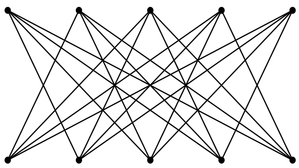
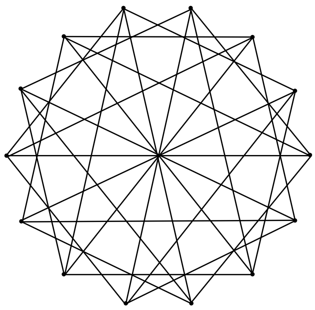
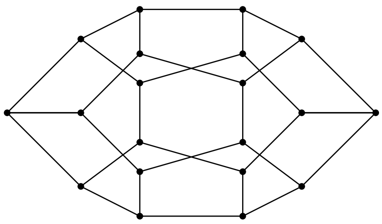
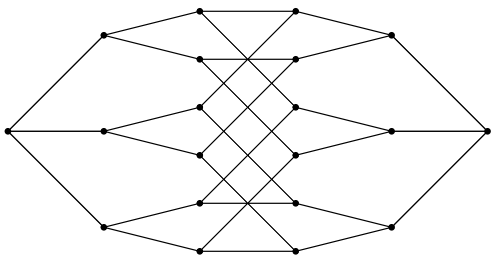
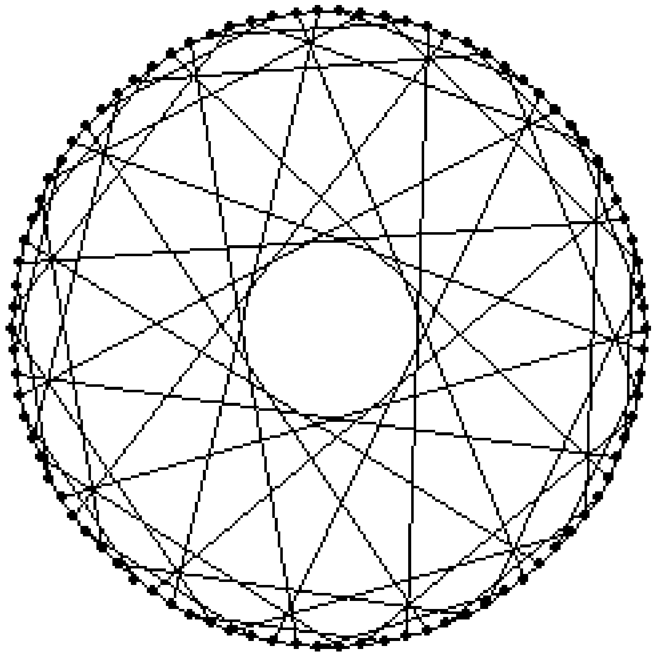
Imprimitive Distance-Transitive Graphs











Monther Rashed Alfuraidan

Imprimitive Distance-Transitive Graphs

Monther Rashed Alfuraidan

This thesis is presented for
the Degree of Doctor of Philosophy of
Michigan State University

Department Of Mathematics
Michigan State University
East Lansing, MI 48824
USA

02,16 2004

To my parents and my wife

Abstract

We present the conjectured list of primitive distance transitive graphs of diameter at least 3. For each graph G on this list we attempt to classify all imprimitive distance-transitive graphs that are antipodal covers of G or have G as halved graph. This classification is successful in all cases except when G is a generalized $2d$-gon, where the distance-transitive antipodal covers of diameter $2d$ remain unclassified.

Acknowledgements

Praise be to ALLAH, Lord of the Word, the Almighty, with Whose gracious help it was possible to accomplish this task.

First and foremost, I have to thank my thesis advisor, Professor Jonathan I. Hall, whom I consider to be my father in a foreign land. He has given me valuable insight, both personally and academically. I am indebted to him for his continual, unwavering support of my work and his incredible patience with me, especially during stressful times like my journey into fatherhood.

I am also grateful to my thesis committee in the Mathematics Department at Michigan State University: Professors Ulrich Meierfrankenfeld, Bruce Sagan, Jeanne Wald, and Michael Frazier. I am particularly thankful for the hours and hours Michael and Jeanne spent practicing and preparing me for my qualifying exams.

I am forever indebted to Professor Mohammed Albar, former Chairman of the Mathematics Department at King Fahd University of Petroleum and Minerals in Saudi Arabia. He is the first person who saw my potential as a mathematician and encouraged me to change my major from engineering to mathematics. He has always supported my academic and teaching efforts. I am also indebted to KFUPM for sponsoring me and providing me with the financial resources necessary to complete my studies in the United States.

I would be remiss if I did not mention the support of my mother, Munairh Ali Alsaqar. Her love, care and prayers have given me the strength to endure through times of trouble. I am especially grateful for her patience; it was very difficult for her to be separated from her youngest child for over 6 years while I studied abroad. I only hope that she is proud to be called the mother of Dr. Monther Alfuraidan.

He died when I was 2 years old, but I am so grateful for my father's legacy to me. I have no memory of him, but I have known him all my life. Everyone who has ever spoken of him has told me what a caring, generous, intelligent, sensitive man he was. I only hope that I can live up to his memory, and I pray to see him someday in heaven.

The rest of my family has also been very supportive with prayers and best wishes for my success. I am grateful every day for my brothers, Mohammed and Salah, and my sister Fatima. They have waited a long time for my return, and I am happy to see them soon.

To my brothers in the United States, Muhannad Yousef and Awad Alqahtani, thank you for being my family away from home. You reminded me of my roots and kept our culture alive.

To all of my friends, both in the United States and in Saudi Arabia (you know who you are), I love you all for your continual faith in me and cheerful phone calls.

Finally, to my beautiful wife, Awatif Mohammed Alnowibit, I owe everything I am and everything I have to her. She has bravely sacrificed her life, her education and her family to be with me as I completed my studies in a foreign land, completely unknown to her. And in that land, she gave me my first-born son, Rashed. Together, they are all things wonderful and I love them more than life.

Contents

List of Figures

List of Tables

List of Symbols

Group Theory

A, B	groups
X	set
$A \wr B$	wreath product of A of B
$B \leq A$	B is a subgroup of A
$B \trianglelefteq A$	B is a normal subgroup of A
$N_A(B)$	normalizer of B in A, where $B \leq A$
$soc(A)$	socle of A
A_x	stabilizer of x in A
A_X	setwise stabilizer of X in A
$Sym(X)$	symmetric group on X
S_n	symmetric group of degree n
A_n	alternating group of degree n
$\mathbb{Z}_n$	additive group of integers modulo n
$M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$	Mathieu group
$PGL(n, q)$	projective general linear group
$PSL(n, q$	projective special linear group
$P\Gamma L(n, q)$	semilinear projective group
$AGL(n, q)$	affine group
$A\Gamma L(n, q)$	semilinear affine group
$GF(q), \mathbb{F}_q$	finite filed with q elements
$V(n, q)$	n-dimensional linear space over $GF(q)$
$PG(n, q)$	projective geometry
$AG(n, q)$	affine geometry
$\sim$	equivalence relation

Graph and Design Theory

G, H	graphs
I	incidence relation
$\mathcal{B}$	set of blocks
$\mathcal{L}$	set of lines
G_i	distance graph
$V(G)$	vertex set of G
$E(G)$	edge set of G
$G(u)$	neighborhood of u in G
$L(G)$	line graph of G
$Aut(G)$	full automorphism group of G
k	valency of a regular graph

$d, d(G), D$	diameter of G
$d(u, v)$	distance between u and v
$g, g(G)$	girth of G
$\overline{G(V, E)}$	complement of G
$\overline{G}$, $\overline{G}(V, E)$, G/P	quotient graph of G (with respect to P)
$G^+, G^-, \frac{1}{2}G$	halved graphs of G
K_n	complete graph with n vertices
$K_{n,n}$	complete bipartite graph with n vertices in each part
$K_{t_1,t_2,\ldots,t_n}$	complete n-partite graph with t_i vertices in the i part
C_n	cycle with length n
P_n	path with length n
P	partition of $V(G)$
r	covering index (fibre size)
a_i, b_i, c_i	intersection numbers
$i(G)$	intersection array of the distance-regular graph G
DRG	distance-regular graph
DTG	distance-transitive graph
$J(n, k)$	Johnson graph
$2J(n, k)$	double Johnson graph
$J_q(n, k)$	Grassmann graph
$2J_q(n, k)$	Grassmann graph
$Q_n, H(n, 2)$	n-cube
$T(n)$	triangular graph
$\otimes$	direct products of graphs
$\sim$	is adjacent to
$H(n, q)$	Hamming graph
O_k	odd graph
$2O_k$	double odd graph
$Alt(n, q)$	alternating forms graph over $\mathbb{F}_q$
$H_q(n, q)$	bilinear forms graph
$Her(n, q)$	Hermitian forms graph over $\mathbb{F}_q$
$S(t, k, v)$	Steiner system

Chapter 1

Introduction

1.1 Basic Definitions

1.1.1 Graph theory

A **graph** $G = (V, E)$ is a finite nonempty set of elements called **vertices** together with a (possibly empty) set of elements called **edges**. Each edge is identified with a pair of vertices. The vertices v_i and $v_j$associated with an edge e are called the end vertices of e. The edge e is then denoted by $e = (v_i, v_j)$ or simply $e = v_iv_j$. If $e = v_iv_i$, then the edge e is called a self-loop at vertex v_i. All edges having the same pair of end vertices are called parallel edges. A graph is **simple** if it has no parallel edges or self-loops. In this thesis, we will always be considering simple graphs. An **r-matching** in a graph G is a set of r edges, no two of which have a vertex in common.

As usual, $|X|$ denotes the number of elements in a set X. For a graph G, if $|V| = n$ and $|E| = m$, then G is called an (n, m) graph; the number n is also referred to as the **order** of G and m as the **size** of G. A graph with no edges is called an empty graph. A graph with no vertices (and hence no edges) is called a null graph.

The edge $e = v_iv_j$ is said to join the vertices v_i and v_j. If $e = v_iv_j$ is an edge of a graph G, then v_i and v_j are **adjacent** vertices, while v_i and e are incident, as are v_j and e. Furthermore, if e_1 and e_2 are distinct edges of G incident with a common vertex, then e_1 and e_2 are adjacent edges. The number of edges incident on a vertex v_i is called the degree of the vertex, and it is denoted by $deg(v_i)$. A vertex of degree 1 is called a pendant vertex. A vertex of degree 0 is called an isolated vertex.

Given a nonempty graph G, the **line graph** $L(G)$ of G is the graph whose vertices can be put in one to one correspondence with the edges of G in such a way that two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent.

A graph G is said to be **regular** if $deg(u) = deg(v)$ for all $u, v \in V(G)$. It is k-regular if $deg(v) = k$ for all $v \in V(G)$; in this case, the number k is also referred to as the **valency**(degree) of G. A 3-regular graph is frequently described as a cubic graph, or sometimes as a trivalent graph.

A regular graph with v points and valency k is called **edge-regular** with parameters

(v, k, λ) if any two adjacent vertices have exactly λ common neighbors.

A graph $G' = (V', E')$ is a **subgraph** of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$ such that an edge $v_i v_j$ is in E' only if v_i and v_j are in V'. If v_i is a vertex of a graph $G = (V, E)$, then the graph $G - v_i = (V', E')$ is the graph obtained after removing from G the vertex v_i and all the edges incident to v_i. If e_i is an edge of a graph $G = (V, E)$, then $G - e_i$ is the subgraph of G obtained after removing from G the edge e_i.The graph $\overline{G} = (V, E')$ is called the **complement** of graph $G = (V, E)$ if the edge $v_i v_j$ is in E' if and only if it is not in E. Hence, if G is an (n, m) graph, then $\overline{G}$ is an $(n, \overline{m})$ graph, where $m + \overline{m} = \binom{n}{2}$.

A **walk** in a graph $G = (V, E)$ is a finite alternating sequence of vertices and edges $v_0, e_1, v_1, e_2, ..., v_{k-1}, e_{k-1}, v_k$ beginning and ending with vertices such that v_{i-1} and v_i are the end vertices of the edge e_i, $1 \leq i \leq k$. A walk is open or closed depending on whether its end vertices are distinct or are not distinct. A **trail** is a walk in which no edge is repeated, while a **path** is a walk in which no vertex is repeated. A closed trail is a **cycle** if all its vertices except the end vertices are distinct. The number of edges in a path (cycle) is called the length of the path (cycle). A graph G is **connected** if there exists a path between every pair of vertices in G. Otherwise, we say G is disconnected. A maximal connected subgraph of a disconnected graph G is called a **component** of G.

The **distance** $d(u, v)$ between two vertices u and v in G is the length of a shortest path joining them if any; otherwise $d(u, v) = \infty$. A shortest $u - v$ path is often called a **geodesic**. The **diameter** $d(G)$ (or simply d) of a connected graph G is the length of any longest geodesic. The **girth** $g(G)$ (or simply g) of G is the length of the shortest cycle in G. The k-regular graph of smallest positive integer order with girth g is called an **(k,g)-cage**. The $(3, g)$-cages are commonly referred to simply as g-cages.

A $(k, 2d+1)$-cage is better known as a **Moore graph** and a $(k, 2d)$-cage as a **generalized polygon** where d is the diameter of the cage.

The **complete graph** K_n has every pair of its n vertices adjacent. Thus K_n has $\binom{n}{2}$ edges and is a $(n-1)$-regular.

A graph G is **n-partite**, $n \geq 1$, if it is possible to partition $V(G)$ into n subsets $V_1, V_2, ..., V_n$ such that every edge of G joins a vertex of V_i to a vertex of V_j, $i \neq j$. A 1-partite graph of order n is the complement of the complete graph K_n. For $n = 2$, such graphs are called **bipartite** graphs. **A complete n-partite** graph G is an n-partite with partite sets $V_1, V_2, ..., V_n$ such that $uv \in E(G)$ for all $u \in V_i$ and $v \in V_j$ with $i \neq j$. If $|V_i| = t_i$, then this graph is denoted by $K(t_1, t_2, ..., t_n)$. For $n = 2$, the **complete bipartite** graph with partite sets V_1 and V_2, where $|V_1| = m$ and $|V_2| = n$, is then denoted by $K(m, n)$.

Two graphs G_1 and G_2 are said to be **isomorphic** if there exists a one to one mapping ϕ, called an isomorphism, from $V(G_1)$ onto $V(G_2)$ such that ϕ preserves adjacency, that is $uv \in E(G_1)$ if and only if $\phi u \phi v \in E(G_2)$.

1.1.2 Design theory

A **design** is an ordered pair $(X, \mathfrak{B})$ with point set X and set of blocks $\mathfrak{B}$ such that $\mathfrak{B}$ is a collection of subsets of X. More generally, it is an ordered triple $(X, \mathfrak{B}, I)$, where X and $\mathfrak{B}$ are sets and I is a subset of $X \times \mathfrak{B}$, called the incidence relation.

The **point graph** of the design $(X, \mathfrak{B}, I)$ is the graph whose vertex set is X and in which two points are adjacent whenever there is a block containing both. The **dual** of the design $(X, \mathfrak{B}, I)$ is the design $(\mathfrak{B}, X, I')$, where $I' = \{(B, x)|(x, B) \in X \times \mathfrak{B}\}$. A design is called a **self-dual** if it is isomorphic to its dual. The **incidence graph** of a design $(X, \mathfrak{B}, I)$ is the bipartite graph with vertex set $X \cup \mathfrak{B}$ and edge set $\{\{x, \mathfrak{B}\}|(x, \mathfrak{B}) \in I\}$.

A **t-(v,k,λ)-design** is a pair $\mathfrak{D} = (X, \mathfrak{B})$, where X is a set of points of cardinality v, and $\mathfrak{B}$ a collection of k-element subsets of X called blocks with the property that any t points are contained in precisely λ blocks.

A **Steiner system S(t,k,v)** is a t-$(v, k, 1)$ design. A **square (or symmetric) 2-design** is a 2-(v, k, λ) design with just as many points as blocks.

A **partial linear space** is a design $(X, \mathfrak{L})$ in which the blocks are called lines such that the line have size at least 2 and two distinct points are joined by at most one line.

If u is a vertex of a graph G, we define $\mathbf{u}^{\perp} = G_{\leq 1}(u) := \{u\} \cup \{v \in V(G)|\ d(u, v) = 1\}$, and if X is a set of vertices of G, we define $\mathbf{X}^{\perp} = \{\cap u^{\perp}|u \in X\}$. If G is edge-regular, then G is the point graph of the partial linear space whose lines are the subsets $\{u, v\}^{\perp\perp}$ for all adjacent vertices $u, v \in G$. These lines are known as **singular lines**.

A graph with the property that each edge lies in a unique maximal clique is a **collinearity graph** of the partial liner space, formed by the vertices of the graph as points and the maximal cliques of it as lines. In other words, the collinearity graph is the point graph of a partial linear space.

A **finite projective plane of order** n, denoted $PG(2, n)$, consists of a set X of n^2+n+1 elements called points, and a set $\mathfrak{B}$ of $(n + 1)$-element subsets of X called lines, having the property that any two points lie on a unique line. In other words, $PG(2, n)$ is a square 2-$(n^2 + n + 1, n + 1, 1)$ design. A **finite affine plane of order** n consists of a set X of n^2 points, and a set $\mathfrak{B}$ of n-element subsets of X called lines, such that two points lie on a unique line, i.e., a 2-$(n^2, n, 1)$ design. A projective or affine plane is **Desarguesian** if it is coordinatized by a division ring.

In a similar way, one can define the **n-dimensional projective geometry** over $GF(q)$, denoted $PG(n, q)$, by means of an $(n+1)$-dimensional vector space $V = V(n+1, GF(q))$.The points are the 1-dimensional subspaces of V; the lines are the 2-dimensional subspaces; planes are 3-dimensional subspaces and so on. The **finite n-dimensional affine geometry** $AG(n, q)$ over $GF(q)$ is the projective geometry $PG(n, q)$ minus its hyperplane H (a subspace of codimension 1) together with all the subspaces it contains.

1.2 Graphs and Groups

It is assumed that the reader is already familiar with basic group theory such as groups, cosets, direct product, normal subgroups, homomorphisms, isomorphism and factor groups. In this thesis, we are particularly concerned with the concepts of group actions, orbits, transitive groups, primitive and imprimitive groups and wreath products. We will include these only to demonstrate our terminology and notation which varies considerably between texts.

Let X be a set. The **symmetric group** on X, written $Sym(X)$, is the set of all permutations of X. It forms a group, with the operation of composition. If X is a finite set with n elements, we write S_n for the symmetric group $Sym(X)$. The subgroup of S_n consisting of the even permutations of n letters is the **alternating group** A_n on n letters .

An **automorphism** of a graph G is an isomorphism of G with itself, that is, a permutation on $V(G)$ that preserves adjacency. The set of all automorphisms of G, with the operation of composition, is the **automorphism group** of G, denoted by $Aut(G)$, which is thus a subgroup of the symmetric group $Sym(V(G))$.

An **action of a group** A on a set X is a map $\phi : A \times X \to X$ such that

1. $1.x = x$ for all $x \in X$ where 1 is the identity element of A;
2. $(a_1a_2)(x) = a_1(a_2x)$ for all $x \in X$ and all $a_1, a_2 \in A$.

Under these conditions, X is called an **A-set**. An action is said to be faithful if the identity is the only element of A that leaves every element of X fixed. The order of an arbitrary permutation group A is $|A|$ and if X is an A-set, then the degree of A is $|X|$.

Let X be an A-set. For $x, y \in X$, let $x \sim y$ if and only if there exists $a \in A$ such that $ax = y$. Then $\sim$ is an equivalence relation on X. The equivalence classes of $\sim$ are the **orbits** of A; and we say that A is **transitive** if there is just one orbit. A group A acting transitively on X is said to act **regularly** if $A_x = 1$ for each $x \in X$. An **orbital** of A is an orbit of A on the set $X \times X$. The number of orbitals is the **rank** of A.

Let A be a group acting transitively on a set X. A **block** is a subset Y of X such that $Y^a = Y$ or $Y^a \cap Y = \emptyset$ for all $a \in A$. Every group acting transitively on X has X and the singletons as blocks; these are called the trivial blocks. Any other block is called nontrivial. A group that acts transitively on a set X with no nontrivial blocks is **primitive**; otherwise, it is **imprimitive**. A graph G is said to be primitive or imprimitive according as the group $Aut(G)$ acting on $V(G)$ has the corresponding property.

