King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences

Semester(Term 041)

MATH 311

Examination # 1

October, 9, 2004

(Time: 90 Minutes)

Solution

Name:______________________ ID#__________ Section ______

1. (i) State what is meant by an "inductive set". Give two examples.

A Set S of numbers is said to be inductive iff:

(a) $1 \in S$

(b) $x+1 \in S$ whenever $x \in S$.

Examples are: $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Q}$.

(ii) State the principle of of mathematical induction.

Suppose that for each $n \in \mathbb{N}$, $P(n)$ is a Statement about a natural number n such that:

(a) $P(1)$ is true;

(b) If $P(k)$ is true for $k \in \mathbb{N}$, then $P(k+1)$ is true.

Then $P(n)$ is true $\forall n \in \mathbb{N}$.

(iii) Prove that between any two real numbers there is a rational number.

Let x, y be real numbers such that $x < y$.

W.L.O.G. assume that $x > 0$. By the Archimedean Property, [Since $y-x > 0$], $\exists\, n \in \mathbb{N}$ such that $n(y-x) > 1$ i.e. $ny > nx+1$ --- (1)

Since $nx > 0$, $\exists\, m \in \mathbb{N}$ such that $m-1 \leq nx < m$ --- (2)

$\Rightarrow \quad m \leq nx+1 < m+1$ --- (3)

Combining the above 3 inequalities, we get

$$nx < m \leq nx+1 < ny$$

i.e. $$nx < m < ny$$

$$x < \frac{m}{n} < y$$

Take $r = \frac{m}{n}$, then $r \in \mathbb{Q}$ and $x < r < y$.

2. Consider the field of real numbers $\mathbb{R}$. Let $a, b, \in \mathbb{R}$. Prove each of the following:

(i) $ab = 0$ if and only if $a = 0$ or $b = 0$ or both.

($\Rightarrow$) Suppose that $a \cdot b = 0$.

We Consider two possibilities: $a = 0$ or $a \neq 0$

If $a = 0$, we are done.
If $a \neq 0$, then a^{-1} exists and

$$ab = 0 \Rightarrow a^{-1}(ab) = a^{-1} \cdot 0$$
$$\Rightarrow (a^{-1}a)b = 0 \Rightarrow 1 \cdot b = 0 \Rightarrow b = 0.$$

($\Leftarrow$) If $b = 0$ then $ab = a.0 = a(0+0) = a.0 + a.0 = ab + ab$
$$\Rightarrow ab = 0$$
Similarly if $a = 0$ we get $ab = 0$

(ii) If $a^2 + b^2 = 0$ then $a = b = 0$.

Suppose that $a^2 + b^2 = 0$.

[Recall, if $x \in \mathbb{R}$, then $x^2 \geq 0$. And if $x \neq 0$, then $x^2 > 0$].

Assume that $a \neq 0$. Then $a^2 > 0$.

$$\Rightarrow a^2 + b^2 > b^2 \geq 0$$

i.e. $a^2 + b^2 > 0$, which is a contradiction.

Similarly when we assume $b \neq 0$, we have a contradiction.

Hence $a = b = 0$.

(iii) If $v > 0$ and $a > b$, then $\frac{a}{v} > \frac{b}{v}$.

Note that $v^{-1} > 0$, Since $v > 0$; otherwise we have $v \cdot v^{-1} < 0 \neq 1$. $\rightarrow\leftarrow$.

Now, $a-b > 0$ Since $a > b$. Also $v^{-1} > 0$. Hence

$$(a-b)v^{-1} > 0$$

$$\Rightarrow \quad av^{-1} - bv^{-1} > 0$$

$$\Rightarrow \quad av^{-1} > bv^{-1}$$

$$\Rightarrow \quad \frac{a}{v} > \frac{b}{v}.$$

(iv) $|a + b| \leq |a| + |b|$.

$$\begin{aligned} 0 \leq |a+b|^2 &= (a+b)^2 \\ &= a^2 + 2ab + b^2 \\ &= |a|^2 + 2ab + |b|^2 \\ &\leq |a|^2 + 2|ab| + |b|^2 \\ &= |a|^2 + 2|a||b| + |b|^2 \\ &= (|a| + |b|)^2 \end{aligned}$$

Hence $|a+b| \leq |a| + |b|$.

3. **Show that the set of complex numbers forms a field.**

(1) $(\mathbb{C}, +)$ is a Commulative group.

(i) The operation "+" is closed: for $x = a+bi, y = c+di \in \mathbb{C}$, we have
$x+y = (a+bi)+(c+di) = (a+c)+(b+d)i \in \mathbb{C}$.

(ii) It is easy to see that "+" is associative (verify!)

(iii) "+" is Commutative (verify!)

(iv) The identity element w.r.t. "+" is $0+0i \in \mathbb{C}$, with $x+0=0+x=x \ \forall x \in \mathbb{C}$

(v) The additive inverse exists, since for $x \in \mathbb{C}$, where $x = a+bi$ $\exists y = -a-bi \in \mathbb{C}$ such that $x+y = (a+bi)+(-a-bi) = 0+0i = 0$

(2) $(\mathbb{C}^*, \cdot)$ is a Commulative group

(i) The operation "·" is closed: for $x = a+bi, y = c+di \in \mathbb{C}$, we have
$x \cdot y = (a+bi)(c+di) = (ac-bd)+(ad+bc)i \in \mathbb{C}$

(ii) The operation "·" is associative (verify!)

(iii) 〃 〃 〃

(iv) The multiplicative identity is $1 = 1+0i \in \mathbb{C}$.

