10.1 Taylor & Maclaurin Polynomial

[Read Examples 1 to 6 p.639-645]

1. ***Review***

<u>Local Linear Approximation of *f*(*x*) at *x* = *a*</u>:

$$p(x) = f(a) + f'(a)(x-a).$$

Properties [LLA]**:**

i. [***Degree***] $p(x)$ is a polynomial of degree 1.

ii. [**Graph**] The graph of $p(x)$ is a line touching the graph of $f(x)$ at a.

iii. [***Matching***] $p(a) = f(a)$; $p'(a) = f'(a)$

iv. [***Role***] $p(x) \approx f(x)$ when $x \approx a$.

Ex. 1: Find LLA of $f(x) = \sin x$ about $x = \pi/6$

2. ***<u>New Name for LLA</u>:***

1^{st} Taylor Polynomial for *f* about *x* = *a*:

$$p(x) = f(a) + f'(a)(x-a).$$

Ex.2: Find 1^{st} Taylor Polynomial for $f(x) = \sin x$ about $x = \pi/6$ and approximate $\sin 31^\circ$.

3. ***<u>New Concept</u>*****:**

n^{th} Taylor Polynomial of *f* about *x* = *a*:

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Properties:

i. [***Degree***] $p_n(x)$ is a polynomial of degree n.

ii. [**Graph**] The graph of $p_n(x)$ is a curve touching the graph of $f(x)$ at a.

iii. [***Matching***] $p_n(a) = f(a), p'(a) = f'(a), \ldots, p^{(n)}(a) = f^{(n)}(a)$

iv. [***Role***] $p(x) \approx f(x)$ when $x \approx a$.

4. ***<u>Special Cases</u>*****:**

i. 2^{nd} Taylor Polynomial of *f* about $x = a$
= Local Quadratic Approx. about *a*

ii. 3^{rd} Taylor Polynomial of *f* about $x = a$
= Local Cubic Approx. about *a*

Ex.3: Find 3^{rd} Taylor Polynomial for $f(x) = \sin x$ about $x = \pi/6$ and approximate $\sin 31^\circ$

5. ***<u>New Concept</u>*****:**

n^{th} Maclaurin Polynomial of *f*
= n^{th} Taylor Polynomial of *f* about $x = 0$.

Ex.3: (a) Find the Local Quadratic Approximation of $f(x) = \dfrac{1}{x+2}$ about $x = 0$.

(b) Also, find the n^{th} Maclaurin polynomial in sigma notation.

Ex.4: Find the 2^{nd} Maclaurin polynomial of $f(x) = \dfrac{1}{x+2}$. [Same as Ex. 3(a)]

6. ***<u>New Concepts</u>*****:**

i. Approximation Error due to n^{th} Taylor Polynomial of *f*:

$$R_n(x) = f(x) - p_n(x).$$

ii. Taylor's Formula with Remainder:

$$f(x) = p_n(x) + R_n(x).$$

7. ***<u>New Concept</u>*****:**

Remainder Estimation Theorem for n^{th} Taylor Poly. of f defined on an interval I about $x = a$

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}.$$

where $M = \max_{x\in I}\left[f^{(n+1)}(x)\right]$

8. ***<u>New Concept</u>*****:**

Problem: Use Remainder Estimation Theorem to approximate a given irrational number with accuracy up to 2 (or 3, 4, 5,….) decimal places.

Solution: For this do as follows:

i. Identify the function f required for approximation.

ii. Find suitable a for Taylor approx.

iii. Use Remainder Estimaion Theorem to find n so that
$$|R_n(x)| \le .005.$$

iv. Find $\boldsymbol{n^{th}}$ Taylor Polyn. of *f* about a

v. Approximate the given number using $\boldsymbol{n^{th}}$ Taylor Polyn. (where n is known).

Ex.5: Use the remainder estimation theorem to approximate $\sqrt{e}$ to 4 decimal places accuracy.

Ex.6: Find the Maclaurin polynomial of $f(x) = \ln(1+x)$. [Same as Ex. 3(a)]