Let A be a permutation group of order $n_1 = |A|$ and degree m_1 acting on the set $X = \{x_1, x_2, ..., x_{m_1}\}$, and let B be another permutation group of order $n_2 = |B|$ and degree m_2 acting on the set $Y = \{y_1, y_2, ..., y_{m_2}\}$. The **wreath product** $A \wr B$ is a permutation group of order $n_1n_2^{m_1}$ acting on $X \times Y$ whose elements are formed as follows: For each $a \in A$ and any sequence $(b_1, b_2, ..., b_{m_1})$ of m_1 permutations in B, there is a unique permutation in $A \times B$ written $(a : b_1, b_2, ..., b_{m_1})$ such that for $(x_i, y_j) \in X \times Y$;

$$\Big(a : b_1, b_2, ..., b_{m_1}\Big)(x_i, y_j) = (ax_i, b_iy_j).$$

In this case, A is frequently described as the **top group** and B as the **bottom group**.

For example, let $A = C_3$, the cyclic group of degree 3, which acts on $X = \{1, 2, 3\}$ and $B = S_2$, the symmetric group of degree 2, acting on $Y = \{a, b\}$. The three permutations of C_3 may be written as (1)(2)(3), (123), and (132). For S_2, we have the permutations $(a)(b)$ and (ab). The wreath product $C_3 \wr S_2$ has degree $6 = 3.2$ but its order is $3.2^3 = 24$. Note that $S_2 \wr C_3$ has order $2.3^2 = 18$ and so is not isomorphic to $C_3 \wr S_2$.

1.3 Transitivity in Graphs

This thesis considers mainly a class of graphs that have special conditions on their automorphism groups. In this section, we will define these conditions in turn, beginning with the weakest one.

A graph G is **vertex-transitive** (**edge-transitive**) if given any pair of its vertices (edges), there is an automorphism which transforms one into the other, that is, if $Aut(G)$ acts transitively on $V(G)$ $(E(G))$.

These two properties are not interchangeable; there exist graphs that are vertex-transitive but not edge-transitive, vice-versa, also graphs that are both vertex- and edge-transitive and graphs that satisfy neither property.

To show that vertex-transitive does not imply edge-transitive, we construct a graph G as follows: take two copies of C_5 with the vertices of one labelled 1 through 5 and those of the other labelled $1'$ through $5'$; then join i to i', $1 \leq i \leq 5$.(See the figure below)

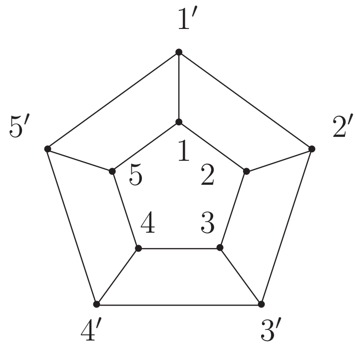


Figure 1.1: a vertex-transitive graph that is not edge-transitive

G is vertex-transitive, since any vertex i can be mapped to any other vertex j by the automorphism which maps $i \to i'$ and $i' \to j$. However, G is not edge-transitive since the edge $11'$ is in two quadrilaterals while the edge $1'2'$ is in only one.

To show that edge-transitive does not imply vertex-transitive, consider $K_{m,n}$ $n \neq m$. This graph is edge-transitive, but it is not vertex transitive, because it is not regular. Folkman (1967) constructed a regular edge-transitive graph which is not vertex-transitive. (see the figure below)

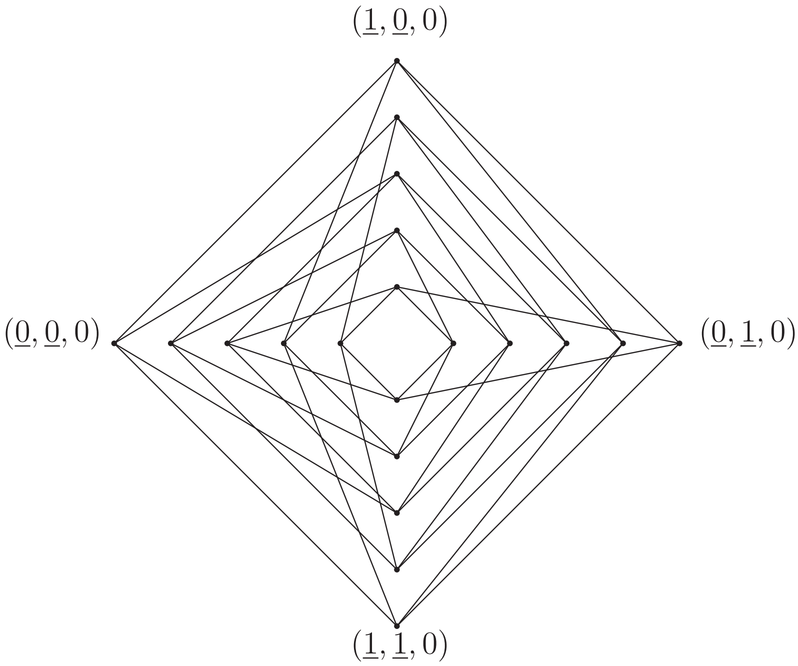


Figure 1.2: Folkman graph

The complete graph K_3 is an example of a graph that are both vertex- and edge-transitive. The figure below gives a graph that is neither vertex- nor edge-transitive.

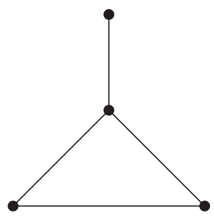


Figure 1.3: a graph that is neither vertex- nor edge-transitive

We now turn to the definition of graphs which have a higher degree of transitivity than either vertex- or edge- transitive graphs. An **n-arc** in a graph G is a walk of length n with a specified initial vertex in which no edge succeeds itself. A graph G is **n-transitive** if $Aut(G)$ acts transitively on the set of all n-arcs of G. A 1-transitive graph is often known as a **symmetric graph**.

We will now define a class with a symmetry condition that is stronger than any of the above, namely distance-transitive graphs. A detail information together with the classification problem of such graphs will be considered in the rest of this thesis.

A graph G is distance-transitive (**DTG**) if, for all vertices u, v, x, y of G such that $d(u, v) = d(x, y)$, there is an automorphism $\sigma \in Aut(G)$ satisfying $\sigma(u) = x$ and $\sigma(v) = y$.

We conclude this introductory chapter with the following hierarchy of conditions:

Distance-Transitive $\Rightarrow$ Symmetric $\Rightarrow$ Vertex-Transitive

Chapter 2

Distance-Transitive Graphs

In this chapter we discuss the basic properties and examples of distance-transitive graphs and the related distance-regular graphs. The fundamental reference is the book of Brouwer, Cohen, and Neumaier [17].

2.1 Definitions and Examples

For any connected graph G with diameter d, we define $G_i := \{(u,v)|d(u,v) = i\}$, the set of all pairs of vertices at distance i, where $0 \leq i \leq d$. The sets G_i $(0 \leq i \leq d)$ are better known as the distance partition graphs of G. Then, for $A \leq Aut(G)$, we say that G is an A-distance-transitive graph if A is transitive on each of the distance partition graphs $G_0, ..., G_d$; G is distance-transitive if it is $Aut(G)$-distance-transitive.

Notice that, our definition here for a distance-transitive graph G is equivalent for that given last chapter. Also, the distance partition graphs G_i are actually the orbitals of G in $V(G)$.

Examples of distance-transitive graphs are the complete graphs K_n, the complete bipartite graphs $K_{n,n}$, the cycle C_n and the Petersen graph. More interesting examples are provided by infinite families of distance-transitive graphs. We introduce five of them below.

Johnson graphs $J(n,k)$ (where $1 \leq k < n$) form our first infinite family of finite distance-transitive graphs. The vertices of $J(n,k)$ are the k-subsets of an n-set, with two k-subsets adjacent if and only if they intersect in exactly $k-1$ elements. The valency of $J(n,k)$ is $k(n-k)$, the diameter is $min(k, n-k)$, and

$$Aut(J(n,k)) \cong \begin{cases} S_n \times \mathbb{Z}_2, & \text{if } n = 2k \geq 4; \\ S_n, & \text{otherwise.} \end{cases} \quad \text{(see [17, section 9.1])}$$

The second family, the **odd graphs** O_k (with $k \geq 2$) have the $(k-1)$-subsets of a $(2k-1)$-set as vertices, with two $(k-1)$-subsets joined by an edge if and only if they disjoint. The valency of O_k is k, the diameter is $k-1$, and its automorphism is S_{2k-1}. O_2 is the complete graph K_3 and O_3 is better known as Petersen graph. (see [17, section 9.1])

The **Hamming graphs** $H(n,q)$ (where $n,q > 1$) form our third family. They have vertex set $\mathbb{Z}_q^n$ and two vertices are adjacent if and only if they differ in just one position. The valency of $H(n,q)$ is $n(q-1)$, the diameter is n, and $Aut(H(n,q)) = S_q \wr S_n$. $H(n,q)$ is primitive distance-transitive if and only if $q \geq 3$. If $q = 2$, it is bipartite and better known as the n-cube. (see [17, section 9.2])

The fourth infinite family of finite distance-transitive, the **Grassmann graphs** $J_q(n,k)$ (where $1 \leq k < n$) have the k-dimensional subspaces of an n-dimensional vector space over a field $\mathbb{F}_q$ as vertices, with two of the k-subspaces joined by an edge if and only if they intersect in a subspace of dimension $k-1$. The valency of $J_q(n,k)$ is $\frac{(q^k-1)(q^{n-k+1}-q)}{(q-1)^2}$ and its diameter is $min(k, n-k)$.

The **bilinear forms graphs** $H_q(n,d)$ (where $n \geq d$) have as vertices the $n \times d$ matrices over $\mathbb{F}_q$ with two matrices joined by an edge if and only if their difference has rank 1. (see [17, section 9.5])

Some others related families will be introduced next chapter.

2.2 Parameters and Feasibility of Intersection Arrays

Distance-transitive graphs have rich combinatorial structure. This structure alone enables one to develop an interesting theory and to carry out a classification. For this reason and many others, it is natural to study such properties in the class of DTG's.

For a vertex v of a connected graph G with diameter d, and $i \leq d$, define $G_i(v) := \{u \in V(G) | d(u,v) = i\}$, the set of vertices at distance i from v. For each v in $V(G)$, $V(G)$ is partitioned into the disjoint subsets $G_0(v), ..., G_d(v)$, the **distance partition** of V with respect to v.

For any connected graph G, any vertices u, v of G, and any non-negative integers h and i, define $s_{hi}(u,v)$ to be the number of vertices of G whose distance from u is h and whose distance from v is i. That is,

$$s_{hi}(u,v) = |G_h(u) \cap G_i(v)|$$

If G is distance-transitive graph, then the numbers $s_{hi}(u,v)$ do not depend on the particular vertices u, v one choose, but only on the distance j between them. So, if $d(u,v) = j$ we write s_{hij} for $s_{hi}(u,v)$.

Clearly there are $(d+1)^3$ of these numbers, but it turns out that there are many identities relating them and just $2d$ of them are sufficient to determine the rest.

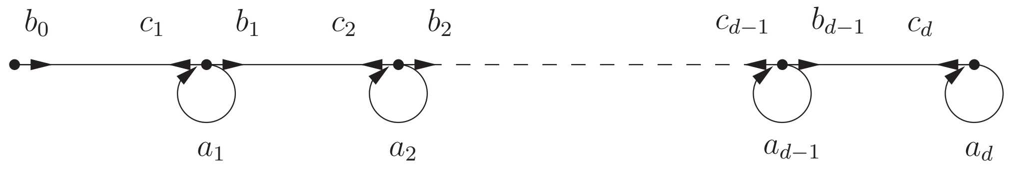
For the numbers s_{1ij} which are not zero we will use the notation

$$c_i = s_{1,i-1,i},\ a_i = s_{1,i,i},\ b_i = s_{1,i+1,i}$$

where $0 \leq i \leq d$ and c_0 and b_d are undefined.

These numbers (c_i, a_i, b_i) have the following simple interpretation in terms of the distance partition graphs G_i $(0 \leq i \leq d)$. Let $v \in V(G)$. For each i, pick a vertex $u \in G_i(v)$. Then a_i, b_i, c_i are the numbers of vertices adjacent to u and lying in $G_i(v)$, $G_{i+1}(v)$ (if $i < d$), and

$G_{i-1}(v)$ (if $i > 0$), respectively. By distance transitivity these numbers are independent of the choices of the vertices u and v, provided that $d(u, v) = i$. It is sometimes convenient to picture these parameters as follows:



or assemble them as an array

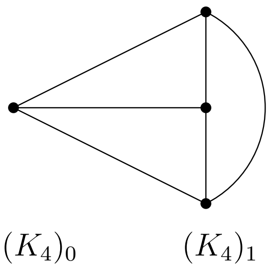
$$\left\{ \begin{array}{cccccccc} * & c_1 & c_2 & . & . & . & c_{d-1} & c_d \\ a_0 & a_1 & a_2 & . & . & . & a_{d-1} & a_d \\ b_0 & b_1 & b_2 & . & . & . & b_{d-1} & * \end{array} \right\}$$

It is easy to see that $a_i + b_i + c_i = k$ for $1 \leq i \leq d-1$ and $c_d + a_d = k$ where k is the valency of G. Hence the middle row in the array can be omitted. Thus the array can be written as

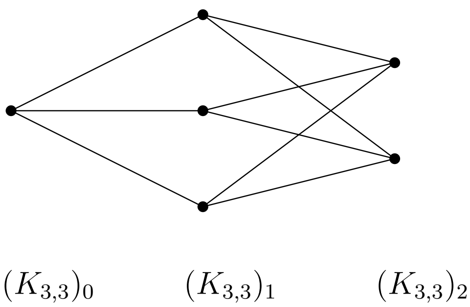
$$\{k, b_1, b_2, ..., b_{d-1}\ 1, c_2, ..., c_d\}.$$

This array is known as the **intersection array** and is denoted by $i(G)$.

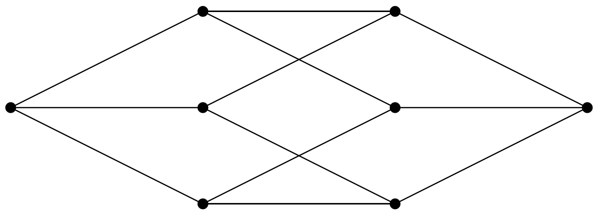
Examples:

1. 

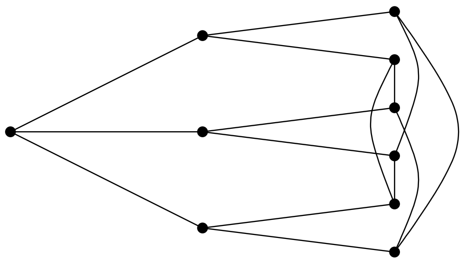
$i(K_4) = \{3; 1\}$

2. 

$i(K_{3,3}) = \{3, 2; 1, 3\}$

3.  $i(H(3,2)) = \{3,2,1;1,2,3\}$

$(H(3,2))_0$ $(H(3,2))_1$ $(H(3,2))_2$ $(H(3,2))_3$

4.  $i(O_3) = \{3,2;1,1\}$

$(O_3)_0$ $(O_3)_1$ $(O_3)_2$

Denote by k_i $(0 \leq i \leq d)$ the number of vertices in $G_i(v)$ for any vertex v; in particular $k_0 = 1$ and $k_1 = k$.

There is an important purely combinatorial analogue to distance transitivity, which simply asks the numerical regularity properties, namely that the numbers a_i, b_i, and c_i, are well-defined, regardless of whether there are any automorphisms that force this to occur. A connected graph G is called **distance-regular graph** (**DRG**) if it is regular of valency k, diameter d, and if for any two vertices $u, v \in V(G)$ at distance $i = d(u,v)$, there are natural numbers

$$b_0 = k,\ b_1, ..., b_{d-1},\ c_1 = 1,\ c_2, ..., c_d$$

such that

1. $c_i = |G_{i-1}(v) \cap G_1(u)|$ $(1 \leq i \leq d)$;
2. $b_i = |G_{i+1}(v) \cap G_1(u)|$ $(0 \leq i \leq d-1)$.

Clearly, any distance-transitive graph is distance-regular, but the converse is certainly not true. Although many familiar examples of distance-regular graphs are distance-transitive, Adel'son-Velskii et al. (1969) construct the following example.

Let G be the graph with vertex set the 26 symbols a_i, b_i (where $0 \leq i \leq 12$), and in which:

$$a_i \sim a_j \Leftrightarrow |i-j| = 1, 3, 4$$
$$b_i \sim b_j \Leftrightarrow |i-j| = 2, 5, 6$$
$$a_i \sim b_j \Leftrightarrow i - j = 0, 1, 3, 9.$$

Then G is distance-regular with intersection array $\{10, 6; 1, 4\}$. But G is not distance-transitive since there is no automorphism taking a vertex a_i to a vertex b_j.

The parameters of a distance-regular graph are subject to many simple but still very useful constraints. We prove some of the basic restrictions that may be needed later. (see [17, chapter 5] for more details)

Proposition 2.2.1. *Let G be a distance-regular graph with valency k and diameter d. Then the following hold:*

1. $k_{i-1}b_{i-1} = k_ic_i$ *(*$1 \leq i \leq d$*),*
2. *If k_i is odd then a_i is even,*
3. $1 \leq c_2 \leq ... \leq c_d$*,*
4. $k \geq b_1 \geq ... \geq b_{d-1}$*,*
5. *If $i + j \leq d$ then $c_j \leq b_i$*

Proof. (1) For any vertex $v \in V(G)$, there are k_{i-1} vertices in $G_{i-1}(v)$ and each is joined to b_{i-1} vertices in $G_i(v)$. Also, there are k_i vertices in $G_i(v)$ and each is joined to $c_i(v)$ vertices in $G_{i-1}(v)$. Thus the number of edges joining a vertex in $G_{i-1}(v)$ to a vertex in $G_i(v)$ is $k_{i-1}b_{i-1} = k_ic_i$.

(2) Let v be a fixed vertex in G. The subgraph of G induced by the vertices in $G_i(v)$ is regular with valency a_i and has k_i vertices. Hence k_ia_i must be even.

(3) Suppose u is in $G_{i+1}(v)$ $(1 \leq i \leq d-1)$. Pick a path $v, x, ..., u$ of length $i+1$; then $d(x, u) = i$. Then $u \in G_i(x)$ and any vertex adjacent to u and at distance $i-1$ from x is at distance i from v. Hence $G(u) \cap G_{i-1}(x)$ is contained in $G(u) \cap G_i(v)$. But the cardinality of the first is c_i, while of the second is c_{i+1}.

(4) Suppose u is in $G_i(v)$ $(0 \leq i \leq d-2)$. Pick a path $x, v, ..., u$ of length $i+1$; then $d(x, u) = i+1$. Then $u \in G_{i+1}(x)$ and any vertex adjacent to u and at distance $i+2$ from x is at distance $i+1$ from v. Hence $G(u) \cap G_{i+2}(x)$ is contained in $G(u) \cap G_{i+1}(v)$. But the cardinality of the first is b_{i+1}, while of the second is b_i.

(5) Suppose $u \in G_i(v)$ and $w \in G_j(v)$ with $d(u, w) = i + j$. Any vertex at distance $j-1$ from w and adjacent to v is at distance $i+1$ from u. Hence $G(v) \cap G_{j-1}(w)$ is contained in $G(v) \cap G_{i+1}(u)$. Thus $c_j \leq b_i$ if $i + j \leq d$. ■

The following results are due to Brouwer, Cohen and Neumaier (see Theorem 5.4 1 & Corollary 5.4.2[17]).

Theorem 2.2.2. *Let G be a distance-regular graph of diameter $d > 2$. If $c_2 > 1$, then either $c_3 \geq \frac{3}{2}c_2$ or $c_3 \geq c_2 + b_2$ and $d = 3$.*

Corollary 2.2.3. *Let G be a distance-regular graph of diameter $d > 2$. If $c_2 \geq 2$, then $c_3 \geq c_2 + 2$ unless the intersection array is $\{k, k-1, 1; 1, k-1, k\}$.*

Corollary 2.2.4. *(Remark (iii) of Theorem 5.4.1[17]) Let G be a distance-regular graph of diameter $d > 2$.*

1. *If $1 < c_3 < 2c_2$, then G contains a quadrangle.*

2. *If $c_3 = c_2 = \omega$, then $\omega = 1$.*

The following result is due to Meredith (see (5) [33]).

