King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences

Math 101 (Calculus I)

Major Exam # 1

Semester I, 2004-2005 (041)

Name: Solution Section #: ______

ID #: ______________________

- Show complete work for full credit.
- Use of graphic calculators and mobile phones or any other equipment is not allowed in this exam.

Question	Score
1	
2	
3	
4	
Total:	

1. Find each of the following limits, if it exists. (5 points each)

(a) $\lim_{x\to 3}(x+2)\cdot\frac{3-x}{|3-x|}$.

$$|3-x| = \begin{cases} 3-x & \text{if } 3-x \geq 0 \\ -(3-x) & \text{if } 3-x<0 \end{cases}$$

$$= \begin{cases} 3-x & \text{if } x\leq 3 \\ -(3-x) & \text{if } x>3 \end{cases}$$

$\boxed{2}$ $\lim_{x\to 3^+}(x+2)\frac{3-x}{|3-x|} = \lim_{x\to 3^+}(x+2)\frac{3-x}{-(3-x)} = -\lim_{x\to 3^+} x+2 = \boxed{-(5)}$

$\boxed{3}$ $\lim_{x\to 3^-}(x+2)\frac{3-x}{|3-x|} = \lim_{x\to 3^-}(x+2)\frac{3-x}{3-x} = \lim_{x\to 5^-} x+2 = \boxed{5}$

$\boxed{4}$ $\lim_{x\to 3^-}(x+2)\frac{3-x}{|3-x|} \neq \lim_{x\to 3^+}(x+2)\frac{3-x}{|3-x|}$

$\Rightarrow \lim_{x\to 3}(x+2)\frac{3-x}{|3-x|}$ DNE

(b) $\lim_{x\to 2}\frac{x^4-16}{x^2-x-2}$. $\left(\frac{0}{0}\right)$

$$= \lim_{x\to 2}\frac{(x^2-4)(x^2+4)}{(x-2)(x+1)} = \lim_{x\to 2}\frac{\cancel{(x-2)}(x+2)(x^2+4)}{\cancel{(x-2)}(x+1)} = \lim_{x\to 2}\frac{(x+2)(x^2+4)}{x+1}$$

$$= \frac{4\cdot 8}{3} = \boxed{\frac{32}{3}}$$

(c) $\lim_{h\to 0}\frac{\sqrt{2+h}-\sqrt{2}}{h} = \lim_{h\to 0}\frac{\sqrt{2+h}-\sqrt{2}}{h}\cdot\frac{\sqrt{2+h}+\sqrt{2}}{\sqrt{2+h}+\sqrt{2}}$

$$= \lim_{h\to 0}\frac{(2+h)-2}{h(\sqrt{2+h}+\sqrt{2})} = \lim_{h\to 0}\frac{h}{h(\sqrt{2+h}+\sqrt{2})}$$

$$= \lim_{h\to 0}\frac{1}{\sqrt{2+h}+\sqrt{2}} = \boxed{\frac{1}{2\sqrt{2}}}$$

(d) $\lim_{x\to -\infty}(-3x+1)^3(2x+1)^2(x+1)$.

$$= \lim_{x\to -\infty}(-3x)^3(2x)^2x = \lim_{x\to -\infty} -108x^6 = \boxed{-\infty}$$

(e) $\lim_{x\to 0^+}\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right) = \lim_{x\to 0^+}\sqrt{x} - \lim_{x\to 0^+}\frac{1}{\sqrt{x}}$

$$= 0-(+\infty) = \boxed{-\infty}$$

(f) $\lim_{x\to 0} \frac{\sin x - 7x}{x\cos x} = \lim_{x\to 0} \frac{\sin x}{x\cos x} - \lim_{x\to 0} \frac{7\cancel{x}}{\cancel{x}\cos x}$

$= \lim_{x\to 0} \frac{\sin x}{x}\cdot\frac{1}{\cos x} - \lim_{x\to 0} \frac{7}{\cos x}$

$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0}\frac{1}{\cos x} - \lim_{x\to 0}\frac{7}{\cos x}$

$= 1\cdot 1 - 7 = \boxed{-6}$

(g) $\lim_{x\to+\infty} \frac{2x + x\sin 3x}{5x^2 - 2x + 1}$.