(v) The multiplicative inverse exists. To see this,
Let $x = a+bi \in \mathbb{C}$. If $y = c+di$ is the inverse of x, then $xy = 1$
$\Rightarrow (a+bi)(c+di) = 1 \Rightarrow ax-by+(bx+ay)i = 1$
$\Rightarrow \left\{\begin{matrix} ac-bd=1 \\ bc+ad=0 \end{matrix}\right\}$ Solving this system, we get $c = \frac{a}{a^2+b^2}$, $d = \frac{-b}{a^2+b^2}$
$\therefore y = \frac{a}{a^2+b^2} + \left(\frac{-b}{a^2+b^2}\right)i \in \mathbb{C}$, is the inverse of $x = a+bi$.

(3) $(\mathbb{C}^*, +, \cdot)$ satisfies the distributive laws, i.e. for $x, y, z \in \mathbb{C}$.
$x(y+z) = xy+xz$,
$(y+z)x = yx+zx$.

4. (i) Find the modified inductive set for

$$P(n): \quad 2n+1 < 2^n, \qquad \text{then prove it.}$$

$S = \{3, 4, 5, \ldots\}$ is the modified inductive set for $P(n)$.

To Prove $P(n)$, $\forall n \geq 3$

(a) $n = 3$: $L.H.S. = 2(3)+1 = 7 < 8 = 2^3 = R.H.S.$

$\Rightarrow P(3)$ is true.

(b) Suppose $P(k)$ is true, $k \geq 3$. i.e. $2k+1 < 2^k$. Then

$$\begin{aligned} 2(k+1)+1 = 2k+2+1 = (2k+1)+2 &< 2^k + 2, \text{ by assumption} \\ &< 2^k + 2^k \\ &= 2^{k+1} \end{aligned}$$

Hence $P(k+1)$ is true.

This shows that $P(n)$ is true $\forall n \geq 3$.

(ii) Use mathematical induction to prove the formula

$$\frac{d^n}{dx^n}[\ln(1+x)] = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}.$$

(a) $n = 1$:

$$L.H.S = \frac{d}{dx}\left[\ln(1+x)\right] = \frac{1}{1+x} = \frac{(-1)^{1-1}(1-1)!}{1+x} = R.H.S.$$

$\Rightarrow P(1)$ is true.

b) Suppose $P(k)$ is true, $k \geq 1$. i.e. $\dfrac{d^k}{dx^k}\left[\ln(1+x)\right] = \dfrac{(-1)^{k-1}(k-1)!}{(1+x)^k}$.

Then

$$\frac{d^{k+1}}{dx^k}\left[\ln(1+x)\right] = \frac{d}{dx}\left(\frac{d^k}{dx^k}\left[\ln(1+x)\right]\right) = \frac{d}{dx}\left[\frac{(-1)^{k-1}(k-1)!}{(1+x)^k}\right], \text{ by assumption}$$

$$= \frac{-(-1)^{k-1}(k-1)!\,k\,(1+x)^{k-1}}{(1+x)^{2k}}$$

$$= \frac{(-1)^k\,k!}{(1+x)^{k+1}}$$

Hence $P(k+1)$ is true, and so $P(n)$ is true $\forall n$.

5. Let $f(x) = \begin{cases} \dfrac{x-4}{\sqrt{x}-2} & , \ x \geq 0, \quad x \neq 4 \\ 4 & , \ x = 4. \end{cases}$

Given $\epsilon = 0.01$. Determine a value δ so that the statement $|f(x) - f(4)| < \epsilon$ whenever $0 < |x-4| < \delta$, is valid. Thus the function $f(x)$ is continuous at $x = 4$.

For $\epsilon = 0.01$, we need to find $\delta > 0$ such that

$$0 < |x-4| < \delta \implies |f(x) - f(4)| < \epsilon$$

i.e. $$0 < |x-4| < \delta \implies \left|\frac{x-4}{\sqrt{x}-2} - 4\right| < 0.01$$

Note that if $x \neq 0$, then $f(x) = \dfrac{x-4}{\sqrt{x}-2} = \dfrac{(\cancel{\sqrt{x}-2})(\sqrt{x}+2)}{\cancel{\sqrt{x}-2}} = \sqrt{x}+2$.

So, we need to find $\delta > 0$, such that

$$0 < |x-4| < \delta \implies |\sqrt{x}+2-4| < 0.01$$

i.e. $$0 < |x-4| < \delta \implies |\sqrt{x}-2| < 0.01$$

Now, $$|\sqrt{x}-2| < 0.01$$

$$\iff -0.01 < \sqrt{x}-2 < 0.01$$

$$\iff 1.99 < \sqrt{x} < 2.01$$

$$\iff 3.9601 < x < 4.0401$$

$$\iff -.0399 < x-4 < .0401$$

Taking $\delta = \min\{.0399, .0401\} = .0399$, we get

$$0 < |x-4| < \delta \implies |f(x) - f(4)| < 0.01.$$

6. Answer True (✓) or False (X):

(i) The set of all 2×2 matrices forms a field. (X)

Multiplicative inverse is not satisfied. For example:

$\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$ has no inverse.

(ii) $\sqrt{2}$ is a rational number. (X)

$\sqrt{2}$ can not be expressed in the form $\frac{a}{b}$, $a, b \in \mathbb{R}$, $b \neq 0$.

[See the Proof of this]

(iii) The set of prime numbers is an inductive set (X)

3 is Prime but $3+1=4$ is not Prime.

(iv) The function $f(x) = \frac{1}{x}\sin x$ is continuous every where. (X)

$f(x)$ is not defined at $x=0$

(v) The system $(Z_4, \bigoplus, \bigodot)$ forms a field. (X)

2 has no multiplicative inverse in Z_4.

i.e. $\nexists\, y \in Z_4$ such that $2y = 1$.