Theorem 2.2.5. *Let G be a distance-regular graph of diameter $d \geq 3$. If $a_1 = 0$ and $c_2 \geq 2$, then $c_{i+1} > c_i$ for each $1 \leq i \leq d-1$.*

Proof. Since G is distance-regular, given any pair of vertices at distance 2, there are c_2 paths of length 2 joining those vertices. Since $a_1 = 0$ and $c_2 \geq 2$, G has girth 4. Hence each pair of adjacent edges of G is in precisely $(c_2 - 1)$ 4-cycles. Now, let $p \in G_{i+1}(u)$ be adjacent to q_s in $G_i(u)$ for $1 \leq s \leq c_{i+1}$ and q_1 to r_t in $G_{i-1}(u)$ for $1 \leq t \leq c_i$. Then each pair pq_1, q_1r_t is on $(c_2 - 1)$ 4-cycles, so there are $c_n(c_2 - 1)$ such cycles. Each of these contains a pq_s edge for some $2 \leq s \leq c_{i+1}$, so as each pair pq_1, pq_s is on at most $(c_2 - 1)$ 4-cycles, we have

$$(c_{i+1} - 1)(c_2 - 1) \geq (c_i - 1)(c_2 - 1)$$

i.e. $c_{i+1} > c_i$. ■

The study of distance-regular (transitive) graphs often proceeds by constructing a list of possible intersection arrays and then trying to find the actual graphs with those arrays. We can view the above restrictions as examples of **feasibility conditions** that must be satisfied by the intersection array of any distance-regular (transitive) graph. A feasible array may correspond to zero, one, or several distance-regular graphs G. For example, $\{3, 2, 2, 2, 2, 2; 1, 1, 1, 1, 1, 3\}$ is feasible but there is no corresponding distance-regular graph, while $\{6, 4, 4; 1, 1, 3\}$ is realized by exactly two non-isomorphic distance-regular graphs.

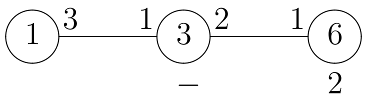
2.3 Regular Partitions

Let $G = (V, E)$ be a graph. A **partition** of V is a set whose elements are disjoint nonempty subsets of V, and whose union is V. A partition $P = (V_1, V_2, ..., V_k)$ of V is called **regular** if, for all $V_i \in P$, the number of neighbors e_{ij} which a vertex in V_i has in V_j is independent of the choice of a vertex in V_i. The partition into singletons is always regular; the partition $\{V\}$ is regular only when G is regular. For any group A of automorphisms of G, the partition of V into A-orbits is regular. This follows since if u and v belong to the same orbit then there is an automorphism in A which maps u to v. Since this automorphism must map each orbit onto itself, it follows that u and v have the same number of neighbors in each orbit. Thus, for a distance-transitive graph G, the distance partition $P(u) = \{G_o(u), G_1(u), ..., G_d(u)\}$ with respect to any vertex u is regular.

The **distribution diagram** of G with respect to a regular partition $P = (V_1, V_2, ..., V_k)$ consists of a number of balloons b_i, one for each element $V_i \in P$, and a number of lines $l_{ij} (= l_{ji})$ joining the two balloons b_i and b_j, one for each pair $\{V_i, V_j\}$ for which $e_{ij} \neq 0$.

This diagram is provided with numbers as follows: in the balloon b_i we write $|V_i|$, and at the V_i-end of l_{ij} we write e_{ij}. The number e_{ii} is just written next to b_i; when $e_{ii} = 0$, we write $-$.

Example. The distance distribution diagram of the Petersen graph is shown by the following figure:



$v = 10.$

Given a regular partition $P = (V_1, ..., V_k)$ of a graph G, the **quotient graph** G/P of G with respect to P is the directed graph with the elements $V_1, ..., V_k$ of P as its vertices with e_{ij} edges going from V_i to V_j. Thus G/P has, in general, both loops and multiple edges. If G/P is actually a (simple) graph then we say that G **covers** G/P.

If G covers G/P then the subgraphs of G induced by the elements of P are all empty, since otherwise G/P would have loops. Suppose that V_1 and V_2 are two elements of P with at least one edge from a vertex in V_1 to a vertex in V_2. Since G/P does not have any multiple edges, each vertex in V_1 must consequently be joined to exactly one vertex in V_2, and vice versa. This shows that V_1 and V_2 have the same cardinality, say r, and that the subgraph of G induced by the vertices in $V_1 \cup V_2$ is an r-matching. Hence all the elements of P have the same size r. The common cardinality of the elements is known as the **index**. In this case G is called an r-cover of G/P. In this thesis we will always require that $r > 1$.

2.4 Imprimitive Distance-Regular Graphs

Recall that a graph G is primitive or imprimitive according as its automorphism group $Aut(G)$ acting on $V(G)$ has the corresponding property.

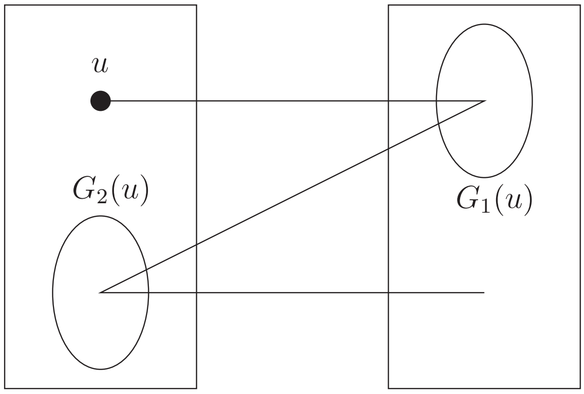
Now suppose that A acts imprimitively on the vertex set V of a distance-regular graph G, that is, there is a nontrivial block B of V. Then each B^a is a block and the distinct blocks B^a form a partition $P = \{V_1, ..., V_k\}$ of V. Let $u, v \in V_1$ and set $i := d(u, v)$. As each $a \in A_u$ fixes u, it must fix V_1 setwise since V_1 is a block of imprimitively for A in V. Hence $G_i(u) \subseteq V_1$. This observation was strengthened by D. H. Smith [62] to show that there are just two different kinds of blocks possible, corresponding to bipartite and antipodal graphs. A DRG G of diameter d is said to be **antipodal** if there is a partition of the vertex set into classes with the property that any two vertices in the same class are at distance d, while two vertices in different classes are at distance less than d. In other words, a DRG G of diameter $d \geq 2$ is antipodal if G_d is disjoint union of complete graphs. We often refer to the antipodal classes as the **fibres** of the antipodal graph. Notice that each complete graph is both antipodal and primitive. Moreover the complete graphs exhaust the antipodal DRG's of diameter 1. An antipodal graph of diameter two is complete multipartite.

The following theorem was proved by Smith [62] for distance-transitive graphs; his proof is easily extended to arbitrary distance-regular graphs (see [17] pg. 140).

Theorem 2.4.1. *Let G be a distance-regular graph with valency $k \geq 3$. If G is imprimitive, it is either bipartite or antipodal. (Both possibilities can occur in the same graph.)*

Thus if V_1 is a nontrivial block of imprimitivity containing u, then either G is bipartite and $V_1 = \{u\} \cup G_2(u) \cup ... \cup G_{d'}(u)$, where d' is the largest even integer not exceeding the diameter d, or G is antipodal of diameter d and $V_1 = \{u\} \cup G_d(u)$. In each of these imprimitive cases it is possible to produce smaller distance-regular graphs.

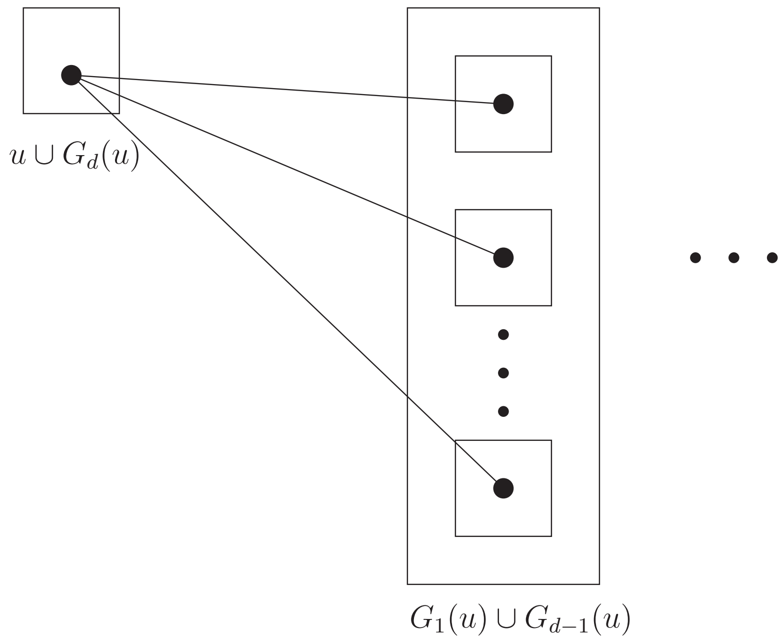
If G is a bipartite distance-regular graph with diameter d, then we may represent G by the following diagram:



It turns out that is helpful to consider the distance graph G_2. If G is bipartite, connected, and of diameter $d > 1$ then G_2 has two components and the graphs induced on these connected components are denoted by G^+ and G^- (or $\frac{1}{2}G$ for an arbitrary one of these) and are known as the **halved graphs** of G. In this case, we say that G is a **bipartite distance-regular(transitive) double** (or simply a **bipartite double**) of $\frac{1}{2}G$. The vertices in one halved graph correspond to a class of cliques in the other. Smith [62] proved that, if G is distance-transitive graph then its two halved graphs are isomorphic distance-transitive graphs.

As an example, consider a set X with $2k-1$ elements. The **doubled odd graph** $2O_k$ on X is the graph G whose vertices are the $(k-1)$-subsets and k-subsets of X, where two vertices u, v are adjacent if and only if $u \neq v$ and $u \subset v$ or $v \subset u$. It is not hard to show that $2O_k$ is a bipartite distance-transitive graph. Its halved graphs are copies of $J(2k-1, k)$.

Now, if G is a distance-regular antipodal graph, then we can represent G by the following diagram:



Smith [62] proved that, if G is distance-transitive graph then the quotient graph $\overline{G}$ of G, define by taking the fibres of G as its vertices with two such fibres join by an edge in $\overline{G}$ if they contain adjacent vertices of G, is a distance-transitive graph. Thus $\overline{G}$ is actually the quotient graph G/P with respect to the partition P consists of the blocks $\{u\} \cup G_d(u)$ of G. Hence G is an r-cover of $\overline{G}$, where r is the common cardinality of the fibres (the index of G). In this case, we say that G is an **r-antipodal distance-regular (transitive) cover** (or simply an **r-antipodal cover**) of $\overline{G}$.

The **doubled odd graph** $2O_k$ is also antipodal. Each fibre consists of a $k-1$-subset of X, together with its complementary (k)-subset. The quotient graph $\overline{G}$ of $2O_k$ is the Odd Graph O_k.

In order to gain more insight into the structure of the imprimitive distance-regular graphs let us consider the following theorem which is due to Biggs & Gardiner (see pg. 141 [17]).

Theorem 2.4.2. *Given a distance-regular graph G, we obtain a primitive distance-regular graph in at most two steps (except in the case of $8n$-gons). More precisely, suppose that G is distance-regular graph of valency $k \geq 3$.*

1. *If G is antipodal with quotient graph $\overline{G}$, then $\overline{G}$ is not antipodal, except when G has diameter $d \leq 3$, in which case $\overline{G}$ is complete, or when G is bipartite of diameter $d = 4$, in which case $\overline{G}$ is complete bipartite.*

2. *If G is bipartite with halved graph $\frac{1}{2}G$, then $\frac{1}{2}G$ is not bipartite.*

3. *If G is antipodal, and either has odd diameter d or is not bipartite, then $\overline{G}$ is primitive.*

4. *If G is bipartite, and either has odd diameter d or is not antipodal, then $\frac{1}{2}G$ is primitive.*

5. *If G has even diameter $d = 2e$, and is both bipartite and antipodal, then the graphs $\frac{1}{2}G$ are antipodal, $\overline{G}$ is bipartite, and the graphs $\overline{\frac{1}{2}G} \cong \frac{1}{2}\overline{G}$ are primitive.*

Proof. 1. If $\overline{G}$ has diameter e and is antipodal, then having distance in $\{0, e, d-e, d\}$ is an equivalence relation in G.
2. If $\frac{1}{2}G$ is bipartite, then $0 = a_1(\frac{1}{2}G) \geq b_1 - 1$, so $b_1 = 1$, $k = 2$.
3. If $\overline{G}$ is bipartite of diameter e, then having distance in $\{0, 2, ..., 2[\frac{1}{2}e], d-2[\frac{1}{2}e], ..., d-2, d\}$ is an equivalence relation in G, so $d = 2e$ is even and G is bipartite.
4. If G is antipodal and bipartite of odd diameter, then it is a 2-antipodal cover, and $\frac{1}{2}G \cong \overline{G}_2$ unless $d \leq 3$, in which case both $\frac{1}{2}G$ and $\overline{G}$ are complete.
5. If $\frac{1}{2}G$ is antipodal of diameter $e \geq 2$, then having distance in $\{0, 2e\}$ is an equivalence relation in G, so $d = 2e$ and G is antipodal. ■

Thus, starting with an imprimitive distance-regular graph we can always construct a primitive distance-regular graph after halving at most once and taking at most one quotient. Sometimes the problem of constructing all imprimitive graphs corresponding to a given primitive one is very nontrivial.

2.5 Bounding the Diameter

One of the first major results in the scope of the classification of distance-transitive graphs was the classification of the cubic graphs given by Biggs & Smith [8]. Subsequently, Biggs, Gardiner, Faradjev, A.A. Ivanov, A.V. Ivanov, Praeger and Smith gave similar determinations of distance-transitive graphs of valency $k = 4, 5, ..., 13$ (see [29],[33],[36],[37],[38],[48],[64],[65]). In each case, the essential steps in the determination are as follows:

1. Bound the order of the vertex stabilizer
2. Bound the diameter
3. Test all the finitely many possible intersection arrays for feasibility.

The first step is closely related to the **Sims Conjecture** which was proved in the early 1980s [24] as a consequence of the classification of the finite simple groups and uses the O'Nan-Scott theorem which is another major achievement of late 20th century group theory.

Theorem 2.5.1. *(Sims Conjecture) There is a function f such that if A is a finite primitive group in which A_x has an orbit of length $k > 1$, then $|A_x| \leq f(k)$.*

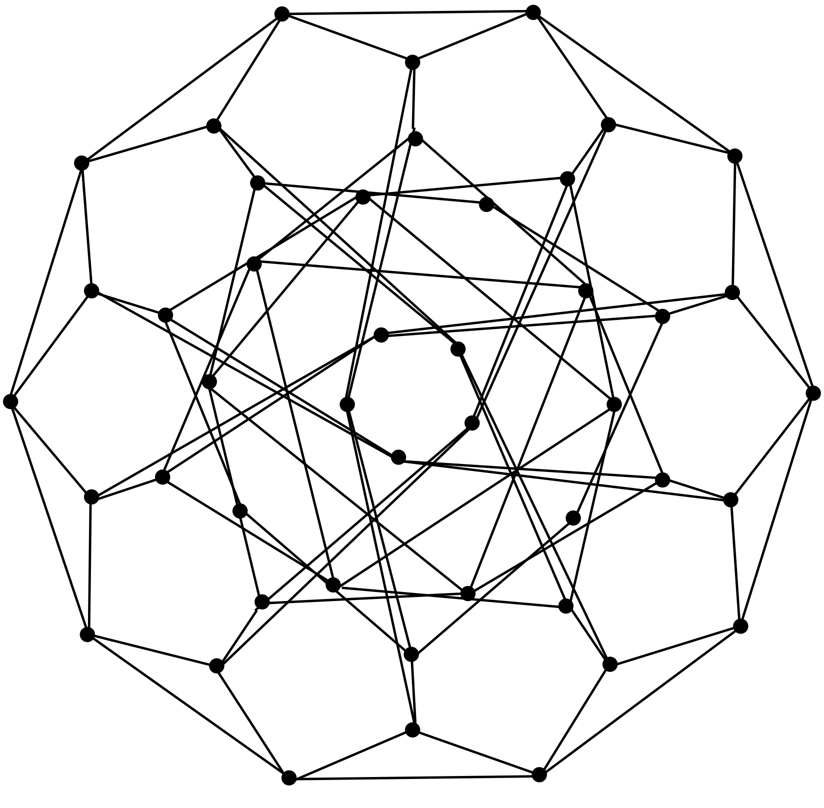
The truth of the Sims conjecture makes it feasible to extend the classification of distance-transitive graphs having a given small valency k to the classification of distance-transitive graphs of arbitrary valency k.

Theorem 2.5.2. *(Cameron [21], Weiss [71]) There are, up to isomorphism, only finitely many finite distance-transitive graphs of given valency k greater than or equal 3.*

To complete this section we formulate a bound on what the diameter actually is. Unfortunately it is not practical for performing calculations. This result is due to Cameron [24] and Weiss [71].

Theorem 2.5.3. *The diameter d of a distance-transitive graph of valency $k > 2$ is at most $(k^6)!4^k$.*

Primitive Distance-Transitive Graphs



Chapter 3

Primitive Distance-Transitive Graphs

Attention now turns to the classification of distance-transitive graphs. This project is not yet complete. However, the truth of Sims conjecture gives the green light for such a process to be completed.

The problem can be divided into two stages: first finding the primitive distance-transitive graphs, and next, for each individual primitive example found, determining all antipodal covers and bipartite doubles associated to it. In the current chapter, we collect all available results on primitive distance-transitive graphs of diameter at least 3. (In the following chapter, we present a conjectured list of all such graphs.) Our main work, the problem of determining those imprimitive DTG's which correspond to a given primitive one of diameter at least 3 will be discussed in detail in the rest of the chapters.

In view of the determination of all rank 3 groups, for the classification of distance-transitive graphs, we may assume $d \geq 3$. Also, since the only connected distance-transitive graphs with valency $k = 2$ and diameter d are the $2d$-gon and $2d+1$-gon, we may also assume $k \geq 3$.

3.1 The Starting Point in the Classification

The first analysis of finite primitive distance-transitive graphs using O'Nan-Scot theorem was given by Praeger, Saxl and Yokoyama [60]. Their result is the first step toward the classification of finite primitive DTG's.

Theorem 3.1.1. ***(Praeger, Saxl, Yokoyama)*** *Let A act distance transitively on a primitive distance-transitive graph G. Then one of the following holds.*

- *G is a Hamming graph $H(n, q)$ or the complement of a Hamming graph $H(2, q)$ and A is a wreath product. (In this case, the graph G is well known but the possibilities for the group A are not completely determined).*

- *A has an elementary abelian normal subgroup which is regular on $V(G)$. (This case is referred to as the* ***affine type****).*

- *A has a simple socle. That is, there is a simple nonabelian normal subgroup N of A such that A canonically embeds in Aut(N) (that is, the centralizer $C_G(N)$ of N in A is trivial). (This case is referred to as* ***the simple socle or almost simple type****).*

As a summary, we have the following tree.

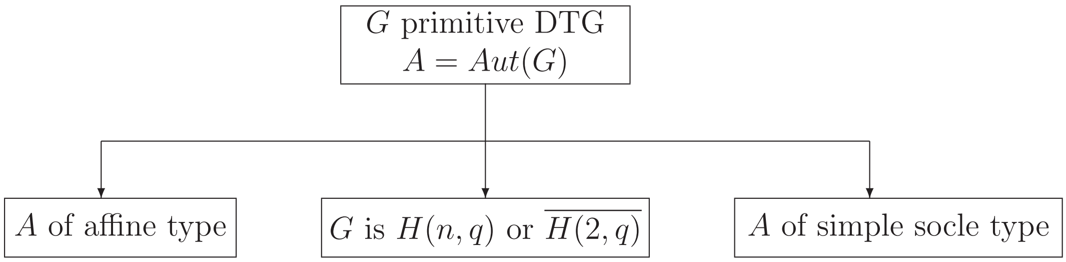


Figure 3.1: primitive main tree

3.2 Primitive DTGs of Affine Type

In this section, we discuss the classification of the primitive affine DTGs.

The main example of graphs admitting distance-transitive action of affine type is the Hamming graph $H(n, q)$. The bilinear forms graph $H_q(n, m)$ gives another classical example of an affine DTG. Further examples can be constructed as follows.

The **Hermitian forms graphs** $Her(n, q)$ (where $n, q > 1$) have as vertices the $n \times n$ Hermitian matrices over $\mathbb{F}_q$ (where $q = p^2$, p a prime power) with two matrices joined by an edge if and only if their difference has rank 1. (see [17, section 9.5])

The **alternating forms graphs** $Alt(n, q)$ (where $n, q > 1$) have as vertices the $n \times n$ alternating matrices over $\mathbb{F}_q$,that is, all $n \times n$ matrices $(a_{ij})_{n \times n}$ with $a_{ij} = -a_{ji}$ for $1 \leq i, j \leq n$, with two matrices joined by an edge if and only if their difference has rank 1. (see [17, section 9.5])

Now let us consider the general classification scheme for the primitive DTGs of affine type. Let G be an affine DTG with $Aut(G) = A$. Then $V(G)$ can be identified with a vector space V over the field $\mathbb{F}_s$ of order s for some power $s = r^b$ of a prime r, maximal with respect to $A_0 \leq GL(V)$, where A_0 is the stabilizer in A of $0 \in V$. Van Bon in [10], has proved the following result.