$= \lim_{x\to+\infty} \frac{2x}{5x^2-2x+1} + \lim_{x\to+\infty} \frac{x\sin 3x}{5x^2-2x+1}$

$= \lim_{x\to+\infty} \frac{\frac{2}{x}}{5-\frac{2}{x}+\frac{1}{x^2}} + \lim_{x\to+\infty} \frac{\frac{\sin 3x}{x}}{5-\frac{2}{x}+\frac{1}{x^2}}$

$= \frac{0}{5-0+0} + \frac{0}{5-0+0} = \boxed{0}$

NOTE $\lim_{x\to-\infty} \frac{\sin 3x}{x} = 0$ why?

let $t = \frac{1}{x}$

$x\to+\infty \Rightarrow t\to 0$

$\therefore \lim_{x\to-\infty} \frac{\sin 3x}{x} = \lim_{t\to 0} t\sin\frac{3}{t}$

Now $-1 \le \sin\frac{3}{t} \le 1 \Rightarrow -|t| \le t\sin\frac{3}{t} \le |t|$

$\lim_{t\to 0} -|t| = \lim_{t\to 0} |t| = 0$ $\therefore$ by Squeezing Th

$\lim_{t\to 0} t\sin\frac{3}{t} = 0$.

(h) $\lim_{x\to 3} \frac{\sin(\pi x)}{x-3}$. (Hint: let $t = \pi x - 3\pi$)

$t = \pi x - 3\pi$ $\quad x\to 3 \Rightarrow t\to 0$

$t = \pi(x-3)$

$\boxed{\frac{t}{\pi} = x - 3}$

Hence $\lim_{x\to 3} \frac{\sin(\pi x)}{x-3} = \lim_{t\to 0} \frac{\sin(t+3\pi)}{\frac{t}{\pi}} = \pi \cdot \lim_{t\to 0} \frac{\sin(t+\pi)}{t} = \pi \lim_{t\to 0} \frac{-\sin t}{t}$

$= -\pi \lim_{t\to 0} \frac{\sin t}{t}$

$= -\pi\cdot 1 = \boxed{-\pi}$

Note: $\sin(t+\pi) = \sin t\cos\pi + \cos t \overset{0}{\sin\pi} = \boxed{-\sin t}$

(i) $\lim_{x\to+\infty} \frac{\sin x}{x}$. (Hint: Use the Squeezing Theorem)

let $t = \frac{1}{x}$ $\quad x\to+\infty$

$t\to 0$

$\therefore \lim_{x\to+\infty} \frac{\sin x}{x} = \lim_{t\to 0} t\sin(\frac{1}{t}) = \boxed{0}$ (why?)

$-1 \le \sin(\frac{1}{t}) \le 1$

$-|t| \le t\sin(\frac{1}{t}) \le |t|$

$\lim_{t\to 0} -|t| = \lim_{t\to 0} |t| = 0$

$\therefore$ by the Squeezing Th

$\lim_{t\to 0} t\sin(\frac{1}{t}) = 0$

(j) $\lim_{x\to+\infty} \sqrt[3]{\frac{8+x^2}{x(x+1)}} = \sqrt[3]{\lim_{x\to+\infty} \frac{8+x^2}{x^2+1}} = \sqrt[3]{\lim_{x\to+\infty} \frac{\frac{8}{x^2}+1}{1+\frac{1}{x^2}}} = \sqrt[3]{\frac{0+1}{1+0}} = \boxed{1}$

2. (a) Use the limit definition to prove that

$$\lim_{x\to 15}\sqrt{x+1}=4.$$

Given $\epsilon>0$, we must show that there is a positive # δ such that $\left|\sqrt{x+1}-4\right|<\epsilon$ if $0<|x-15|<\delta$

