10.2 Sequences
[Read Examples 1 to 10 p.648-656]

1. *Review*

Real Valued Function: $f : D \to R$.

where $D :=$ **Domain** of f.

Functions defined by a Rule (Formula): $f(x) = y$

i. $f(x) = \sqrt{x}$, $x \in D = [0, \infty)$

ii. $f(x) = x^2/(2+3x^4)$, $x \in D = R$

iii. $f(x) = (x-1)/\sin x$, $x \in D = R \setminus \{n\pi : n \text{ is an integer}\}$

2. *New Name* : "Special Case of Function"

"Sequence"

when

Domain of $f :=$ the set of **positive integers "I_+", i.e.,**

$f : I_+ \to R$.

Sequences defined by a Rule (Formula):

$f(n) = a_n,\ n \in I_+$

i. $f(n) = \sqrt{n}$, $n = 1, 2, 3, \ldots$

Here: $a_n = \sqrt{n}$

ii. $f(n) = n^2/(2+3n^4)$, $n = 1, 2, 3, \ldots$

Here: $a_n = n^2/(2+3n^4)$

iii. $f(n) = (n-1)/\sin n$, $n = 1, 2, 3, \ldots$

Here: $a_n = (n-1)/\sin n$

3. *Notations for a sequence*:

i. Bracket form: $\{a_n\}_{n=1}^{\infty}$.

ii. Expanded form: $a_1, a_2, a_3, \ldots$

3. *Terminology*:

i. **1st term of sequence $\{a_n\}_{n=1}^{\infty}$: a_1**

ii. **2nd term of sequence $\{a_n\}_{n=1}^{\infty}$: a_2**

iii. **General or nth term of $\{a_n\}_{n=1}^{\infty}$: a_n**

Ex 1. Find the general term of the sequences:

i. $\frac{5(1)^2}{3}, \frac{5(2)^3}{3}, \frac{5(3)^4}{3}, \ldots$

ii. $\frac{-2}{3.4}, \frac{3}{4.5}, \frac{-4}{5.6}, \ldots$

iii. $\frac{\ln 3}{2.3}, \frac{\ln 5}{2.3.4}, \frac{\ln 7}{2.3.4.5}, \ldots$

4. Graph of a Sequence:

A succession of isolated points in xy-Plane.

5. *Review*:

$$\lim_{x\to\infty} f(x).$$

6. *Limit of a Sequence*: $\lim_{n\to\infty} a_n = L.$

For $\varepsilon > 0$, there is a positive integer N such that

$$n > N \Rightarrow |a_n - L| < \varepsilon$$

7. **Similarity:** Most of the rules for finding $\lim_{n\to\infty} a_n$ are same as we note in case of finding $\lim_{x\to\infty} f(x)$,

e.g., *Techniques like*

algebraic manipulation,	simplification,
rationalization,	sandwich theorem,
specific formulas,	l'Hopital Rule

8. Method for Finding $\lim_{n\to\infty} a_n$:

i. Change: $a_n \to f(n) \to f(x)$.

ii. Find: $\lim_{x\to\infty} f(x)$ (Using known techniques)

iii. Answer: If $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$.

9. Exercises: Evaluate the following limits:

i. $\left\{\dfrac{4n^3 - 2n^8}{1+3n^8 - 7n}\right\}_{n=1}^{\infty}$, ii. $\left\{n \sin \dfrac{6\pi}{5n}\right\}_{n=1}^{\infty}$, iii. $\left\{\left(\dfrac{4+n}{7n}\right)^n\right\}_{n=1}^{\infty}$,

iv. $\{5k^2 e^{-k}\}_{k=1}^{\infty}$, v. $\left\{\sqrt{m^2 - m} + m\right\}_{m=1}^{\infty}$, vi. $\left\{\frac{5e^k}{3^k}\right\}_{k=1}^{\infty}$.

10. Additional Theorems on Limit:

i. $\lim_{n\to\infty} a_n = L \Leftrightarrow \lim_{n\to\infty} a_{2n-1} = L \,\&\, \lim_{n\to\infty} a_{2n} = L$

ii. $\lim_{n\to\infty} |a_n| = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$.

$\left[\lim_{n\to\infty} |a_n| = L \neq 0 \text{ does not imply } \lim_{n\to\infty} a_n = L \text{ :See } \{(-1)^n 5\}_{n=1}^{\infty}\right]$

11. Exercise: Evaluate the limit if it exists:

i. $\{3 + (-3)^n\}_{n=1}^{\infty}$, ii. $\left\{\sum_{k=1}^{n} \frac{k^3}{5n^4}\right\}_{n=1}^{\infty}$, iii. $\left\{\sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)\right\}_{n=1}^{\infty}$

12. (Use of Definite Integral) Evaluate $\lim_{n\to\infty} a_n$:

$$a_n = \frac{1}{n} \sum_{k=1}^{\infty} \frac{1}{1 + k/n}$$

13. (Sequence defined recursively) Do Ex. 36.

14. For $f(x) = (2n+1)/(n+2)$, and $\varepsilon = 0.01$, find L and N that satisfies the definition given in (6) above.