Theorem 3.2.1. *Let A be an affine primitive group acts distance-transitively on a connected noncomplete graph G of valency $k \geq 3$ and diameter $d > 2$. Then with s and V as above, we have one of the following.*

- *(1) G is a Hamming graph $H(n, q)$*

- *(2) G is a bilinear forms graph $H_q(n,d)$.*
- *(3) V is 1-dimensional and A_0 is a subgroup of $GL(1,s)$.*
- *(4) The generalized Fitting subgroup $K := F^*(A_0/Z(A_0 \cap GL(V)))$ of the central quotient $A_0/Z(A_0 \cap GL(V))$ of A_0 is nonabelian and simple, its projective representation on V is absolutely irreducible and can be realized over no proper subfield of F_s.*

In a diagram:

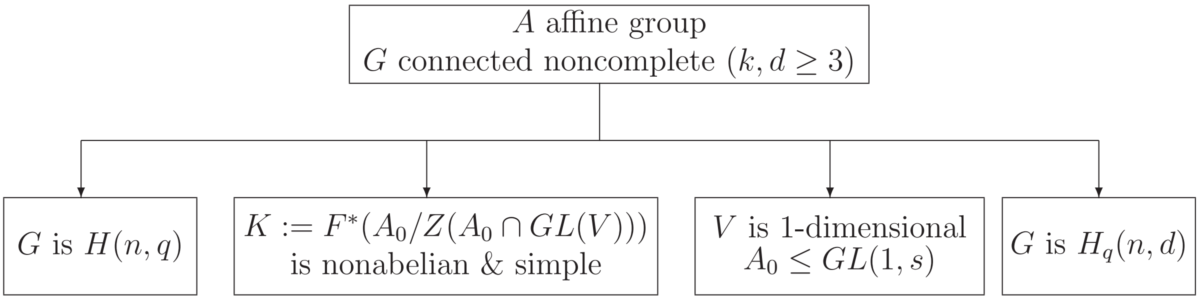


Figure 3.2: affine tree

Cohen and others have pursued Van Bon's strategy to complete the classification of DTGs of affine type. We will list all such possibilities of G together with the derived results.

Notice that, in cases (1) & (2) the graphs are fully determined.

3.2.1 The one dimensional affine case

This case is completely done by Cohen, Ivanov and Alexander in [26]. They proved the following result.

Theorem 3.2.2. *Suppose that case (3) holds with $n = 1$ and $d > 2$. Then G is the Hamming graph $H(4,3)$ and $s = 64$, $A_0 \cong \mathbb{Z}_9 \rtimes \mathbb{Z}_3$ or $\mathbb{Z}_9 \rtimes \mathbb{Z}_6$.*

In case (4), The classification of finite simple groups is invoked to make a further subdivision according to various types of simple group K.

$$\begin{cases} (i) & \text{alternating: } K \cong A_n (n \geq 5). \\ (ii) & \text{Lie type same characteristic:(exceptional \& classical)} \\ (iii) & \text{Lie type cross characteristic:} \\ (iv) & \text{sporadic case:} \end{cases}$$

3.2.2 The affine alternating

The affine alternating case is completed in [55] by Liebeck & Praeger. They proved the following result:

Theorem 3.2.3. *Suppose A, $s = r^b$, G and K are as above, with $K \cong A_n$ for some integer $n \geq 5$. Then either the diameter $d \leq 2$, or G is a halved n-cube $\frac{1}{2}H(n,2)$, a quotient n-cube $\overline{H}(n,2)$ or a quotient halved n-cube $\frac{1}{2}\overline{H}(n,2)$.*

3.2.3 The affine groups of exceptional Lie type in same characteristic

In this subsection, K is an exceptional Lie type group over $\mathbb{F}_q$ for some power $q = r^a$ of r. The affine groups of exceptional Lie type in same characteristic are dealt with in [14]. Van Bon & Cohen proved the following result:

Theorem 3.2.4. *Suppose that A, $s = r^b$, V, G and K are as in Theorem 3.2.1(4), with $\hat{K}$ a quasisimple group of exceptional Lie type over $\mathbb{F}_q$ for some power $q = r^a$ of r. Then $\hat{K} \cong \hat{E}_6(q)$, the universal Chevalley group of type E_6 over $\mathbb{F}_q$, V is a 27-dimensional $\mathbb{F}_q\hat{K}$-module (so $q = s$), and G is the affine E_6 graph (see 5.2.8 below for def.) with intersection array $i(G) = \{\frac{(q^{12}-1)(q^9-1)}{q^4-1}, q^8(q^4+1)(q^5-1), q^{16}(q-1); 1, q^8+q^4, \frac{q^{20}-q^8}{q^4-1}\}$.*

3.2.4 The affine groups of classical Lie type in same characteristic

The affine groups of classical Lie type in same characteristic are handled by Van Bon, Cohen & Cuypers[15]. They proved the following result:

Theorem 3.2.5. *Suppose that A, $s = r^b$, V, G and K are as in Theorem 3.2.1(4), with K a classical simple group of characteristic r. Then one of the following holds, where $\epsilon_m = -1$ if m even and $\epsilon_m = 1$ otherwise.*

- *G is the alternating forms graph and $K = SL(m,s)/\langle \epsilon_m I_m \rangle$ or $K = SL(m,r^a)/\langle \epsilon_m I_m \rangle$ $(b|a)$ and $n = m(m-1)/2$.*
- *G is the Hermitian forms graph and $K = SL(m,s^2)/\langle \epsilon_m I_m \rangle$ with $n = m(m+1)/2$.*
- *G is the quotient cube $\overline{H}(9,2)$ and $K = PSL(2,8)$.*
- *G is the halved cube $\frac{1}{2}H(9,2)$ and $K = PSL(2,8)$.*

3.2.5 The affine groups of Lie type of cross characteristic

The affine groups of Lie type of cross characteristic are dealt with in [27]. Cohen, Magaard and Shpectorov proved the following result:

Theorem 3.2.6. *Suppose that A, $s = r^b$, V, G and K are as in Theorem 3.2.1(4) with K a simple group of Lie type over $\mathbb{F}_q$ for some power $q = p^a$ of a prime p distinct from r and that K cannot be defined as a group of Lie type over a field of characteristic s. If G is not a Hamming graph, a quotient cube, a half-cube, or a quotient half-cube, then G is the coset graph of the extended ternary Golay code with intersection array $\{24, 22, 20; 1, 2, 12\}$, full automorphism group $3^6.2.M_{12}$ and $K \cong PSL(2,11)$.*

3.2.6 The affine sporadic groups case

This case is completely classified in [16] by Van Bon, Ivanov and Saxl. They proved the following result:

Theorem 3.2.7. *Suppose A, s, V, G and K are as in Theorem 3.2.1(4) with K a sporadic simple group. Then G and K are described in the following table*

Table 3.1: affine sporadic distance-transitive graphs

$\mid V \mid$	array	name	K
3^6	{24, 22,20; 1, 2, 12}	extended ternary Golay	$2.M_{12}$
2^{10}	{22, 21, 20; 1, 2, 6}	truncated binary Golay	M_{22}
2^{10}	{231, 160, 6; 1, 48, 210}	(truncated binary Golay)$_2$	M_{22}
2^{11}	{23, 22, 21; 1, 2, 3}	perfect Golay	M_{23}
2^{11}	{253, 210, 3; 1, 30, 231}	(perfect Golay)$_2$	M_{23}

3.3 Primitive DTGs of Simple Socle Type

In this section, we discuss the present state in the classification of primitive DTGs of simple socle type.

The classification of finite simple groups can be invoked to make further subdivision of the possibilities for $F^*(A)$.

- (i) Alternating groups.
- (ii) Groups of Lie type.
- (iii) Sporadic groups.

In a diagram:

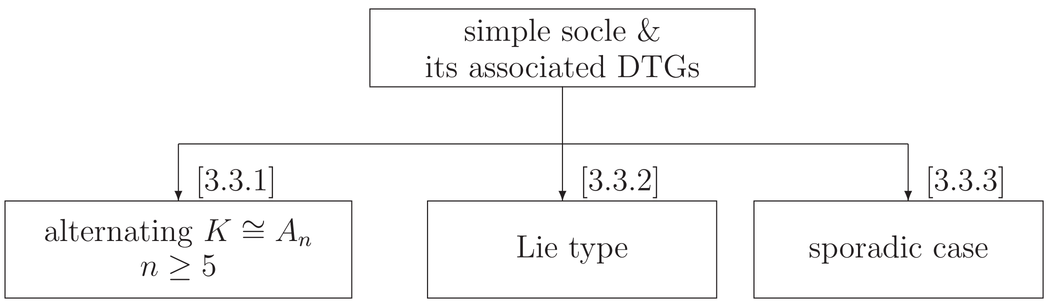


Figure 3.3: simple socle tree

3.3.1 The alternating simple socle

The main examples of graphs admitting distance-transitive action of the alternating simple socle are the Johnson graphs $J(n,k)$, $n \geq 2k$ and the Odd graphs O_k

This subcase where $F^*(A) \cong A_n (n \geq 5)$ is dealt with in [56]. Liebeck, Praeger and Saxl proved the following result:

Theorem 3.3.1. *Let A act primitively and distance-transitively on a graph G with valency and diameter at least three. If $F^*(A) \cong A_n$ for some $n \geq 5$, then G is one of the following graphs:*

- *Johnson graph $J(n,d)$ where $n > 2d$*
- *odd graph O_k where $n = 2k - 1$*
- *quotient Johnson graph $\overline{J}(2d,d)$ where $n = 2d$ and $d \geq 4$*
- *The complement of the quotient Johnson graphs $\overline{J}(8,4)$ & $\overline{J}(10,5)$*

3.3.2 The simple socle of Lie type

This is the only open case in the classification. The classification is expected to follow the pattern of $PSL(n,q)$, which is handled in [13].

Here the main examples are the Grassmann graphs $J_q(n,k)$, $1 \leq k < n$. The group $PSL(n,q)$ acts distance transitively on this graph and the full automorphism group of $J_q(n,k)$ is $P\Gamma L(n,q)$ if $n \neq 2k$ and $Aut(PSL(n,q))$ otherwise.

Theorem 3.3.2. ***(Van Bon & Cohen)*** *Let A be a group satisfying $PSL(n,q) \trianglelefteq A \trianglelefteq Aut(PSL(n,q))$ for $n \geq 2$ and $(n,q) \neq (2,2),(2,3)$. Suppose that A acts primitively and distance transitively on a graph G having diameter $d \geq 3$. Then either G is a Grassmann graph $J_q(n,k)$ or G is as listed in the following table.*

Table 3.2: graphs with distance-transitive groups A such that $F^*(A) \simeq PSL(n, q)$

$\mid V \mid$	array	name	(n,q)
28	{3,2,2,1;1,1,1,2}	Coxeter	(2,7)
36	{5,4,2;1,1,4}	Sylvester	(2,9)
45	{4,2,2,2;1,1,1,2}	gen. 8-gon (2,1)	(2,9)
68	{12,10,3;1,3,8}	Doro	(2,16)
102	{3,2,2,2,1,1,1;1,1,1,1,1,1,3}	Biggs-Smith	(2,17)
57	{6,5,2;1,1,3}	Perkel	(2,19)
65	{10,6,4;1,2,5}	Locally Petersen	(2,25)
$\frac{q^4+q^3-q-1}{q-1}$	{2q,q,q;1,1,2}	gen. 6-gon (q,1)	(3,q)
280	{9,8,6,3;1,1,3,8}	$(Her(3,4))_3$	(3,4)
56	{15,8,3;1,4,9}	J(8,3)	(4,2)

We close this subsection by listing all known primitive distance-transitive graphs of diameter $d > 2$ and simple socle of Lie type automorphism groups that are not $PSL(n, q)$.

Table 3.3: the rest(known) DTGs with simple socle Lie type groups

G	$Aut(G)$
Dual polar graphs $[C_d(q)]$	$P\Sigma p(2d, q)$
Dual polar graphs $[B_d(q)]$	$P\Gamma O(2d + 1, q)$
Dual polar graphs $[^2D_{d+1}(q)]$	$P\Gamma O^-(2d + 2, q)$
Dual polar graphs $[^2A_{2d}(r)]$	$P\Gamma U(2d + 1, r)$
Dual polar graphs $[^2A_{2d-1}(r)]$	$P\Gamma U(2d, r)$
Dual polar graphs $[\frac{1}{2}D_n(q)]$, $n = 4$	$P\Gamma O^+(2n, q)$
Dual polar graphs $[\frac{1}{2}D_n(q)]$, $n > 4$	$P\Gamma\Omega^+(2n, q)$
E_7 graphs	$F^*(Aut(G)) = E_7(q)$
unitary nonisotropics graph on 208 points	$P\Gamma U(3, 4^2)$
line graph of the Hoffman-Singleton graph	$P\Sigma U(3, 5^2)$
generalized hexagons (q, q)	$F^*(Aut(G)) = G_2(q)'$
generalized hexagons $(q, q^3), (q^3, q)$	$F^*(Aut(G)) = {}^3D_4(q)$
generalized octagons $(q, 1)$	$F^*(Aut(G)) = Sp(4, q)$ with $q = 2^a$
generalized octagons $(q, q^2), (q^2, q)$	$F^*(Aut(G)) = {}^2F_4(q)'$ with $q = 2^{2a+1}$
generalized dodecagons $(q, 1)$	$F^*(Aut(G)) = G_2(q)$ with $q = 3^a$

3.3.3 The sporadic simple socle

For $F^*(A)$ sporadic simple socle, the possible graphs G are determined in [50]. Ivanov, Linton, Lux, Saxl and Soicher have proved the following result:

Theorem 3.3.3. *Let A be a primitive distance-transitive group of automorphisms of a graph G with $d \geq 3$ and sporadic simple generalized Fitting subgroup K. Then G and K are as described in the following table.*

Table 3.4: simple socle sporadic distance-transitive graphs

$\mid V \mid$	array	name	K
266	{11,10,6,1;1,1,5,11}	Livingstone	J_1
315	{10,8,8,2;1,1,4,5}	near octagon	J_2
759	{30,28,24;1,3,15}	Witt	M_{24}
506	{15,14,12;1,1,9}	truncated from Witt	M_{23}
330	{7,6,4,4;1,1,1,6}	doubly truncated Witt	M_{22}
22880	{280,243,144,10;1,8,90,280}	Patterson	Suz

Chapter 4

Statement of Main Results

In this brief chapter we state the main results of this thesis. Specifically, we give in Table 4.1 a list which, we believe, contains all primitive distance-transitive graphs of diameter at least 3, as discussed in the previous chapter. We then give, under Theorem 4.1, a second list (Table 4.2) which, for each primitive case from Table 4.1 except one, describes all associated imprimitive graphs.

The exceptional case is that of distance-transitive generalized $2d$-gons, where we have left open the determination of all distance-transitive antipodal covers of diameter $2d$. For bipartite imprimitive distance-transitive graphs, our results are complete.

4.1 Known Primitive Distance-Transitive Graphs of Diameter at Least Three

In Table 4.1 below we give a list of the known distance-transitive graphs G of diameter at least 3, as discussed in chapter 3, and some information about the group $A = Aut(G)$.

Table 4.1: primitive distance-transitive graphs

G	A
Polygons P_n, $n \geq 6$	D_{2n}
Johnson graphs $J(n,k)$, $n > 2k$	S_n
quotient Johnson graphs $\overline{J}(2k,k)$, $k \geq 6$	S_{2k}
odd graphs O_k, $k \geq 4$	S_{2k-1}
Hamming graphs $H(n,q)$, $n > 2$	$S_q \wr S_n$
$\frac{1}{2}H(n,2)$, $n \geq 6$	$2^{n-1}.S_n$
quotient n-cube $\overline{H}(n,2)$, $n \geq 6$	$2^{m-1}.S_m$, $m = \lfloor \frac{n}{2} \rfloor$
quotient halved cube $\frac{1}{2}\overline{H}(n,2)$, even $n \geq 12$	$2^{n-2}.S_n$
Grassmann graphs $J_q(n,k)$, $n > 2k > 4$	$P\Gamma L(n,q)$

G	A
dual polar graphs $[C_d(q)]$	$P\Sigma p(2d,q)$
dual polar graphs $[B_d(q)]$	$P\Gamma O(2d+1,q)$
dual polar graphs $[{}^2D_{d+1}(q)]$	$P\Gamma O^-(2d+2,q)$
dual polar graphs $[{}^2A_{2d}(r)]$	$P\Gamma U(2d+1,r)$
dual polar graphs $[{}^2A_{2d-1}(r)]$	$P\Gamma U(2d,r)$
halved graphs $[\frac{1}{2}D_d(q)]$, $d \geq 6$	$P\Gamma\Omega^+(2d,q)$
bilinear forms graphs $H_q(n,d)$, $n \geq d > 2$	$P\Gamma L(n+d,q)_d$
alternating forms graphs $Alt(n,q)$, $n \geq 6$	$\mathbb{F}_q^n.(\mathbb{F}_q^*.P\Gamma L(n,q))$
Hermitean forms graphs $Her(n,q)$, $n > 2$, $q = p^2$	$\mathbb{F}_p^{n^2}.(\Gamma L(n,q)/\{x \in \mathbb{F}_q \mid x^{p+1} = 1\})$
E_7 graphs	$F^*(A) = E_7(q)$
affine E_6 graph	$\mathbb{F}_q^{27}.\mathbb{F}_q^*\widehat{E}_6(q)(Aut(\mathbb{F}_q))$
extended ternary Golay	$3^6.2.M_{12}$
truncated Golay	$2^{10}.M_{22}.2$
distance 2 graph of truncated Golay	$2^{10}.M_{22}.2$
perfect Golay	$2^{11}.M_{23}$
distance 2 graph of perfect Golay	$2^{11}.M_{23}$
Coxeter graph	$PSL(2,7).2$
Sylvester graph	$Aut(S_6)$
Doro graph	$P\Sigma L(2,16)$
Biggs-Smith graph	$PSL(2,17)$
Perkel graph	$PSL(2,19)$
Locally Petersen graph	$PSL(2,25).2$
$(Her(3,4))_3$	$P\Gamma L(3,4).2$
unitary nonisotropics graph on 208 points	$P\Gamma U(3,4^2)$
line graph of the Hoffman-Singleton graph	$P\Sigma U(3,5^2)$
Livingstone graph	J_1
Hall-Janko near octagon	$Aut(J_2)$
Witt	M_{24}
truncated from Witt	M_{23}
doubly truncated from Witt	M_{22}
Patterson graph	Suz
generalized hexagons $(q,1)$	$F^*(A) = PSL(3,q)$
generalized hexagons (q,q)	$F^*(A) = G_2(q)'$
generalized hexagons $(q,q^3),(q^3,q)$	$F^*(A) = {}^3D_4(q)$
generalized octagons $(q,1)$	$F^*(A) = Sp(4,q)$ with $q = 2^a$
generalized octagons $(q,q^2),(q^2,q)$	$F^*(A) = {}^2F_4(q)'$ with $q = 2^{2a+1}$
generalized dodecagons $(q,1)$	$F^*(A) = G_2(q)$ with $q = 3^a$

4.2 Covers and Doubles

Theorem 4.2.1. *For each graph G in Table 4.1 above, Table 4.2 below lists all distance-transitive imprimitive graphs that are antipodal covers of G and all distance-transitive bipartite graphs that have G as their halved graph.*

The only case not covered is that of diameter $2d$, distance-transitive, antipodal covers of generalized $2d$-gons.

In the following table, all (known) primitive distance-transitive graphs of diameter at least 3 are presented in the first column. In the second (fifth) column, it is indicated whether the graph G has antipodal (bipartite) distance-transitive covers (doubles) or not. If no covers (doubles) exist we write none. If such a cover (double) exists we write it down. Moreover, if this is the only cover (double), we write only in front of such a cover (double). In the third (sixth) column we give the section number where such a detail about the existence and uniqueness of the covers (doubles) can be found. In the fourth (seventh) column we give the reference where such information was found. Our main works are indicated by '-' in the reference column. That means the conclusions of such covers (doubles) are completely derived and proved in this thesis. The only remain unsolve problem is finding the antipodal distance-transitive covers of even diameter of the generalized $2d$-gons (if any exists). We used the question sign (?) for this case in the table.