(5 points)

$$\left|\sqrt{x+1}-4\right|=\left|\sqrt{x+1}-4\cdot\frac{\sqrt{x+1}+4}{\sqrt{x+1}+4}\right|=\frac{|x+1-16|}{\sqrt{x+1}+4}=\frac{|x-15|}{\sqrt{x+1}+4}<\frac{|x-15|}{4}<\frac{\delta}{4}$$

$\therefore\ \epsilon=\frac{\delta}{4}\implies \boxed{\delta=4\epsilon}$

(Note: You may assume that $14\le x<15$, then conclude that

$$\frac{1}{\sqrt{x+1}+4}<\frac{1}{\sqrt{15}+4}\implies\left|\sqrt{x+1}-4\right|<\ldots<\frac{|x-15|}{\sqrt{15}+4}<\frac{\delta}{\sqrt{15}+4}\implies\boxed{\delta=(\sqrt{15}+4)}$$

(b) Find all asymptotes for the graph of

$$f(x)=\frac{4x-3}{\sqrt{x^2+1}}.$$

$\boxed{1}$ $\nexists$ vertical asymptotes since $x^2+1=0$ has no real solution.

(12 points)

$\boxed{2}$ $\displaystyle\lim_{x\to+\infty}\frac{4x-3}{\sqrt{x^2+1}}=\lim_{x\to+\infty}\frac{(4x-3)/x}{\sqrt{x^2+1}/x}=\lim_{x\to+\infty}\frac{4-\frac{3}{x}}{\sqrt{x^2+1}/|x|}=\lim_{x\to+\infty}\frac{4-\frac{3}{x}}{\sqrt{x^2+1}/\sqrt{x^2}}$

$\displaystyle=\lim_{x\to+\infty}\frac{4-\frac{3}{x}}{\sqrt{1+\frac{1}{x^2}}}=\frac{4-0}{\sqrt{1+0}}=4$

$\boxed{3}$ $\displaystyle\lim_{x\to-\infty}\frac{4x-3}{\sqrt{x^2+1}}=\lim_{x\to-\infty}\frac{(4x-3)/-x}{\sqrt{x^2+1}/-x}=\lim_{x\to-\infty}\frac{-4+\frac{3}{x}}{\sqrt{x^2+1}/|x|}$

$\displaystyle=\lim_{x\to-\infty}\frac{-4+\frac{3}{x}}{\sqrt{x^2+1}/\sqrt{x^2}}=\lim_{x\to-\infty}\frac{-4+\frac{3}{x}}{\sqrt{1+\frac{1}{x^2}}}=\frac{-4+0}{\sqrt{1+0}}=\boxed{-4}$

②+③ $\implies$ $\boxed{y=-4}$ and $\boxed{y=4}$ are horizontal asymptotes

(c) Prove that the equation

$$x^5-3x^4-2x^3-x+1=0$$

has a solution between 0 and 1.

(5 points)

① $f(0)=\boxed{0}$

② $f(1)=1-3-2-1+1=\boxed{-4}$

③ $x^5-3x^4-2x^3-x+1$ is a continuous function on $[0,1]$

Hence by the intermediate Value Theorem there is a real number $c\in[0,1]$ s.t $\boxed{f(c)=0}$. Thus c is a solution of.

3. (a) Find all number(s) at which f is discontinuous, where

$$f(x) = \begin{cases} -x^2 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ -\dfrac{1}{x} & \text{if } x > 1 \end{cases}$$

(8 points)

$\boxed{1}$ $-x^2$ is continuous for all $x < 1$

$\boxed{2}$ $\frac{-1}{x}$ is continuous for all $x > 1$

$\boxed{3}$ $\left.\begin{array}{l} \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} -x^2 = -1 \\ \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} -\frac{1}{x} = -1 \end{array}\right\} \Rightarrow \boxed{\lim_{x\to 1} f(x) = -1}$

but $f(1) = 2$. Thus $\lim_{x\to 1} f(x) \neq f(1)$

Hence f is continuous everywhere except at $\boxed{x = 1}$.

