2R/3</td><td>H/3</td><td>$4\pi R^2 H/27$</td><td>Maxim*</td></tr>
<tr><td>R</td><td>0</td><td>0</td><td></td></tr>
</table></td></tr>
</table>

Q 38. A trapezoid is inscribed in a semicircle of radius 2 so that one side is along the diameter. Find the maximum possible area of the trapezoid.

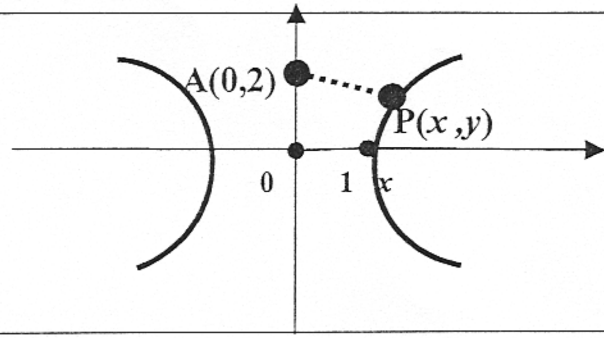
Solution:

<table>
<tr><td>1</td><td>Draw the Diagram:
[Place the Diameter of the Semicircle on x – axis with its center at the Origin 0]
Join O to the Upper Right Vertex D of Trapezium.</td><td>
</td></tr>
<tr><td>2</td><td>Identify & Label the Unknowns</td><td>• <u>Trapezium</u>: Larger Side Length = L
Smaller Side Length = l; Height = h
• Area of Trapezium = A
• Angle COD = θ</td></tr>
<tr><td>3</td><td>Identify the Variable to be Optimized</td><td>A</td></tr>
<tr><td>4</td><td>Relation among the Variables</td><td>$A = h(l + L)/2$</td></tr>
<tr><td>5</td><td>Side Conditions (See Fig.)</td><td>L= 4; h = 2 sin θ ; l = 2 (OE) = 4 cos θ
$0 \le \theta \le \pi/2$;</td></tr>
<tr><td>6</td><td>Convert Main Equation To Equation of One Variable</td><td>A= sin θ (4+4 cos θ)= 4(sin θ – cos θ sin θ)</td></tr>
<tr><td>7</td><td>Find C.N. of A</td><td>$dA/d\theta = 4(\cos\theta - \sin^2\theta + \cos^2\theta) = 0$;
$\theta = \pi/3$</td></tr>
<tr><td>8</td><td>Use 2nd Deriv. To test C.N</td><td>$A'' = 4\sin\theta\,(1 - 4\cos\theta)$
$\Rightarrow A''(\pi/3) < 0 \Rightarrow$ Maxim at C.N.*</td></tr>
<tr><td>9</td><td>Test the C.N. and 0 , $\pi/3$ for Max/Mini Values</td><td>
<table>
<tr><td>θ</td><td>L</td><td>l</td><td>h</td><td>A</td><td><u>A will be</u></td></tr>
<tr><td>0</td><td>2</td><td>2</td><td>0</td><td>0</td><td></td></tr>
<tr><td>$\pi/3$</td><td>4</td><td>2</td><td>$\sqrt{3}$</td><td>$3\sqrt{3}$</td><td>Maxim*</td></tr>
<tr><td>$\pi/2$</td><td>0</td><td>0</td><td>2</td><td>0</td><td></td></tr>
</table>
</td></tr>
</table>

Q 51. Find all points on the curve $x^2 - y^2 = 1$ closest to (0,2).

Solution:

<table>
<tr><td>1</td><td>Draw the Diagram:

{Sketch the cuve $x^2 - y^2 = 1$ in xy-plane. The point A (0,2) will be on y-axis.}
Note: Midpoint of the base of Rectangle must be on the Origin.</td><td>
</td></tr>
<tr><td>2</td><td>Identify & Label the Unknowns</td><td>• Point on the Curve: P (x,y)
• Fixed Point: A (0,2)
• (Distance between A and P)^2: S</td></tr>
<tr><td>3</td><td>Identify the Variable to be Optimized</td><td>S</td></tr>
<tr><td>4</td><td>Relation among the Variables</td><td>$S = x^2 + (y-2)^2$</td></tr>
<tr><td>5</td><td>Side Conditions (See Fig.)</td><td>$x^2 = 1 + y^2$</td></tr>
<tr><td>6</td><td>Convert Main Equation To Equation of One Variable</td><td>$S = (1+y^2) + (y-2)^2$
$= 2y^2 - 4y + 5$</td></tr>
<tr><td>7</td><td>Find C.N. of S</td><td>$dS/dy = 4y - 4 = 0$; $y = 1$</td></tr>
<tr><td>8</td><td>Use 2nd Deriv. To test C.N</td><td>$S'' = 4 \Rightarrow S''(1) > 0 \Rightarrow$ Minim at C.N.*</td></tr>
<tr><td>9</td><td>Test the C.N. and 0 , 4 for Max/Mini Values</td><td><table><tr><th>y</th><th>x</th><th>S</th><th><u>A will be</u></th></tr><tr><td>1</td><td>$\pm\sqrt{2}$</td><td>3</td><td>Minim*</td></tr></table>Ans. Shortest Distance between P and A(x,y) will be $\sqrt{3}$ when $x = \sqrt{3}$ or $-\sqrt{3}$ and $y = 1$</td></tr>
</table>