Table 4.2: associated imprimitive distance-transitive graphs

G	Antipodal covers	Sec.	Ref.	Bipartite doubles	Sec.	Ref.
P_n, $n \geq 6$	Only P_{2n}	5.1	-	Only P_{2n}	6.2.1	[44]
$J(n,d)$, $n > 2d > 6$	None	5.2.1	[12]	Only $2.O_d$	6.2.2	[43]
$\overline{J}(2d,d)$ $d \geq 6$	Only $J(2d,d)$	5.2.1	[12]	None	6.2.3	[44]
O_{d+1}	Only $2.O_{d+1}$	5.2.2	[12]	None	6.2.4	[44]
$H(n,q)$, $n > 2$	None	5.2.3	[12]	None	6.2.5	[43]
$\frac{1}{2}H(n,2)$, $n \geq 6$	None	5.2.3	[12]	Only $H(n,2)$	6.2.5	[44]
$\overline{H}(n,2)$, $n \geq 6$	Only $H(n,2)$	5.2.3	[12]	None	6.2.6	[44]
$\frac{1}{2}\overline{H}(n,2)$, even $n \geq 12$	Only $\frac{1}{2}H(n,2)$	5.2.3	[12]	Only $\overline{H}(n,2)$	6.2.6	[44]
$J_q(n,d)$, $n > 2d$	None	5.2.4	[12]	Only $2.J_q(2d+1,d)$	6.2.7	[44]
dual polar graphs, $d \geq 3$	None	5.2.5	[12]	None	6.2.8	[44]
$[\frac{1}{2}D_d(q)]$, $d = 6$ or $d = 7$	None	5.2.5	[12]	Only $[D_d(q)]$	6.2.8	-
$[\frac{1}{2}D_d(q)]$, $d \geq 8$	None	5.2.5	[12]	Only $[D_d(q)]$	6.2.8	[44]
$H_q(n,d)$, $n \geq d > 2$	None	5.2.6A	[12]	None	6.2.9	[44]
$Alt(n,q)$, $n \geq 6$	None	5.2.6B	[12]	None	6.2.10	[44]

G	Antipodal covers	Sec.	Ref.	Bipartite doubles	Sec.	Ref.
$Her(n,q)$, $n \geq 3$ &$q = p^2$	Only !2- and !4-covers of diameter 6 when $n = 3$ & $q = 4$	5.2.6C	[12] & [17]	None	6.2.11	[44]
E_7 graphs	None	5.2.7	[12]	None	6.3.2K	-
affine E_6 graph	None	5.2.8	[12]	None	6.3.2L	-
extended ternary Golay	None	5.3.2	[12]	None	6.3.1A	-
truncated Golay	Only shortened binary Golay code & its double	5.3.3B	-	None	6.3.1B	-
(truncated Golay)$_2$	None	5.3.3C	-	Only double of truncated Golay	6.3.1C	-
perfect Golay	Only its double	5.3.3D	-	None	6.3.1D	-
(perfect Golay)$_2$	None	5.3.3E	-	Only double of binary Golay	6.3.1E	-
Coxeter graph	None	5.3.4A	-	None	6.3.2A	-
Sylvester graph	None	5.3.4B	-	None	6.3.2B	-
Doro graph	None	5.3.4C	-	None	6.3.2C	-
Biggs-Smith graph	None	5.3.4D	-	None	6.3.2D	-
Perkel graph	None	5.3.4E	-	None	6.3.2E	-
Locally Petersen graph	None	5.3.4F	-	None	6.3.2F	-
$(Her(3,4))_3$	None	5.3.4G	-	None	6.3.2G	-
unitary nonisotropics graph on 208 points	None	5.3.4I	-	None	6.3.2I	-
line graph of the Hoffman-Singleton graph	None	5.3.4J	-	None	6.3.2J	-
Livingstone graph	None	5.3.5A	-	None	6.3.3A	-
Hall-Janko near octagon	None	5.3.5B	-	None	6.3.3B	-
Witt	None	5.3.1	[12]	None	6.3.3C	-
truncated from Witt	None	5.3.1	[51]	None	6.3.3D	-
doubly truncated from Witt	Only Faradjev-Ivanov-Ivanov 3-cover with diameter 8	5.3.1	[18] & [29]	None	6.3.3E	-
Patterson graph	None	5.3.5F	-	None	6.3.3F	-

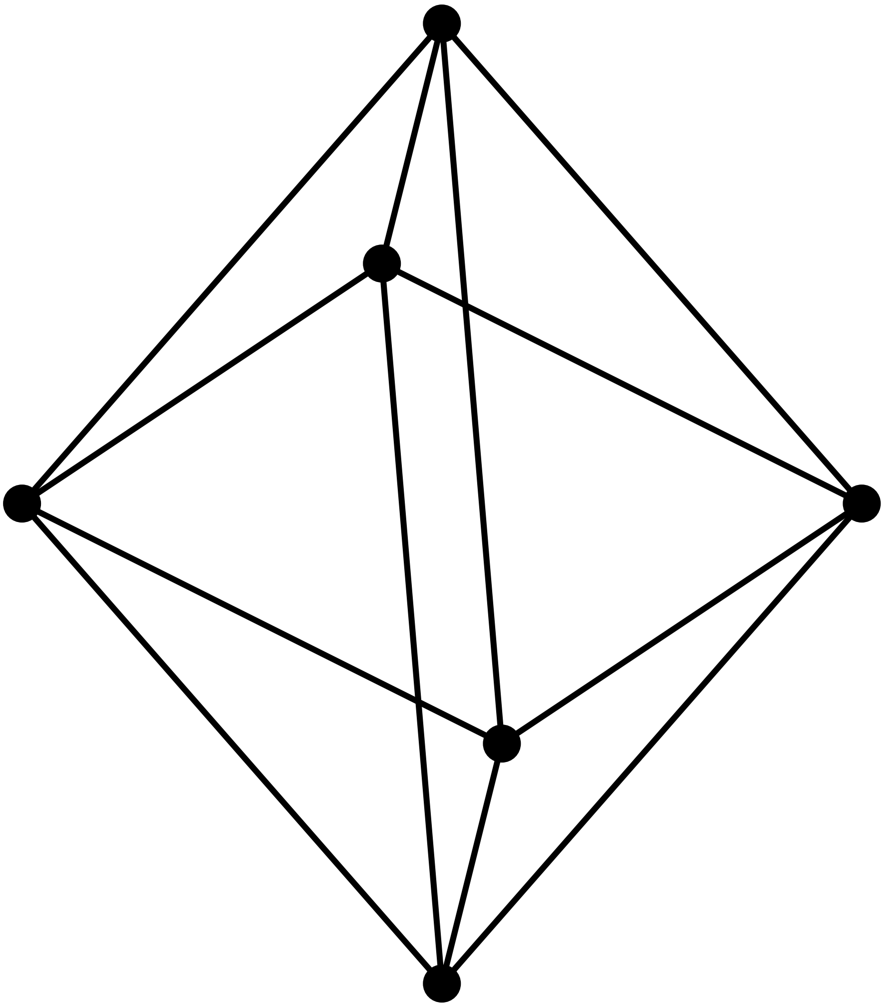
G	Antipodal covers	Sec.	Ref.	Bipartite doubles	Sec.	Ref.
generalized hexagons of order $(q, 1)$	(odd diameter) None	5.4	[12]	None	6.4	-
	(even diameter) ?					
generalized hexagons of order (q, q)	(odd diameter) None	5.4	[12]	Only gen. dodecagons $(1, q)$	6.4	-
	(even diameter) ?					
generalized hexagons of order (q, q^3)	(odd diameter) None	5.4	[12]	None	6.4	-
	(even diameter) ?					
generalized hexagons of order (q^3, q)	(odd diameter) None	5.4	[12]	None	6.4	-
	(even diameter) ?					
generalized octagons of order $(q, 1)$	(odd diameter) None	5.4	[12]	None	6.4	-
	(even diameter) ?					
generalized octagons of order (q, q^2)	(odd diameter) None	5.4	[12]	None	6.4	-
	(even diameter) ?					
generalized octagons of order (q^2, q)	(odd diameter) None	5.4	[12]	None	6.4	-
	(even diameter) ?					
generalized dodecagons of order $(q, 1)$	(odd diameter) None	5.4	[12]	None	6.4	-
	(even diameter) ?					

We actually prove somewhat more. In particular, in almost all cases, Table 4.2 classifies the imprimitive distance-regular graphs associated with the graphs of Table 4.1.

Also, in situations where there are diameter 2 distance-transitive graphs belonging to the same infinite family as distance-transitive graphs of larger diameter from Table 4.1, we sometimes consider their covers as well. However, we have made no effort to cover the distance 2 case uniformly. Although all distance-transitive graphs of diameter 2 are known (see, for instance [54]), there are many more families and isolated examples than for larger diameter. Also, in contrast to the general case, the complement of a distance-transitive graph of diameter 2 is again distance-transitive of diameter 2; so imprimitive graphs associated with both the graph and its complement must be considered.

The imprimitive distance-transitive graphs associated with primitive graphs of diameter 1 have all been classified. For distance-transitive antipodal covers of complete graphs see [40,41], and for those of complete bipartite graphs see [49]. For distance-transitive bipartite graphs whose halved graph is complete, see [48]. We will have no more to say about these cases.

Antipodal Covers



Chapter 5

Antipodal Covers

In this chapter we prove results that, in particular, prove Theorem 4.1 in the case of antipodal covers. For this chapter, the primary references are the book of Brouwer, Cohen, and Neumaier [17] and the paper of Van Bon and Brouwer [12].

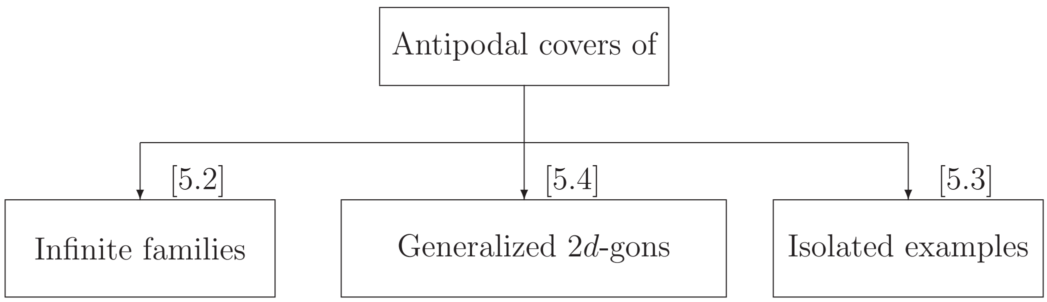


Figure 5.1: antipodal main tree

5.1 Quotient Graphs

Recall that a distance-regular graph G with diameter d is antipodal if for all distinct vertices $v, w \in \{u\} \cup G_d(u)$, we have $d(v, w) = d$. The quotient graph $\overline{G}$ of G is defined by taking the fibres (antipodal classes) of G as its vertices, with two such fibres join by an edge in $\overline{G}$ if they contain adjacent vertices of G. In this case, we say that G is an r-antipodal cover of $\overline{G}$, where r is the common cardinality of the fibres (the index of G).

We can give an equivalent definition of a cover H of G using the projection map ρ from $V(H)$ to $V(G)$. Let H be an antipodal distance-regular graph of diameter $D \geq 3$ with quotient graph G. We say that H is a cover of G if there is a map $\rho : V(H) \to V(G)$ called a covering map which is a graph morphism, i.e., preserves adjacency, and a local isomorphism, i.e., for each vertex u of H the restriction of ρ to the set $u^{\perp} = H_{\leq 1}(u)$ is bijective. Then $\rho^{-1}(u), u \in G$ is the set of fibres and $r = |\rho^{-1}(u)|$ is the index of the cover.

The following result is due to Gardiner (see Proposition 4.2.2 [17] & [31]).

Proposition 5.1.1. *Let G be a distance-regular graph with intersection array $i(G) = \{b_0, b_1, ..., b_{d-1}; c_1, c_2, ..., c_d\}$ and diameter $d \in \{2m, 2m+1\}$. Then G is antipodal if and only if $b_i = c_{d-i}$ for $i = 0, ..., d$, $i \neq m$. In this case G is an r-antipodal cover of its quotient graph $\overline{G}$, where $r = 1 + \frac{b_m}{c_{d-m}}$. Moreover, for $d > 2$, $\overline{G}$ is a distance-regular of diameter m with intersection array*

$$i(\overline{G}) = \{b_0, b_1, ..., b_{m-1}; c_1, c_2, ..., c_{m-1}, \gamma c_m\}.$$

where $\gamma = \begin{cases} r, & \text{if } d=2m \text{ ;} \\ 1, & \text{if } d=2m+1. \end{cases}$

The following theorem is due to Gardiner [31].

Theorem 5.1.2. *Let G be a distance-regular graph of diameter d with intersection numbers b_i, c_i and suppose H is an r-antipodal cover of G of diameter $D > 2$. Then H has diameter $D = 2d$ or $D = 2d+1$ and $i(H) = \{b_0, b_1, ..., b_{d-1}, \frac{r-1}{r}c_d, c_{d-1}, ..., c_1; c_1, ..., c_{d-1}, \frac{1}{r}c_d, b_{d-1}, ..., b_0\}$ for even D, and $i(H) = \{b_0, b_1, ..., b_{d-1}, t(r-1), c_d, c_{d-1}, ..., c_1; c_1, ..., c_{d-1}, c_d, t, b_{d-1}, ..., b_0\}$ for odd D and some integer t.*

Remarks:

- For D even the intersection array properties implies that $r \mid c_d$ and $r \leq c_d/max(c_{d-1}, c_d - b_{d-1})$.
- For D odd the integer t satisfies the conditions $t(r-1) \leq min(b_{d-1}, a_d)$ and $c_d \leq t$. Clearly, given $i(G)$, there are only finitely many possibilities for r and t, and if $c_d > min(b_{d-1}, a_d)$, there are none.

Corollary 5.1.3. *If H is an antipodal distance-regular cover of G, then H has an antipodal distance-regular cover only if G(and hence H) is either a cycle, a complete graph or a complete bipartite graph.*

Proof. Let G be a noncomplete nor a complete bipartite, DRG of diameter d and valency k with intersection numbers a'_i, b'_i, c'_i and suppose H is an r-antipodal cover of G. Then by theorem 5.1.2, we have $c'_1 = 1 = b'_{d-1}$.

If the diameter D of H is even, then $b'_{d-1} = 1$ implies $r = 2$ and $1 = c_d/2 = c_{d-1} = ... = c_1 = 1$. But if H is an antipodal graph, then $c_d = k = 2$, and so H is a cycle.

If the diameter D of H is odd, then $1 = b_{d-1} = c_d = k$, which is impossible. ■

Corollary 5.1.4. *For an r-antipodal distance-regular cover of a graph G of valency k to exist, the index r is at most k.*

Now let us consider the reconstruction problem by which an antipodal graph is obtained from its quotient graph.

Van Bon proved the following geometric criteria that are necessary conditions for the existence of antipodal cover of DRGs (see sec. 2 [47]). In what follows, if $u, v \in V(G)$ with $d(u, v) = d$, then $C(u, v)$ is the union of all geodesics between u and v.

Proposition 5.1.5. *Suppose that G is distance-regular graph of diameter $d \geq 2$ and has an r-antipodal distance-regular cover of diameter $2d$. Then for any two vertices u, v in G with $d(u, v) = d$, the subgraph induced by $C(u, v) \setminus \{u, v\} = \bigcup_{j=1}^{d-1}(G_j(u) \cap G_{d-j}(v))$ in G is the disjoint union of r subgraphs of equal size.*

Proof. Let H be an r-antipodal distance-regular cover of diameter $2d$ of G, with covering map $\rho : H \to G$. Let $u_1 \in \rho^{-1}(u)$, and let $\rho^{-1}(v) = \{v_1, ..., v_r\}$. Let C_j be the union of all geodesics in H between u_1 and $v_j (1 \leq j \leq r)$ and $C = \bigcup_j C_j \ \{u_1, v_j\}$. Then $\rho \mid_C : C \to C(u, v)$ is an isomorphism.

Proposition 5.1.6. *Suppose that G is a distance-regular graph of diameter $d \geq 2$ and has an r-antipodal distance-regular cover of diameter $2d+1$. Let u, v be vertices of G with $d(u, v) = d$, and put $E = \{v\} \cup \{G(v) \cap G_d(u)\}$. Then the collection of sets $C(u, w) \setminus \{u, w\} (w \in E)$ can be partitioned into r nonempty parts such that sets from different parts are disjoint, and all edges joining vertices in different parts are contained in $G(u)$.*

Proof. Let H be an r-antipodal distance-regular cover of diameter $D = 2d + 1$ of G, with covering map $\rho : H \to G$. Let u, v be two vertices of G with $d(u, v) = d$. Further, let $u_1 \in \rho^{-1}(u)$ and C_i be the union of all geodesics in G of length at most i with origin and terminal both in u. If C is a cycle in G of length at most $D - 1 = 2d$ then, since ρ is an antipodal covering, $\rho^{-1}(C)$ is the disjoint union of r copies of C. This implies that $H(u_1)$ contains C_{D-1}. Since ρ is a graph morphism and a local isomorphism, the subgraph induced by $C(u, v) \setminus \{u, v\}$ in G is the disjoint union of r subgraphs of equal size. ■

Thus the problem of finding antipodal covers of a given DRG is related to the study of the structure of geodesics joining the vertices at maximal distance.

A **near polygon** is a connected graph G of diameter $d \geq 2$ with the following two properties:

- There are no induced subgraphs of the shape $K_{1,1,2}$
- If $u \in V(G)$ and L is a singular line with $d(u, L) < d$, then there is a unique point on L nearest to u

Such a graph is also called a **near (2d+1)-gon** if there is a point at distance d from some singular line and a **near (2d)-gon** otherwise.

Corollary 5.1.7. *If $d \geq 2$, and any two adjacent vertices v, w in $G_d(u)$ have a common neighbour in $G_{d-1}(u)$, then G have no antipodal distance-regular covers of diameter $2d + 1$. In particular, this holds for the collinearity graph of a regular near $2d$-gon.*

Proof. Clearly, if $G(v) \cap G(w) \cap G_{d-1}(u) \neq \emptyset$, then $C(u,v) \setminus \{u,v\}$ and $C(u,w) \setminus \{u,w\}$ are not disjoint. But if G is a near $2d$-gon, and vw is an edge in $G_d(u)$, then by definition of regular near $2d$-gon, the line containing vw has a (unique) point in $G_{d-1}(u)$.

Corollary 5.1.8. *If $d \geq 2$ and for any two adjacent vertices v, w in $G_d(u)$ there is a point in $G(u) \cap G_{d-1}(v) \cap G_{d-1}(w)$, then G have no antipodal distance-regular cover of diameter $2d+1$. Similarly, If $d \geq 3$ and for any two vertices u, v at distance d and any two vertices $x, y \in G(v) \cap G_{d-1}(u)$ there is a vertex $w \in G(u) \cap G_{d-2}(x) \cap G_{d-2}(y)$, then G have no antipodal distance-regular covers of diameter $2d$.*

Trivially, bipartite graphs have no antipodal distance-regular covers of odd diameter. As was already remarked, antipodal graphs of diameter $d \geq 3$ have no antipodal distance-regular covers.

Given a distance-regular graph G with diameter d and parameters a_i, b_i, c_i and k_i, **its intersection matrix** B is the tridiagonal $(d+1) \times (d+1)$ matrix

$$\begin{pmatrix} 0 & k & & & & \\ 1 & a_1 & b_1 & & & \\ & c_2 & a_2 & \cdot & & \\ & & c_3 & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & b_{d-1} \\ & & & & & & a_d \end{pmatrix}.$$

We shall say that a vector $x = [x_0, x_1, ..., x_n]^t$ is standard when $x_0 = 1$.

Theorem 5.1.9. *(see Theorem 2.4.5 [53]) Let $u = [u_1, u_2, ..., u_d]^t$ be a standard right eigenvector corresponding to the eigenvalue λ of B. Then the multiplicity of the eigenvalue λ of a distance-regular graph with diameter d and n vertices is*

$$m(\lambda) = \frac{n}{\sum_{i=0}^{d} k_i (u_i)^2}.$$

We close this section by proving the well-known result that the only antipodal distance-transitive cover of the polygon P_n with $n \geq 6$ is P_{2n}.

Proposition 5.1.10. *The only antipodal distance-transitive cover of the polygon P_n, $n \geq 6$, is P_{2n}.*

Proof. Case(1) n is even, the conclusion follows from corollary 5.1.3 above.
Case(2) $n \geq 7$ is odd. Let H be an r-antipodal cover of P_n, $n = 2d+1$, of diameter D. Then by 5.1.2, either $D = 2d$ and in this case $i(H) = \{2, 1, ..., 1, \frac{r-1}{r}(1), 1, ..., 1; 1, ..., 1, \frac{1}{r}(1), 1, ..., 1, 2\}$ or $D = 2d+1$ and $i(H) = \{2, 1, ..., 1, t(r-1), 1, ..., 1; 1, ..., 1, 1, t, 1, ..., 2\}$.

(1) $D = 2d$. By the remarks of 5.1.2, $r|1$ and $r \leq \frac{1}{max(1,1-1)}$. Thus the only possibility is $r = 1$. Hence no such a cover exists.

