10.4 Infinite Series

[Read Examples 1 to 5 p.658-669]

1. *New Concept*: "Infinite Series"

- Sigma Notation: $\sum_{n=1}^{\infty} a_n$ or $\sum_{k=1}^{\infty} a_k$
- Expanded Form: $a_1 + a_2 + a_3 + \cdots$

2. General Term of the Series $\sum_{m=1}^{\infty} a_m$: "a_m"

3. *Important*: Recognize the Difference between a Sequence & a Series

$$\{a_n\}_{n=1}^{\infty}; \qquad \sum_{n=1}^{\infty} a_n$$

4. *New Concept*:

Sequence of n^{th} Partial Sums of $\sum_{n=1}^{\infty} a_n$:

$$\{s_n\}_{n=1}^{\infty} \text{ where } s_n = \sum_{k=1}^{n} a_k .$$

5. Important Examples:

i. Geometric Series (G.S.):

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots .$$

1^{st} Term = a; Common Ratio = r

Formula for n^{th} Partial Sum of G.S.: $s_n = a.\frac{1-r^n}{1-r}$.

Examples:

a. $\sum_{n=1}^{\infty} \frac{5}{3^n} = \frac{5}{3} + \frac{5}{3^2} + \frac{5}{3^3} + \cdots$

b. $\sum_{k=1}^{\infty} 3(0.1)^k = .3 + .03 + .003 + \cdots$

ii. Telescoping Series (T.S.): (***Terms Go on Cancelling***)

Examples:

a. $\sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+1}\right] = \left[1 - \frac{1}{2}\right] + \left[\frac{1}{2} - \frac{1}{3}\right] + \left[\frac{1}{3} - \frac{1}{4}\right] + \cdots$.

b. $\sum_{n=1}^{\infty} \left[\frac{1}{2n-1} - \frac{1}{2n+3}\right] = \left[1 - \frac{1}{5}\right] + \left[\frac{1}{3} - \frac{1}{7}\right] + \left[\frac{1}{5} - \frac{1}{9}\right] + \left[\frac{1}{7} - \frac{1}{11}\right] + \cdots$

iii. Harmonic Series (H.S.):

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots .$$

6. Exercise:

Find the sequence of n^{th} partial sum of the above series and express it in compact form if possible.

7. Convergent Series:

i. An infinite series $\sum_{n=1}^{\infty} a_n$ **converges** if its sequence of n^{th} Partial Sums $\{s_n\}_{n=1}^{\infty}$ has a finite limit.

In this case if $\lim_{n\to\infty} s_n = s$, we say that s is the **sum of the series** $\sum_{n=1}^{\infty} a_n$.

ii. If the sequence of n^{th} Partial Sums $\{s_n\}_{n=1}^{\infty}$ does not have finite limit, we say that the series $\sum_{n=1}^{\infty} a_n$ **Diverges.**

8. Exercises: Check if the following series converge. If so, find the sum.

i. $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{7}{6^{k-1}}$; ii. $\frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \cdots$; iii. $\sum_{k=1}^{\infty} \frac{1}{9k^2+3k-2}$.

9. *New Concept*: "Repeating Decimals"

i. 0.333333…; ii. 1.002002002…..;
iii. 0.451141414… iv. 0.782178217821…

Exercises: Express 0.451141414… as a fraction.

Method:

i. Identify the repeating decimal numbers:

0.451+0.000141414…

ii. Express the part of repeating decimals as a Geometric Series:

0.000141414…= 0.00014+0.0000014+…..

$$= 14\left[\frac{1}{10^5} + \frac{1}{10^7} + \frac{1}{10^9} + \cdots\right]$$

iii. Apply Geometric Series Formula to find Sum.

iv. Simplify the whole answer to a fraction.

10. Not all the series have n^{th} Partial Sum in compact form.

Exercise: Can we find **n^{th} Partial Sum of** H.S. $\sum_{n=1}^{\infty} \frac{1}{n}$ in compact form. Does this series converge?

Exercise: Can we find **n^{th} Partial Sum of** $\sum_{n=1}^{\infty} (-1)^n$ in compact form. Does this series converge?

Exercise: Find the **n^{th} Partial Sum** of series $\sum_{n=1}^{\infty} x^n$. For what values of x this series is convergent?

Exercise: Find the **n^{th} Partial Sum** of the series $\sum_{k=1}^{\infty} \ln(1 - \frac{1}{(k+1)^2})$. Does this series converge?