(b) Show that $f(x) = \sqrt{16 - x}$ is continuous on the interval $(-\infty, 16]$. (5 points)

$\boxed{1}$ Let $c \in (-\infty, 16)$

$\lim_{x\to c} f(x) = \lim_{x\to c} \sqrt{16 - x} = \sqrt{\lim_{x\to c} 16 - x} = \sqrt{16 - c} = f(c)$

$\boxed{2}$ $\lim_{x\to 16^-} f(x) = \lim_{x\to 16^-} \sqrt{16 - x} = \sqrt{\lim_{x\to 16^-} x - 16} = \sqrt{0} = 0 = f(16)$

$\therefore$ $\boxed{1}$ & $\boxed{2}$ $\Rightarrow$ $f(x)$ is continuous on $(-\infty, 16]$.

(c) Suppose that f is a continuous function, $\lim_{x\to 4} f(x) = 0$ and $\lim_{x\to 4} g(x) = -3$. Find $\lim_{x\to 4} \dfrac{xg(x)}{\sqrt{f(x) + 1}}$. (5 points)

$$\lim_{x\to 4} \frac{xg(x)}{\sqrt{f(x)+1}} = \frac{\lim_{x\to 4} x \cdot \lim_{x\to 4} g(x)}{\lim_{x\to 4} \sqrt{f(x)+1}} = \frac{\lim_{x\to 4} x \cdot \lim_{x\to 4} g(x)}{\sqrt{\lim_{x\to 4} f(x) + \lim_{x\to 4} 1}} = \frac{4 \cdot (-3)}{\sqrt{0+1}} = \boxed{-12}$$

(d) Consider the function $f(x) = x^2 + 1$ and the point $P(2, f(2))$.

(i) Find the slope of the graph of $y = f(x)$ at the point P.

(ii) Find the instantaneous rate of change of y with respect to x at the general point $x = x_0$.

(10 points)

(i) $m_{curve} = \lim_{x \to 2} \frac{f(x) - f(2)}{x-2} = \lim_{x \to 2} \frac{x^2+1-5}{x-2}$

$= \lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}}$

$= \lim_{x \to 2} x+2 = \boxed{4}$

(ii) $r_{inst} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{(x^2+1) - (x_0^2+1)}{x - x_0}$

$= \lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0} = \lim_{x \to x_0} \frac{\cancel{(x-x_0)}(x+x_0)}{\cancel{(x-x_0)}}$

$= \lim_{x \to x_0} x + x_0 = \boxed{2x_0}$

4. Extra Credits (10 points)

(a) If $\lim_{x \to a}[f(x) + g(x)] = 4$ and $\lim_{x \to a}[f(x) - g(x)] = 1$, find $\lim_{x \to a} f(x)g(x)$.

$\boxed{1}$ $\lim_{x \to a} (f(x)+g(x))^2 = \left(\lim_{x \to a} (f(x)+g(x))\right)^2 = 4^2 = 16$

$\boxed{2}$ $\lim_{x \to a} (f(x)-g(x))^2 = \left(\lim_{x \to a} (f(x)-g(x))\right)^2 = 1^2 = 1$

$\boxed{3}$ $(f(x)+g(x))^2 - (f(x)-g(x))^2 = \left[(f(x))^2 + 2f(x)g(x) + (g(x))^2\right] - \left[(f(x))^2 - 2f(x)g(x) + (g(x))^2\right] = 4f(x)g(x)$

$\Rightarrow \lim_{x \to a} (f(x)+g(x))^2 - (f(x)-g(x))^2 = \lim_{x \to a} 4f(x)g(x)$

$\Rightarrow 16 - 1 = 4 \lim_{x \to a} f(x)g(x)$

$\Rightarrow \boxed{\frac{15}{4} = \lim_{x \to a} f(x)g(x)}$

(b) Evaluate $\lim_{x \to 1} \frac{\sqrt[3]{x} - 1}{\sqrt{x}+1}$.

$= \frac{1-1}{1+1} = \frac{0}{2}$

$= \boxed{0}$