(2) $D = 2d+1$. By the remarks of 5.1.2, $t(r-1) \leq min(1,1)$ and $1 \leq t$. Thus the only possibility is $t=1$ and $r=2$. Hence $i(H) = \{2,1,...,1,1,1,...,1;1,...,1,1,1,1,...,2\}$ which is uniquely determined by the polygon $P_{2D} = P_{2n}$. It is distance-transitive. Hence the only antipodal distance-transitive double of P_n is P_{2n}. ■

5.2 Infinite Families

In this section we discuss all distance-transitive antipodal covers of examples belonging to infinite families (with the exception of the generalized $2d$-gons). These are essentially those examples with classical parameters. This case is handled by Van Bon and Brouwer [12] (see also Terwilliger [68]). In those cases where Van Bon and Brouwer report results without proof, we provide the corresponding proofs.

We shall say that a distance-regular graph has **classical parameters** (d, b, α, β) if its diameter is d and its intersection array parameters can be expressed in terms of the diameter d and three other parameters b, α, β as follows:

$$b_i = (\begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix})(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}),$$
$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} (1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix})$$

for $i = 0, ..., d$, where

$$\begin{bmatrix} i \\ l \end{bmatrix} = \begin{cases} \prod_{j=0}^{l-1} \frac{i-j}{l-j} = \binom{i}{l} & \text{if b=1}, \\ \prod_{j=0}^{l-1} \frac{b^i - b^j}{b^l - b^j} & \text{if } b \neq 1. \end{cases}$$

are the **Gaussian binomial coefficients** with basis b i.e., the number of l-dimensional subspaces of an i-dimensional vector space over $\mathbb{F}_b$. Using the above two formulas, one can write the valency k and a_i in terms of the d, b, α, β as well:

$$k = \begin{bmatrix} d \\ 1 \end{bmatrix} \beta$$
$$a_i = \begin{bmatrix} i \\ 1 \end{bmatrix} (\beta - 1 + \alpha(\begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} - \begin{bmatrix} i-1 \\ 1 \end{bmatrix})).$$

For a given graph G, we define its double $\widetilde{G}$ as the graph whose vertices are the symbols u^+, u^- $(u \in G)$ where two vertices u^σ, v^τ are adjacent whenever $u \sim v$ and $\sigma \neq \tau$. Then clearly $\widetilde{G}$ is bipartite.

The following tree gives an overall picture of the current section.

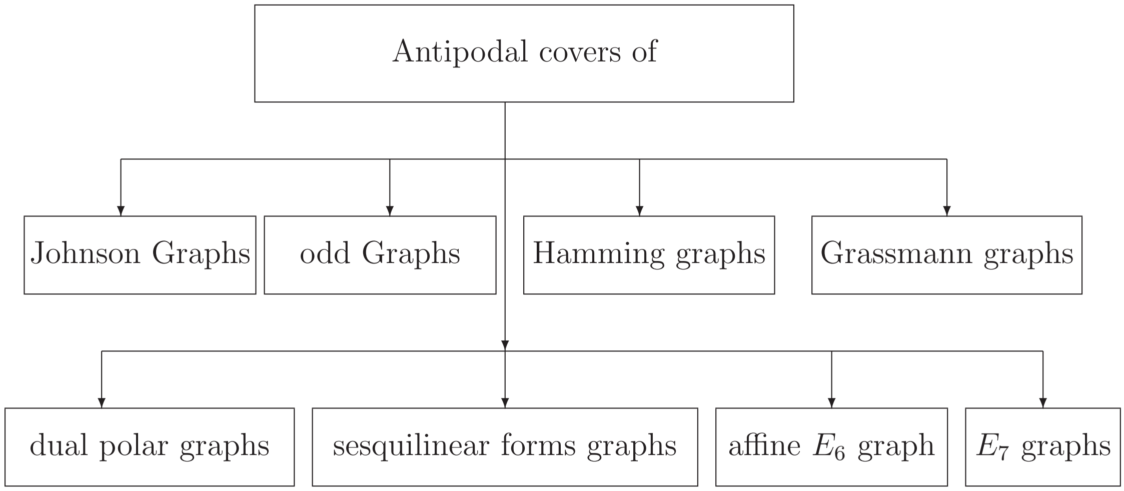


Figure 5.2: antipodal covers of classical DTGs tree

5.2.1 Johnson graphs

Recall that the Johnson graphs $J(n, k)$ (where $1 \leq k < n$) have as vertices the k-subsets of an n-set, with two k-subsets adjacent if and only if they intersect in exactly $k - 1$ elements. The Johnson graphs $J(n, 2)$ are sometimes known as **Triangular graphs** and also denoted by $T(n)$, and the Johnson graphs $J(n, 3)$ are sometimes known as **Tetrahedral graphs**. The Johnson graph $J(n, k)$ is distance-transitive graph with diameter $d = min(k, n-k)$ and intersection array

$$i(J(n, k)) = \{k(n - k), ..., (k - i)(n - k - i), ...; 1^2, ..., i^2, ...\}.$$

Lemma 5.2.1. *(see sec 9.1 [17]) $J(n, k) \cong J(n, n - k)$, for $n \geq k$. Moreover, for $u, v \in J(n, k)$, $d(u, v) = m$ if and only if $|u \cap v| = k - m$.*

Examples

1. $J(4, 2)$ is the line graph $L(K_4)$ of the complete graph K_4. It is strongly regular with vertex set $V = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$. It is isomorphic to the octahedron and has intersection array $i = \{4, 1; 1, 4\}$.

2. $J(5, 2)$ is the line graph $L(K_5)$ of the complete graph K_5. It is strongly regular with vertex set $V = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}\ , \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}$. It has intersection array $i = \{6, 2; 1, 4\}$.

3. $J(6, 3)$ is the first Johnson graph with diameter three. It has 20 vertices and 90 edges with intersection array $i = \{9, 4, 1; 1, 4, 9\}$.

Proposition 5.2.2. *(see sec. 4 [12]) For $n \geq 2d \geq 6$, The Johnson graph $J(n, d)$ has no antipodal distance-regular covers.*

Proof. The Johnson graph $J(n,k)$ has diameter $d = min(n, n-k)$. Since $J(n,k) \cong J(n, n-k)$ (lemma 5.2.1 above), we may assume that $n \geq 2k$, so that $d = k$.

Let $u, v \in V(J(n,d))$ with $d(u,v) = d$, and pick any two distinct vertices $x, y \in G(u) \cap G_{d-1}(v)$. By lemma 5.2.1, we have $|u \cap v| = d - d = 0$ and $|u \cap x| = |u \cap y| = d - 1$. Without loss of generality, we may write u, v and x as $u = \{u_1, u_2, ..., u_d\}$, $v = \{v_1, v_2, ..., v_d\}$ and $x = \{u_1, u_2, ..., u_{d-1}, v_i\}$ for some $i \in \{1, ..., d\}$, where the u_i and v_i are all distinct. So, we must have one of the following possibilities for y:

(i) $y = \{u_1, u_2, ..., u_{d-1}, v_j\}$ for some $j \in \{1, ..., i-1, i+1, ..., d\}$;
(ii) $y = \{u_1, u_2, ..., u_{j-1}, u_{j+1}, ..., u_{d-1}, u_d, v_i\}$ for some $j \in \{1, ..., d\}$;
(iii) $y = \{u_1, u_2, ..., u_{j-1}, u_{j+1}, ..., u_{d-1}, u_d, v_k\}$ for some $k \neq i, j \in \{1, ..., d\}$;

In cases (i) & (ii), x and y are adjacent vertices. In case(iii), the vertex $z = \{u_1, u_2, ..., u_{j-1}, u_{j+1}, ..., u_{d-1}, u_d, v_i\}$ is a common neighbor of x and y in $G(u) \cap G_{d-1}(v)$. Hence $G(u) \cap G_{d-1}(v)$. So, by proposition 5.1.5, there are no antipodal distance-regular covers of even diameter.

Similarly, let $u = \{u_1, u_2, ..., u_d\} \in V(J(n,d))$. Suppose that v, w be two adjacent in $G_d(u)$. Lemma 5.2.1, gives $|u \cap v| = |u \cap w| = d - d = 0$ and $|v \cap w| = d - 1$. Without loss of generality, we can write v and w as $v = \{v_1, v_2, ..., v_d\}$ and $w = \{v_1, v_2, ..., v_{d-1}, a\}$ where the u_i, v_i and a are all distinct. Then $y = \{v_1, v_2, ..., v_{d-1}, u_i\}$, for $i \in \{1, 2, ..., d\}$, is a common neighbor of v and w in $G_{d-1}(u)$, so that by corollary 5.1.7 there are no antipodal distance-regular covers of odd diameter. ■

The only imprimitive Johnson graphs are the graphs $J(2d, d)$ which are antipodal. So, we need to consider their quotients.

The $p \times q$ graph has vertex set $P \times Q$ where $|P| = p$, $|Q| = q$, and (x_1, y_1) is adjacent to (x_2, y_2) if and only if $x_1 = x_2$ or $y_1 = y_2$ (but not both). A graph H is called **locally** G if for each vertex x of H the graph induced by H on the set of neighbors of x is isomorphic to the graph G.

Let us denote by $\mu(x,y)$ (where x and y are two vertices at distance 2 in a graph G) the graph induced on the set of common neighbors of x and y; we shall call subgraphs of G of this form μ-graphs.

Proposition 5.2.3. *(see pg. 147 [12]) For $d \geq 4$, the only antipodal distance-regular cover of the quotient Johnson graph $\overline{J}(2d, d)$ is the Johnson graph $J(2d, d)$.*

Proof. Let H be an antipodal distance-regular cover of G, with covering map $\rho : H \to G$. G is locally $m \times m$ and the connected components of each μ-graph of G are 4-cycles (see Theorem 1 [9]). Since ρ is a graph morphism, i.e., preserves adjacency, and a local isomorphism, i.e., for each u of H the restriction of ρ to the set $u^{\perp} = H_{\leq 1}(u)$ is bijective, then H must be locally $m \times m$ and the connected components of each μ-graph of H are 4-cycles. Hence H is a Johnson graph (see Theorem 1 [9]). ■

The Johnson graphs $J(n, 2) := T(n)$ and the quotient graphs $\overline{J}(8, 4)$ and $\overline{J}(10, 5)$ are strongly regular, and their complementary graphs are also strongly regular. The graph

$\overline{T}(5)$,the complement of $T(5)$, is the Petersen graph and will be treated next subsection. The complement of the quotient graph $\overline{J}(8,4)$ is the Grassmann graph of the lines in $PG(3,2)$, and we will see below that no antipodal covers exist. The complement of the quotient graph $\overline{J}(10,5)$ has no feasible parameters and so no antipodal covers exist.

Proposition 5.2.4. *(see Proposition 4.2 [12]) The complement graphs $\overline{T(n)}$ of $T(n)$ have no distance-regular covers for $n \geq 8$. There is a unique 3-antipodal distance transitive cover with 45 vertices and diameter 4 of $\overline{T(6)}$. Also, there is a unique 3-antipodal distance transitive cover with 63 vertices and diameter 4 of $\overline{T(7)}$.*

5.2.2 Odd graphs

Recall that the odd graphs O_k $(k \geq 2)$ have the $(k-1)$-subsets of a $(2k-1)$-set as vertices, with two $(k-1)$-subsets joined by an edge if and only if they disjoint. O_k is distance-transitive graph with valency k, diameter $d = k-1$ and intersection array

$$i(O_k) = \{k, k-1, k-1, ..., \tfrac{1}{2}(k+1), \tfrac{1}{2}(k+1); 1, 1, ..., \tfrac{1}{2}(k-1), \tfrac{1}{2}(k-1)\}$$

for k odd, and

$$i(O_k) = \{k, k-1, k-1, ..., \tfrac{1}{2}k+1, \tfrac{1}{2}k+1; 1, 1, ..., \tfrac{1}{2}k-1, \tfrac{1}{2}k-1\}$$

for k even.

Proposition 5.2.5. *For $m \geq 0$ and $u, v \in O_k$, we have*

1. *$d(u,v) = 2m$ if and only if $|u \cap v| = (k-1) - m$.*
2. *$d(u,v) = 2m+1$ if and only if $|u \cap v| = m$.*

Proposition 5.2.6. *(see Proposition 4.1 [12]) The double odd graphs $2O_k$ are the only antipodal distance-regular covers of O_k for $k \geq 4$, and they are distance-transitive. The Petersen graph O_3 has two antipodal distance-transitive covers, namely its double $\widetilde{O_3}$(sometimes called the Desargues graph) and the dodecahedron.*

5.2.3 Hamming graphs

Recall that the Hamming graph $H(n,q)$ has the set of all n-tuples from an alphabet of q symbols as its vertex set, where two n-tuples are adjacent when they differ in exactly one coordinate. The Hamming graph is distance-transitive with diameter $d = n$ and intersection array

$$i(H(n,q)) = \{n(q-1), (n-1)(q-1), ..., 1(q-1); 1, 2, ..., n\}.$$

Proposition 5.2.7. *(see Proposition 5.1 [12]) The Hamming graph $H(n,q)$ with $n > 2$ has no antipodal distance-regular covers.*

The only imprimitive Hamming graphs are the $n-$cubes $H(n,2)$ which are both bipartite and antipodal. So, we need to consider their quotient and halved graphs.

Proposition 5.2.8. *(see Proposition 5.2 [12]) The only antipodal distance-regular cover of the quotient n-cube $\overline{H}(n,2)$ with $n \geq 6$ is the n-cube $H(n,2)$, and it is distance-transitive.*

The only antipodal distance-transitive covers of the quotient 4-cube are the 4-cube and a unique cover with intersection array $\{4,3,3,1;1,1,3,4\}$. The only antipodal distance-transitive covers of the quotient 5- are the 5-cube and a unique cover with intersection array $\{5,4,1,1;1,1,4,5\}$. The quotient n-cubes are characterized by the parameters except when $n = 6$. For $n = 6$, there are three nonisomorphic graphs but none of these has antipodal distance-regular covers (see [12]).

Proposition 5.2.9. *(see Proposition 5.3 [12]) The halved n-cube $\frac{1}{2}H(n,2)$ has no antipodal distance-regular covers for $n \geq 4$.*

Now, if n is even, the halved graph $\frac{1}{2}H(n,2)$ is still antipodal. So, we need to consider its quotient graph. In what follows, $\frac{1}{2}\overline{H}(n,2)$ will denote the quotient halved (or halved quotient) n-cube.

Proposition 5.2.10. *(see Proposition 5.3 [12]) The only antipodal distance-regular cover of the quotient halved n-cube $\frac{1}{2}\overline{H}(n,2)$ (even $n \geq 8$) is $\frac{1}{2}H(n,2)$, and it is distance-transitive.*

The Hamming graphs $H(2,q)$ and the quotient cubes $\overline{H}(4,2)$, $\overline{H}(5,2)$, the halved graphs $\frac{1}{2}H(4,2)$ and $\frac{1}{2}H(5,2)$ and the halved quotient cubes $\frac{1}{2}\overline{H}(8,2)$ and $\frac{1}{2}\overline{H}(n,2)$ are strongly regular graphs. So we should consider the possible covers of their complements.

$\overline{H(2,2)}$ and $\overline{\overline{H}(4,2)}$ and $\overline{\frac{1}{2}H(4,2)}$ are disconnected and so they are no more DTGs. Next, $\overline{H(2,3)} \cong H(2,3)$ and $\overline{\overline{H}(5,2)} \cong \frac{1}{2}H(5,2)$ and $\overline{\frac{1}{2}H(5,2)} \cong \overline{H}(5,2)$, so these have been treated already. Next $\overline{H(2,q)}$ with $q \geq 4$ has $c_d > 2b_{d-1}$ and hence no covers exist. $\overline{\frac{1}{2}\overline{H}(8,2)}$ is the alternating forms on $\mathbb{F}_2^4$ and we will show that no covers exist. Finally, $\overline{\frac{1}{2}\overline{H}(10,2)}$ has $c_2 > 2b_1$ and so no covers exist.

5.2.4 Grassmann graphs

Recall that the Grassmann graph $J_q(n,k)$ $(1 \leq k < n)$, have vertices the k-dimensional subspaces of an n-dimensional vector space over a field $\mathbb{F}_q$, with two of the k-subspaces joined by an edge if and only if they intersect in a subspace of dimension $k-1$. $J_q(n,k)$ is distance-transitive with diameter $d = min(k, n-k)$.

Lemma 5.2.11. *(see sec 9.3 [17]) For $n \geq k$, we have $J_q(n,k) \cong J_q(n,n-k)$.*

Thus we usually assume that $n \geq 2k$ and so, $d = k$.

Proposition 5.2.12. *(see Proposition 6.1 [12]) The Grassmann graphs of diameter at least 2 have no antipodal distance-regular covers.*

The only strongly regular Grassmann graphs are $J_q(n,2)$. No covers exist for their complements except when $n=3$, where the complement is actually the quotient Johnson graph $\overline{J(8,4)}$ which has the unique cover $J(8,4)$.

Let V be a $(2d+1)$-dimensional vector space over $GF(q)$. The **doubled Grassmann graph** $2J_q(2d+1,d)$ is the graph G whose vertices are the (k)-subspaces and $d+1$-subspaces of V, where distinct vertices u, v are joined if and only if $u \subset v$ or $v \subset u$.

Proposition 5.2.13. *(see Proposition 6.2, [12]) The double Grassmann graphs $2J_q(2d+1,d)$ of diameter at least 2 have no antipodal distance-regular covers.*

5.2.5 Dual polar graphs

Let V be one of the following spaces equipped with a specified form with q a prime power.
$[Sp(2d,q)] := [C_d(q)] = \mathbb{F}_q^{2d}$ with a nondegenerate symplectic form;
$[\Omega(2d+1,q)] := [B_d(q)] = \mathbb{F}_q^{2d+1}$ with a nondegenerate quadratic form;
$[\Omega^+(2d,q)] := [D_d(q)] = \mathbb{F}_q^{2d}$ with a nondegenerate quadratic form of (maximal) Witt index d;
$[\Omega^-(2d+2,q)] := [{}^2D_{d+1}(q)] = \mathbb{F}_q^{2d+2}$ with a nondegenerate quadratic form of (non-maximal) Witt index d;
$[U(2d+1,r)] := [{}^2A_{2d}(r)] = \mathbb{F}_q^{2d+1}$ with a nondegenerate Hermitean form $(q=r^2)$;
$[U(2d,r)] := [{}^2A_{2d-1}(r)] = \mathbb{F}_q^{2d}$ with a nondegenerate Hermitean form $(q=r^2)$;

A subspace of V is called **totally isotropic** (in certain cases totally singular) if the form vanishes completely on this subspace. The maximal totally isotropic (singular) subspaces here are the d-spaces of V. The dual polar graph on V has as vertex set the maximal totally isotropic subspaces, where two vertices are joined if they meet in $(d-1)$-dimensional subspace. It has classical parameters $(d,q,0,q^e)$ where $e \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$.

Proposition 5.2.14. *(see Proposition 7.1 [12]) Dual polar graphs of diameter at least 3 have no antipodal distance-regular covers. For $d=2$, no other covers exist except for the complete bipartite $K_{q+1,q+1}$ which have been treated and the generalized quadrangle of order $(2,2)$ which has a unique 3-antipodal distance transitive cover.*

The dual polar graphs on $[D_d(q)]$ are bipartite. So we need to consider their halved graphs $\frac{1}{2}[D_d(q)]$.

Proposition 5.2.15. *(see pg. 151 [12]) The halved graph $\frac{1}{2}[D_d(q)]$ of diameter $m=[d/2] \geq 2$ has no antipodal distance-regular covers.*

Proof. (We shall use the terminology of the theory of near polygons, see [20] & [61].) Let G denotes the dual polar graph $[D_d(q)]$. Suppose that $u, v \in V(G_2)$ with $d(u,v)=m$. Let $x, y \in (G_2)(u) \cap (G_2)_{m-1}(v)$. Choose $x_1, y_1 \in V(G)$ such that $d(x_1,x)=d(y_1,y)=d(x_1,u)=d(y_1,u)=1$ and $d(x_1,v)=d(y_1,v)=2m-1$. Then there is a vertex w joined to x_1 and y_1 with $d(w,v)=2m-2$ (see Theorem 2(ii) [20]). Thus $x_1, y_1 \in G(w) \cap G_{2m-1}(v)$. Hence there is a unique quad Q containing $\{x_1, y_1, w\}$ (see Lemma 2.14 [61]). Since all point-quad relations are classical, then there is a unique point $z \in G_{2m-2}(v) \cap Q$, and $z \sim x_1, y_1$ (see Lemma 2.14

[61]). Thus, there is a common neighbor z of x and y in G_2 with $z \in G_2(u) \cap (G_2)_{m-1}(v)$. Hence $G_2(u) \cap (G_2)_{m-1}(v)$ is connected. It follows from proposition 5.1.5 that no covers of diameter $2m$ exist.

Now, let x, y be two adjacent vertices in $(G_2)_m(u)$. Then $x, y \in G_{2m}(u)$ and $d(x, y) = 2$ in G. Hence there is a vertex w joined to x and y with $d(w, u) = 2m - 1$ (see Theorem 2(ii) [20]). Let $z \in V(G)$ with $d(w, z) = 1$ and $d(z, u) = 2m - 2$. Then $z \sim x, y$ in G_2. Further $z \in G_{2m-2} = (G_2)_{m-1}$. Hence by corollary 5.1.7 no covers of diameter $2m+1$ exist. ■

The six dual polar graphs with $d = 2$ and the halved dual polar graphs $\frac{1}{2}[D_4(q)]$ and $\frac{1}{2}[D_5(q)]$ are strongly regular graphs. However, no covers occur for any of their complements (see sec. 7 [12]).

5.2.6 Sesquilinear forms graphs

A. Bilinear forms graphs

Recall that the bilinear forms graphs $H_q(n, m)$ $(n \geq m)$ have vertices the $n \times m$ matrices over $\mathbb{F}_q$, with two matrices joined by an edge if and only if their difference has rank 1. The bilinear forms graph is distance-transitive with diameter $d = m$.

Proposition 5.2.16. *(see Proposition 8.1 [12]) The bilinear forms graph $H_q(n, d)$, $n \geq d$, of diameter $d \geq 2$ has no antipodal distance-regular covers.*

B. Alternating forms graphs

Recall that the alternating forms graphs $Alt(n, q)$ $(n, q > 1)$ have vertices the $n \times n$ alternating matrices over $\mathbb{F}_q$,that is, all $n \times n$ matrices $(a_{ij})_{n \times n}$ with $a_{ij} = -a_{ji}$ for $1 \leq i, j \leq n$, with two matrices joined by an edge if and only if their difference has rank 1.

Proposition 5.2.17. *(see Proposition 9.1 & 9.2 [12]) The alternating forms graph $Alt(n, q)$ on $\mathbb{F}_q$ with $n \geq 4$ has no antipodal distance-regular covers.*

The alternating forms graphs $Alt(4, q)$ and $Alt(5, q)$ are strongly regular. However, no covers of their complements exist except for the alternating form graph $Alt(4, 2)$ which has complement isomorphic to the quotient halved 8-cube, and was treated.

C. Hermitean forms graphs

Recall that the Hermitian forms graphs $Her(n, q)$ $(n, q > 1)$ have vertices the $n \times n$ Hermitian matrices over $\mathbb{F}_q$ (where $q = p^2$, p a prime power) with two matrices joined by an edge if and only if their difference has rank 1.

Proposition 5.2.18. *(see Propositions 10.1 & 10.2 [12] & sec. 11.3H [17])*

1. *The only antipodal distance-regular covers of even diameter of the Hermitean forms graphs with diameter at least 3 are the unique 2- and 4-covers of diameter 6 in the case where $d = 3$ and $q = 4$, and they are distance-transitive.*

2. *The only antipodal distance-regular cover of odd diameter of the Hermitean forms graph with diameter at least* 2 *is the* 5*-cube in the case where* $d = 2$ *and* $q = 4$*, and it is distance-transitive.*

The Hermitean forms graphs $Her(2, q)$ are strongly regular. However, no covers exist for any of their complements.

5.2.7 E_7 graphs

Let G be the collinearity graph of the points in a building of type E_7 defined over $\mathbb{F}_q$, where the points are those objects whose residue is of type E_6. Then G is classical distance-transitive with intersection array

$$\{q(q^8+q^4+1)\tfrac{q^9-1}{q-1}, q^9(q^4+1)\tfrac{q^5-1}{q-1}, q^{17}; 1, (q^4+1)\tfrac{q^5-1}{q-1}, (q^8+q^4+1)\tfrac{q^9-1}{q-1}\}$$

and parameters

$$(d, b, \alpha, \beta) = (3, q^4, [\tbinom{5}{1}]_q - 1, [\tbinom{10}{1}]_q - 1).$$

Proposition 5.2.19. *(see Proposition 12.1 [12]) The collinearity graph of the points in a finite building of type* E_7 *(either thin or thick) has no antipodal distance-regular covers.*

5.2.8 The affine E_6 graph

Let G be the collinearity graph of a finite thick building of type E_7, and ∞ a vertex of G. Then the subgraph induced on $G_3(\infty)$ is called the **affine** E_6 **graph**. The affine E_6 graph is distance-transitive with intersection array

$$\{\tfrac{(q^{12}-1)(q^9-1)}{q^4-1}, q^8(q^4+1)(q^5-1), q^{16}(q-1); 1, q^4(q^4+1), q^8\tfrac{q^{12}-1}{q^4-1}\}.$$

Furthermore, it has classical parameters

$$(d, b, \alpha, \beta) = (3, q^4, q^4-1, q^9-1).$$

Proposition 5.2.20. *(see Proposition 13.1 [12]) The affine* E_6 *graph over* $\mathbb{F}_q$ *has no antipodal distance-regular covers.*

5.3 Isolated Examples

In this section we will discuss the antipodal distance-transitive covers of all left known primitive distance-transitive graphs with diameter $d > 2$ that are not covered in the previous section. Our conclusions are based on two steps. First, we list all feasible intersection arrays of possible covers. Then we look for the corresponding antipodal covers, if any exists, using the known list of all distance-transitive graphs with small number of vertices or prove the nonexistence of such covers using distance regularity conditions.

5.3.1 Witt graphs

In this subsection, we will discuss graphs related to Witt designs. They are essentially distance-transitive with classical parameters (see sec. 11.4 [17]). So, they are covered by Van Bon and Brouwer [12].

The large Witt graph G is the graph with vertices the 759 blocks of a Steiner system $S(5,8,24)$, where two vertices are adjacent when they are disjoint. G is classical distance-transitive with intersection array $\{30,28,24;1,3,15\}$, parameters $(d,b,\alpha,\beta)=(3,-2,-4,10)$ and full automorphism group M_{24}.

Proposition 5.3.1. *(see Proposition 14.1 [12]) The large Witt graph has no antipodal distance-regular covers.*

The subgraph of the large Witt graph induced by the 506 blocks of $S(5,8,24)$ that miss a fixed symbol is itself classical distance-transitive with intersection array $\{15,14,12;1,1,9\}$, parameters $(d,b,\alpha,\beta)=(3,-2,-2,5)$ and full automorphism group M_{23}.

Proposition 5.3.2. *(see pg. 162 [12] & [51]) The subgraph of the large Witt graph G with intersection array* $\{15,14,$
$12;1,1,9\}$ *has no antipodal distance-regular covers.*

Proof. Let H be an r-antipodal cover of G of diameter D. Then by 5.1.2, either $D=6$ and in this case $i(H)=\{15,14,12,\frac{r-1}{r}(9),1,1;1,1,\frac{1}{r}(9),12,14,15\}$ or $D=7$ and $i(H)=\{15,14,12,t(r-1),9,1,1;1,1,9,t,12,14,15\}$.

Case(1) $D=6$. By the remarks of 5.1.2, $r|\ 9$ and $r\leq\frac{9}{max(1,9-12)}$. Thus we have two possibilities of r, namely 3 and 9.

For $r=3$, $i(H)=\{15,14,12,6,1,1;1,1,3,12,14,15\}$ and for $r=9$, $i(H)=\{15,14,12,8,1,1$ $;1,1,1,12,14,15\}$. However, Ivanov & Shpectorov [51] showed that neither 3-covers nor 9-covers exist .

Case(2) $D=7$. Since $c_d=9>min(b_{d-1},a_d)$, no antipodal distance-regular covers of odd diameter exist (see the remarks of 5.1.2). ■

The subgraph of the large Witt graph induced by the 330 blocks of $S(5,8,24)$ that miss two fixed symbol is itself distance-transitive with intersection array $\{7,6,4,4;1,1,1,6\}$.

Proposition 5.3.3. *(see pg. 163 [12], [18] & [29]) The only antipodal distance-regular cover of the subgraph of the large Witt graph G with intersection array* $\{7,6,4,4;1,1,1,6\}$ *is the Faradjev-Ivanov-Ivanov* 3*-cover with diameter* 8*, and it is distance-transitive. Moreover, such a cover is uniquely determined by its parameters.*

Proof. Let H be an r-antipodal cover of G of diameter D. Then by 5.1.2, either $D=8$ and in this case $i(H)=\{7,6,4,4,\frac{r-1}{r}(6),1,1,1;1,1,1,\frac{1}{r}(6),4,4,6,7\}$ or $D=9$ and

$i(H) = \{7, 6, 4, 4, t(r-1), 6, 1, 1, 1; 1, 1, 1, 6, t, 4, 4, 6, 7\}$.

Case(1) $D = 8$. By the remarks of 5.1.2, $r|(6)$ and $r \leq \frac{6}{max(1,6-4)}$. Thus we have two possibilities of r, namely 2 and 3.

For $r = 2$, $i(H) = \{7, 6, 4, 4, 3, 1, 1, 1; 1, 1, 1, 3, 4, 4, 6, 7\}$. However, Brouwer [18] showed that no such 2-covers exist.

For $r = 3$, $i(H) = \{7, 6, 4, 4, 4, 1, 1, 1; 1, 1, 1, 2, 4, 4, 6, 7\}$. Faradjev, Ivanov, and Ivanov [29] constructed such 3-cover, and Brouwer [18] showed that this graph is uniquely determined by its parameters.

Case(2) $D = 9$. Since $c_d = 6 > min(b_{d-1}, a_d)$, no antipodal distance-regular covers of odd diameter exist (see the remarks of 5.1.2). ■

5.3.2 Golay graphs

In this subsection, we will discuss graphs related to Golay codes. They are essentially distance-transitive with classical parameters (see sec. 11.3 [17]). So, they are covered by Van Bon and Brouwer [12].

Let S be a set and n a natural number. A **code** C of length n over the alphabet S is a subset of S^n. The code is called **binary(ternary)** when $S = \mathbb{F}_2$ (resp, $S = \mathbb{F}_3$). The elements of C are called **words**.

The **Hamming distance** $d_H(u, v)$ between two words u and v is the number of positions in which the entries in u and v differ, $d_H(u, v) = |\{i : u_i \neq v_i\}|$. The **weight** $wt(u)$ of a word u is its number of nonzero coordinates.

Let G be a graph with vertex set V and diameter d. A code in G is a nonempty subset C of V. The **minimum distance** $\delta(C)$ of C is defined as

$$\delta(C) = \begin{cases} 2d + 1, & |C| = 1; \\ min\{d_H(u, v) | u, v \in C, u \neq v\}, & |C| > 1. \end{cases}$$

The number

$$t(C) := max\{d_H(v, C) | v \in V\}$$

is called the **covering radius** of C. $\delta(C)$ and $t(C)$ are related by the inequality $\delta(C) \leq 2t(C) + 1$. The code C is **perfect** if $\delta(C) = 2t(C) + 1$.

If C is a code of length n, then a **truncation** of C is a code of length $n - 1$ obtained by deleting a fixed coordinate position; a shortening of C is a code of word length $n - 1$ obtained by deleting a fixed position and only retaining the code words that were 0 at that position. Conversely, the **extended code** is the code of length $n + 1$ obtained by adding an extra coordinate so as to make the weight even.

The **Golay codes** are the unique codes C with minimum distance δ in $H(n, q)$ with $(n, q, \delta, |C|) = (23, 2, 7, 2^{12})$, $(23, 2, 8, 2^{12})$, $(11, 3, 5, 3^6)$, and $(12, 3, 6, 3^6)$. These are called the **binary Golay code**, the **extended binary Golay code**, the **ternary Golay code** and the **extended ternary Golay code**.

A. The coset graph of the extended ternary Golay code

Let C be the extended ternary Golay code. The **coset graph of C**, denoted $G(C)$, has vertex set the cosets of C, where two vertices are adjacent when they contain words that are at Hamming distance one. Then $G(C)$ is classical distance-transitive with intersection array $\{24, 22, 20; 1, 2, 12\}$, parameters $(d, b, \alpha, \beta) = (3, -2, -3, 8)$ and full automorphism group $Aut(\Gamma(C)) = 3^6.2.M_{12}$.

Proposition 5.3.4. *(see pg. 163 [12]) The coset graph $G(C)$ of the extended ternary Golay code C has no antipodal distance-regular covers.*

Proof. Let H be an r-antipodal cover of the coset graph $G(C)$ of diameter D. Then by 5.1.2, either $D = 6$ and in this case $i(H) = \{24, 22, 20, \frac{r-1}{r}(12), 2, 1; 1, 2, \frac{1}{r}(12), 20, 22, 24\}$ or $D = 7$ and $i(H) = \{24, 22, 20, t(r-1), 12, 2, 1; 1, 2, 12, t, 20, 22, 24\}$.

Case(1) $D = 6$. By the remarks of 5.1.2, $r|12$ and $r \leq \frac{12}{max(2,12-20)}$. Thus we have three possibilities of r, namely 2, 3, and 6.

For $r = 2$, $i(H) = \{24, 22, 20, 6, 2, 1; 1, 2, 6, 20, 22, 24\}$ and so, $|V(H)| = 1 + 24 + 264 + 880 + 264 + 24 + 1 = 1458$. Now using the list of all feasible intersection arrays of distance-regular graphs having diameter 6 and with at most 4096 vertices given in [17], we conclude that no such cover exists.

For $r = 3$, $i(H) = \{24, 22, 20, 8, 2, 1; 1, 2, 4, 20, 22, 24\}$ and so, $|V(H)| = 1 + 24 + 264 + 1320 + 528 + 48 + 2 = 2187$. Now using the list of all feasible intersection arrays of distance-regular graphs having diameter 6 and with at most 4096 vertices given in [17], we conclude that no such cover exists.

For $r = 6$, $i(H) = \{24, 22, 20, 10, 2, 1; 1, 2, 2, 20, 22, 24\}$ and so, $|V(H)| = 1 + 24 + 264 + 2640 + 1320 + 120 + 5 = 2374$. Now using the list of all feasible intersection arrays of distance-regular graphs having diameter 6 and with at most 4096 vertices given in [17], we conclude that no such cover exists.

Case(2) $D = 7$. This graph is a near hexagon, so by corollary 5.1.7 there are no covers of odd diameter.

B. The coset graph of the doubly truncated binary code

If C is a **doubly truncated binary Golay code** ($q = 2, n = 21, |C| = 2^{12}$), then $G(C)$ is distance-transitive graph on 512 vertices with intersection array $\{21, 20, 16; 1, 2, 12\}$. Furthermore, $G(C)$ has classical parameters $(d, b, \alpha, \beta) = (3, -2, -3, 7)$ and hence the same parameters as the Hermitean forms graph with $n = 3$ and $q = 4$. (In fact, these graphs are isomorphic.) Thus $G(C)$ has the coset graph of the code C ($n = 21, |C| = 2^{11}$) obtained by taking all words in the binary Golay code that start with 00 or 11 and deleting these two coordinate positions, as a unique 2-cover. Moreover, $G(C)$ has the coset graph of the code C ($n = 21, |C| = 2^{10}$) obtained by taking all words in the extended binary Golay code that start with 000 or 111 and deleting these three coordinate positions, as a unique 4-cover. No other antipodal covers exist. See Proposition 5.2.18.1.

5.3.3 Affine sporadic graphs

In this subsection, we will discuss in detail the r-antipodal distance-transitive covers of the affine sporadic graphs given in Table 3.1.

A. Extended ternary Golay graph

The extended ternary Golay graph G with 3^6 vertices and intersection array $i(G) = \{24, 22, 20; 1, 2, 12\}$ has no antipodal distance-regular covers. (See sec. 5.3.2.)

B. Truncated binary Golay graph

Proposition 5.3.5. *The only bipartite distance-transitive doubles of the truncated binary Golay graph G with 2^{10} vertices and intersection array $i(G) = \{22, 21, 20; 1, 2, 6\}$ are its double $\widetilde{G}$ as a 2-cover of diameter 6 and the coset graph of the shortened binary Golay code as a 2-cover of diameter 7. The coset graph of the shortened binary Golay code is uniquely determined by its parameter as a distance-regular graph. The double coset graph of the truncated Golay code is uniquely determined by its parameters as a distance-transitive graph.*

Proof. Let H be an r-antipodal cover of G of diameter D. Then by 5.1.2, either $D = 6$ and in this case $i(H) = \{22, 21, 20, \frac{r-1}{r}(6), 2, 1; 1, 2, \frac{1}{r}(6), 20, 21, 22\}$ or $D = 7$ and $i(H) = \{22, 21, 20, t(r-1), 6, 2, 1; 1, 2, 6, t, 20, 21, 22\}$.

Case(1) $D = 6$. By the remarks of 5.1.2, $r|6$ and $r \leq \frac{6}{max(2,6-20)}$. Thus we have two possibilities of r, namely 2 and 3.

For $r = 2$, $i(H) = \{22, 21, 20, 3, 2, 1; 1, 2, 3, 20, 21, 22\}$ and so, $|V(H)| = 1 + 22 + 231 + 1540 + 231 + 22 + 1 = 2048$. This is the coset graph of the shortened binary Golay code. It is distance-transitive and uniquely determined by its parameters. (see 11.3(H) [17]).

For $r = 3$, $i(H) = \{22, 21, 20, 4, 2, 1; 1, 2, 2, 20, 21, 22\}$ and so, $|V(H)| = 1 + 22 + 231 + 2310 + 462 + 44 + 2 = 3072$. Now using the list of all feasible intersection arrays of distance-regular graphs having diameter 6 and with at most 4096 vertices given in [17], we conclude that no such cover exists.

Case(2) $D = 7$. By the remarks of 5.1.2, $t(r-1) \leq min(20, 16)$ and $6 \leq t$. Thus we have the following possibilities of t and r.

t	r	t	r
6	2	10	2
6	3	11	2
7	2	12	2
7	3	13	2
8	2	14	2
8	3	15	2
9	2	16	2

For $r = 2$ and $t \in \{6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$, we have $i(H) = \{22, 21, 20, t, 6, 2, 1; 1, 2, 6, t, 20, 21, 22\}$ with $|V(H)| = 1 + 22 + 231 + 770 + 770 + 231 + 22 + 1 = 2048$. Using the list of all feasible intersection arrays given in (ch14 [17]), we conclude that no such graphs exist.

For $r = 3$ and $t \in \{6, 7, 8\}$, we have, $i(H) = \{22, 21, 20, 2t, 6, 2, 1; 1, 2, 6, t, 20, 21, 22\}$ with $|V(H)| = 1 + 22 + 231 + 770 + 1540 + 462 + 44 + 2 = 3072$. Again using the same list of all feasible intersection array, we conclude that no such graphs exist.

For $r = 2$ and $t = 16$, we have $i(H) = \{22, 21, 20, 16, 6, 2, 1; 1, 2, 6, 16, 20, 21, 22\}$ with $|V(H)| = 1 + 22 + 231 + 770 + 770 + 231 + 22 + 1 = 2048$. Now using the same list of all feasible intersection arrays of distance-regular graphs with diameter 7 and at most 4096 vertices, such a graph exists. It is the double coset graph $\widetilde{G}$ of the truncated Golay code G. It is distance-transitive with automorphism group $(2^{10}.M_{22}.2) \times 2$ (see 11.3F [17]). Moreover, it is uniquely determined by its intersection array as a distance-transitive graph (see the proof of Proposition 6.3.4 below). ■

C. Distance two graph of truncated binary Golay graph

Proposition 5.3.6. *The distance two graph of truncated Golay graph G with 2^{10} vertices and intersection array $i(G) = \{231, 160, 6; 1, 48, 210\}$ has no antipodal distance-regular covers.*

Proof. Let H be an r-antipodal cover of G of diameter D. Then by 5.1.2, either $D = 6$ and in this case $i(H) = \{231, 160, 6, \frac{r-1}{r}(210), 48, 1; 1, 48, \frac{1}{r}(210), 6, 160, 231\}$ or $D = 7$ and $i(H) = \{231, 160, 6, t(r-1), 210, 48, 1; 1, 48, 210, t, 6, 160, 231\}$.

Case(1) $D = 6$. By the remarks of 5.1.2, $r|210$ and $r \leq \frac{210}{max(48, 210-6)}$. Thus no such a cover exists.

Case(2) $D = 7$. Since $c_d = 210 > min(6, 21)$ (see remarks of 5.1.2), no feasible array exists. Hence no such cover exists. ■

D. Perfect Golay graph

Proposition 5.3.7. *The only antipodal distance-transitive cover of the perfect Golay graph G with 2^{11} vertices and intersection array $i(G) = \{23, 22, 21; 1, 2, 3\}$ is its double $\widetilde{G}$ as a 2-cover. Moreover, the double graph $\widetilde{G}$ is uniquely determined by its parameters.*

Proof. Let H be an r-antipodal cover of G of diameter D. Then by 5.1.2, either $D = 6$ and in this case $i(H) = \{23, 22, 21, \frac{r-1}{r}(3), 2, 1; 1, 2, \frac{1}{r}(3), 21, 22, 23\}$ or $D = 7$ and $i(H) = \{23, 22, 21, t(r-1), 3, 2, 1; 1, 2, 3, t, 21, 22, 23\}$.

Case(1) $D = 6$. By the remarks of 5.1.2, $r|3$ and $r \le \frac{3}{max(2,3-21)}$. Thus $r = 1$. Hence no covers exist.

Case(2) $D = 7$. By the remarks of 5.1.2, $t(r-1) \le min(21, 23-3) = 20$ and $3 \le t$. Thus we have the following possibilities of t and r.

t	r	t	r	t	r	t	r	t	r	t	r
3	2	9	2	15	2	3	3	9	3	3	5
4	2	10	2	16	2	4	3	10	3	4	5
5	2	11	2	17	2	5	3	3	4	5	5
6	2	12	2	18	2	6	3	4	4	3	6
7	2	13	2	19	2	7	3	5	4	4	6
8	2	14	2	20	2	8	3	6	4	3	7

For $r = 2$ and $t \in \{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19\}$, we have $i(H) = \{23, 22, 21, t, 3, 2, 1; 1, 2, 3, 3, t, 21, 22, 23\}$ with $|V(H)| = 1 + 23 + 253 + 1771 + 1771 + 253 + 23 + 1 = 4096$. Using the list of all feasible intersection arrays given in (ch14 [17]), we conclude that no such graphs exist.

For $r = 2$ and $t = 20$, we have $i(H) = \{23, 22, 21, 20, 3, 2, 1; 1, 2, 3, 20, 21, 22, 23\}$ with $|V(H)| = 1 + 23 + 253 + 1771 + 1771 + 253 + 23 + 1 = 4096$. Now using the same list of all feasible intersection arrays of distance-regular graphs with diameter 7 and at most 4096 vertices, such a graph exists. It is the double coset graph $\widetilde{G}$ of the binary Golay code G. It is distance-transitive with automorphism group $2^{11}.M_{23}.2$. In addition, $\widetilde{G}$ is uniquely determined by its parameters (see 11.E [17]).

For $r = t = 3$, we have $i(H) = \{23, 22, 21, 6, 3, 2, 1; 1, 2, 3, 3, 21, 22, 23\}$ which is ruled out by the following divisibility condition

if $a_1 = 0$ and $c_2 \ge 2$, then $c_{i+1} > c_i$ for each $i \ge 1$ (see Theorem 2.2.5).

Similarly, for $r = 4, 5, 6, 7$ and $t = 3$, we have $c_3 = c_4 = 3$, hence they are ruled out by the above condition.

For $r = 3$ and $t = 4$, we have $i(H) = \{23, 22, 21, 8, 3, 2, 1; 1, 2, 3, 4, 21, 22, 23\}$ with $|V(H)| = 1 + 23 + 253 + 1771 + 3542 + 506 + 46 + 2 = 6144$. The eigenvalues of H are

$$\{23, 14.02, 7, 5.5507, -1, -1.9318, -9, -9.6386\}.$$

The corresponding eigenvector of the second largest eigenvalue $\lambda_2 = 14.02$ is

$$[1, 0.60956, 0.34299, 0.17093, -8.546710^{-2}, -0.17150, -0.30477, -0.50001]^t$$

and the multiplicity (see theorem 5.1.9 above) $m(\lambda_2) =$

$$\frac{6144}{1*1^2+23*(0.60956)^2+253*(0.34299)^2+1771*(0.17093)^2+3542*(-8.546710^{-2})^2+506*(-0.17150)^2+46*(-0.30477)^2+2*(-0.50001)^2}$$

$= 44.984$, which is not an integer. Thus there is no graph with the given array.

In a similar way, all the remain cases, i.e., $r = 3$ and $t \in \{5, 6, 7, 8, 9, 10\}$, $r = 4$ and $t \in \{4, 5, 6\}$, $r = 5$ and $t \in \{3, 4, 5\}$ and $r = 6$ and $t = 4$ are ruled out by the calculations of the multiplicities of their second largest eigenvalues. Hence the only antipodal distance-transitive cover of the perfect Golay graph G is its double $\widetilde{G}$. ■

E. Distance two graph of perfect Golay graph

Proposition 5.3.8. *The distance two graph of perfect Golay graph G with 2^{11} vertices and intersection array $i(G) = \{253, 210, 3; 1, 30, 231\}$ has no antipodal distance-regular covers.*

Proof. Let H be an r-antipodal cover of G of diameter D. Then by 5.1.2, either $D = 6$ and in this case $i(H) = \{253, 210, 3, \frac{r-1}{r}(231), 30, 1; 1, 30, \frac{1}{r}(231), 3, 210, 253\}$ or $D = 7$ and $i(H) = \{253, 210, 3, t(r-1), 231, 30, 1; 1, 30, 231, t, 3, 210, 253\}$.

Case(1) $D = 6$. By the remarks of 5.1.2, $r|231$ and $r \leq \frac{231}{max(30, 231-3)}$. Thus $r = 1$. Hence no covers exist.

Case(2) $D = 7$. Since $c_d = 231 > min(3, 22)$ (see remarks of 5.1.2), no feasible array exists. Hence no such cover exists. ■

5.3.4 Simple socle graphs of Lie type

In this subsection, we will discuss in detail the r-antipodal distance-transitive covers of the simple socle graphs of Lie type given in Table 3.2 and Table 3.3 (with the exception of the generalized $2d$-gons).

A. Coxeter graph

Proposition 5.3.9. *The Coxeter graph G with 28 vertices and intersection array $i(G) = \{3, 2, 2, 1; 1, 1, 1, 2\}$ has no antipodal distance-regular covers.*

Proof. Let H be an r-antipodal cover of G of diameter D. Then by 5.1.2, either $D = 8$ and in this case $i(H) = \{3, 2, 2, 1, \frac{r-1}{r}(2), 1, 1, 1; 1, 1, 1, \frac{1}{r}(2), 1, 2, 2, 3\}$ or $D = 9$ and $i(H) = \{3, 2, 2, 1, t(r-1), 2, 1, 1, 1; 1, 1, 1, 2, t, 1, 2, 2, 3\}$.

Case(1) $D = 8$. By the remarks of 5.1.2, $r|2$ and $r \le \frac{2}{max(1,2-1)}$. Thus 2 is the only possible value of r. Hence $i(H) = \{3, 2, 2, 1, 1, 1, 1, 1; 1, 1, 1, 1, 1, 2, 2, 3\}$, and so $|V(H)| = 1+3+6+12+12+12+6+3+1 = 55$. Now using the list of all feasible intersection arrays of distance-regular graphs having diameter 8 and with at most 4096 vertices given in [17], we conclude that no such cover exists.

Case(2) $D = 9$. Since $c_d = 2 > min(1, 1)$ (see remarks of 5.1.2), no feasible array exists. Hence no such cover exists. ■

B. Sylvester graph

Proposition 5.3.10. *The Sylvester graph G with 36 vertices and intersection array $i(G) = \{5, 4, 2; 1, 1, 4\}$ has no antipodal distance-regular covers.*

Proof. Let H be an r-antipodal cover of G of diameter D. Then by 5.1.2, either $D = 6$ and in this case $i(H) = \{5, 4, 2, \frac{r-1}{r}(4), 1, 1; 1, 1, \frac{1}{r}(4), 2, 4, 5\}$ or $D = 7$ and $i(H) = \{5, 4, 2, t(r-1), 4, 1, 1; 1, 1, 4, t, 2, 4, 5\}$.

Case(1) $D = 6$. By the remarks of 5.1.2, $r|4$ and $r \le \frac{4}{max(1,4-2)}$. Thus 2 is the only possible value of r. Hence $i(H) = \{5, 4, 2, 2, 1, 1; 1, 1, 2, 2, 4, 5\}$, and so $|V(H)| = 1+5+20+20+20+5+1 = 72$. Now using the list of all feasible intersection arrays of distance-regular graphs having diameter 6 and with at most 4096 vertices given in [17], we conclude that no such cover exists.

Case(2) $D = 7$. Since $c_d = 4 > min(2, 1)$ (see remarks of 5.1.2), no feasible array exists. Hence no such cover exists. ■

C. Doro graph

Proposition 5.3.11. *The Doro graph G with 68 vertices and intersection array $i(G) = \{12, 10, 3; 1, 3, 8\}$ has no antipodal distance-regular covers.*

Proof. Let H be an r-antipodal cover of G of diameter D. Then by 5.1.2, either $D = 6$ and in this case $i(H) = \{12, 10, 3, \frac{r-1}{r}(8), 3, 1; 1, 3, \frac{1}{r}(8), 3, 10, 12\}$ or $D = 7$ and $i(H) = \{12, 10, 3, t(r-1), 8, 3, 1; 1, 3, 8, t, 3, 10, 12\}$.

Case(1) $D = 6$. By the remarks of 5.1.2, $r|8$ and $r \le \frac{8}{max(3,8-3)}$. Thus $r = 1$. Hence no cover exists.

Case(2) $D = 7$. Since $c_d = 8 > min(3, 12-8)$ (see remarks of 5.1.2), no feasible array exists. Hence no such cover exists. ■

D. Biggs-Smith graph

Proposition 5.3.12. *The Biggs-Smith graph G with 102 vertices and intersection array $i(G) = \{3,2,2,2,1,1,1;1,1,1,1,1,1,3\}$ has no antipodal distance-regular covers.*

Proof. Let H be an r-antipodal cover of G of diameter D. Then by 5.1.2, either $D = 14$ and $i(H) = \{3,2,2,2,1,1,1,\frac{r-1}{r}(3),1,1,1,1,1,1;1,1,1,1,1,1,\frac{1}{r}(3),1,1,1,2,2,2,3\}$ or $D = 15$ and $i(H) = \{3,2,2,2,1,1,1,t(r-1),3,1,1,1,1,1,1;1,1,1,1,1,1,3,t,1,1,1,2,2,2,3\}$.

Case(1) $D = 14$. By the remarks of 5.1.2, $r|3$ and $r \leq \frac{3}{max(1,3-1)}$. Thus $r = 1$. Hence no cover exists.

Case(2) $D = 15$. Since $c_d = 3 > min(1, 3-3)$ (see remarks of 5.1.2), no feasible array exists. Hence no such cover exists. ■

E. Perkel graph

Proposition 5.3.13. *The Perkel graph G with 57 vertices and intersection array $i(G) = \{6,5,2;1,1,3\}$ has no antipodal distance-regular covers.*

Proof. Let H be an r-antipodal cover of G of diameter D. Then by 5.1.2, either $D = 6$ and in this case $i(H) = \{6,5,2,\frac{r-1}{r}(3),1,1;1,1,\frac{1}{r}(3),2,5,6\}$ or $D = 7$ and $i(H) = \{6,5,2,t(r-1),3,1,1;1,1,3,t,2,5,6\}$.

Case(1) $D = 6$. By the remarks of 5.1.2, $r|3$ and $r \leq \frac{3}{max(1,3-2)}$. Thus $r = 3$. Hence $i(H) = \{6,5,2,2,1,1;1,1,1,2,5,6\}$ and $|V(H)| = 1+6+30+60+60+12+2 = 171$. Now using the list of all feasible intersection arrays of distance-regular graphs having diameter 6 and with at most 4096 vertices given in [17], we conclude that no such cover exists.

Case(2) $D = 7$. Since $c_d = 3 > min(2, 6-3)$ (see remarks of 5.1.2), no feasible array exists. Hence no such cover exists. ■

F. Locally Petersen graph

Proposition 5.3.14. *The locally Petersen graph G with 65 vertices and intersection array $i(G) = \{10,6,4;1,2,5\}$ has no antipodal distance-regular covers.*

Proof. Let H be an r-antipodal cover of G of diameter D. Then by 5.1.2, either $D = 6$ and in this case $i(H) = \{10,6,4,\frac{r-1}{r}(5),2,1;1,2,\frac{1}{r}(5),4,6,10\}$ or $D = 7$ and $i(H) = \{10,6,4,t(r-1),5,2,1;1,2,5,t,4,6,10\}$.

Case(1) $D = 6$. By the remarks of 5.1.2, $r|5$ and $r \leq \frac{5}{max(2,5-4)}$. Thus $r = 1$. Hence no cover exists.

Case(2) $D = 7$. Since $c_d = 5 > min(4, 10-5)$ (see remarks of 5.1.2), no feasible array exists. Hence no such cover exists. ■

G. The distance three graph $(Her(3,4))_3$

Proposition 5.3.15. *The distance three graph $G = (Her(3,4))_3$ of the Hermitian graph $Her(3,4)$ with 280 vertices and intersection array $i(G) = \{9,8,6,3;1,1,3,8\}$ has no antipodal distance-regular covers.*

Proof. Let H be an r-antipodal cover of G of diameter D. Then by 5.1.2, either $D = 8$ and in this case $i(H) = \{9,8,6,3,\frac{r-1}{r}(8),3,1,1;1,1,3,\frac{1}{r}(8),3,6,8,9\}$ or $D = 9$ and $i(H) = \{9,8,6,3,t(r-1),8,3,1,1;1,1,3,8,t,3,6,8,9\}$.

Case(1) $D = 8$. By the remarks of 5.1.2, $r|8$ and $r \leq \frac{8}{max(3,8-3)}$. Thus $r = 1$. Hence no cover exists.

Case(2) $D = 9$. Since $c_d = 8 > min(3, 9-8)$ (see remarks of 5.1.2), no feasible array exists. Hence no such cover exists. ■

H. The Johnson graph $J(8,3)$

The Johnson graph $J(8,3)$ has no antipodal distance-regular covers. (See sec. 5.2.1.)

I. Unitary nonisotropics graph on 208 points

Proposition 5.3.16. *The unitary nonisotropics graph G with 208 vertices and intersection array $i(G) = \{12,10,5;1,1,8\}$ has no antipodal distance-regular covers.*

Proof. Let H be an r-antipodal cover of G of diameter D. Then by 5.1.2, either $D = 6$ and in this case $i(H) = \{12,10,5,\frac{r-1}{r}(8),1,1;1,1,\frac{1}{r}(8),5,10,12\}$ or $D = 7$ and $i(H) = \{12,10,5,t(r-1),8,1,1;1,1,8,t,5,10,12\}$.

Case(1) $D = 6$. By the remarks of 5.1.2, $r|8$ and $r \leq \frac{8}{max(1,8-5)}$. Thus 2 is the only possible value of r. Hence $i(H) = \{12,10,5,4,1,1;1,1,4,3,5,10,12\}$, and so $|V(H)| = 1+12+120+150+120+12+1 = 416$. Now using the list of all feasible intersection arrays of distance-regular graphs having diameter 6 and with at most 4096 vertices given in [17], we conclude that no such cover exists.

Case(2) $D = 7$. Since $c_d = 8 > min(5,4)$ (see remarks of 5.1.2), no feasible array exists. Hence no such cover exists. ■

J. Line graph of the Hoffman-Singleton graph

Proposition 5.3.17. *The line graph of the Hoffman-Singleton graph G with 175 vertices and intersection array $i(G) = \{12,6,5;1,1,4\}$ has no antipodal distance-regular covers.*

Proof. Let H be an r-antipodal cover of G of diameter D. Then by 5.1.2, either $D = 6$ and in this case $i(H) = \{12, 6, 5, \frac{r-1}{r}(4), 1, 1; 1, 1, \frac{1}{r}(4), 5, 6, 12\}$ or $D = 7$ and $i(H) = \{12, 6, 5, t(r-1), 4, 1, 1; 1, 1, 4, t, 5, 6, 12\}$.

Case(1) $D = 6$. By the remarks of 5.1.2, $r|4$ and $r \leq \frac{4}{max(1,4-5)}$. Thus we have two possibilities of r, namely 2 and 4.

For $r = 2$, $i(H) = \{12, 6, 5, 2, 1, 1; 1, 1, 2, 5, 6, 12\}$ with $|V(H)| = 1+12+72+180+72+12+1 = 350$. Now using the list of all feasible intersection arrays of distance-regular graphs having diameter 6 and with at most 4096 vertices given in [17], we conclude that no such cover exists.

For $r = 4$, $i(H) = \{12, 6, 5, 3, 1, 1; 1, 1, 1, 5, 6, 12\}$ with $|V(H)| = 1 + 12 + 72 + 360 + 216 + 36 + 3 = 700$. Now using the list of all feasible intersection arrays of distance-regular graphs having diameter 6 and with at most 4096 vertices given in [17], we conclude that no such cover exists.

Case(2) $D = 7$. By the remarks of 5.1.2, $t(r-1) \leq min(5, 8)$ and $4 \leq t$. Thus we have the following possibilities of t and r.

t	r
4	2
5	2

For $r = 2$ and $t \in \{4, 5\}$, we have $i(H) = \{12, 6, 5, t, 4, 1, 1; 1, 1, 4, t, 5, 6, 12\}$ with $|V(H)| = 1 + 12 + 72 + 90 + 90 + 72 + 12 + 1 = 350$. Using the list of all feasible intersection arrays given in (ch14 [17]), we conclude that no such graphs exist. ■

5.3.5 Sporadic simple socle graphs

In this subsection, we will discuss in detail the r-antipodal distance-transitive covers of the sporadic simple socle graphs given in Table 3.4.

A. Livingstone graph

Proposition 5.3.18. *The Livingstone graph G with 266 vertices and intersection array $\{11, 10, 6, 1; 1, 1, 5, 11\}$ has no antipodal distance-regular covers.*

Proof. Let H be an r-antipodal cover of G of diameter D. Then by 5.1.2, either $D = 8$ and in this case $i(H) = \{11, 10, 6, 1, \frac{r-1}{r}(11), 5, 1, 1; 1, 1, 5, \frac{1}{r}(11), 1, 6, 10, 11\}$ or $D = 9$ and $i(H) = \{11, 10, 6, 1, t(r-1), 11, 5, 1, 1; 1, 1, 5, 11, t, 1, 6, 10, 11\}$.

Case(1) $D = 8$. By the remarks of 5.1.2, $r|11$ and $r \leq \frac{11}{max(5,11-1)}$. Thus $r = 1$. Hence no cover exists.

Case(2) $D = 9$. Since $c_d = 11 > min(1, 11 - 11)$ (see remarks of 5.1.2), no feasible array exists. Hence no such cover exists. ■

B. Hall-Janko near octagon

Proposition 5.3.19. *The Hall-Janko near octagon G with 315 vertices and intersection array $\{10, 8, 8, 2; 1, 1, 4, 5\}$ has no antipodal distance-regular covers.*

Proof. Let H be an r-antipodal cover of G of diameter D. Then by 5.1.2, either $D = 8$ and in this case $i(H) = \{10, 8, 8, 2, \frac{r-1}{r}(5), 4, 1, 1; 1, 1, 4, \frac{1}{r}(5), 2, 8, 8, 10\}$ or $D = 9$ and $i(H) = \{10, 8, 8, 2, t(r-1), 5, 4, 1, 1; 1, 1, 4, 5, t, 2, 8, 8, 10\}$.

Case(1) $D = 8$. By the remarks of 5.1.2, $r|5$ and $r \leq \frac{5}{max(4,5-2)}$. Thus $r = 1$. Hence no cover exists.

Case(2) $D = 9$. Since $c_d = 5 > min(2, 10 - 5)$ (see remarks of 5.1.2), no feasible array exists. Hence no such cover exists. ■

C. Witt graph

The Witt graph with 759 vertices and intersection array $\{30, 28, 24; 1, 3, 15\}$ has no antipodal distance-regular covers. (See sec. 5.3.1.)

D. Truncated from Witt graph

The truncated Witt graph with 506 vertices and intersection array $\{15, 14, 12; 1, 1, 9\}$ has no antipodal distance-regular covers. (See sec. 5.3.1.)

E. Doubly truncated from Witt graph

The only antipodal distance-transitive cover of the doubly truncated Witt graph with 330 vertices and intersection array $\{7, 6, 4, 4; 1, 1, 1, 6\}$ is the Faradjev-Ivanov-Ivanov 3-cover with diameter 8. (See sec. 5.3.1.)

F. Patterson graph of Suz type

Proposition 5.3.20. *The Patterson graph G of Suz type with 22880 vertices and intersection array $\{280, 243, 144, 10; 1, 8, 90, 280\}$ has no antipodal distance-regular covers.*

Proof. Let H be an r-antipodal cover of G of diameter D. Then by 5.1.2, either $D = 8$ and in this case $i(H) = \{280, 243, 144, 10, \frac{r-1}{r}(280), 90, 8, 1; 1, 8, 90, \frac{1}{r}(280), 10, 144, 243, 280\}$ or $D = 9$ and $i(H) = \{280, 243, 144, 10, t(r-1), 280, 90, 8, 1; 1, 8, 90, 280, t, 10, 144, 243, 280\}$.

Case(1) $D = 8$. By the remarks of 5.1.2, $r|280$ and $r \leq \frac{280}{max(90,280-10)}$. Thus $r = 1$.

Hence no cover exists.

Case(2) $D = 9$. Since $c_d = 280 > min(10, 280 - 280)$ (see remarks of 5.1.2), no feasible array exists. Hence no such cover exists. ■

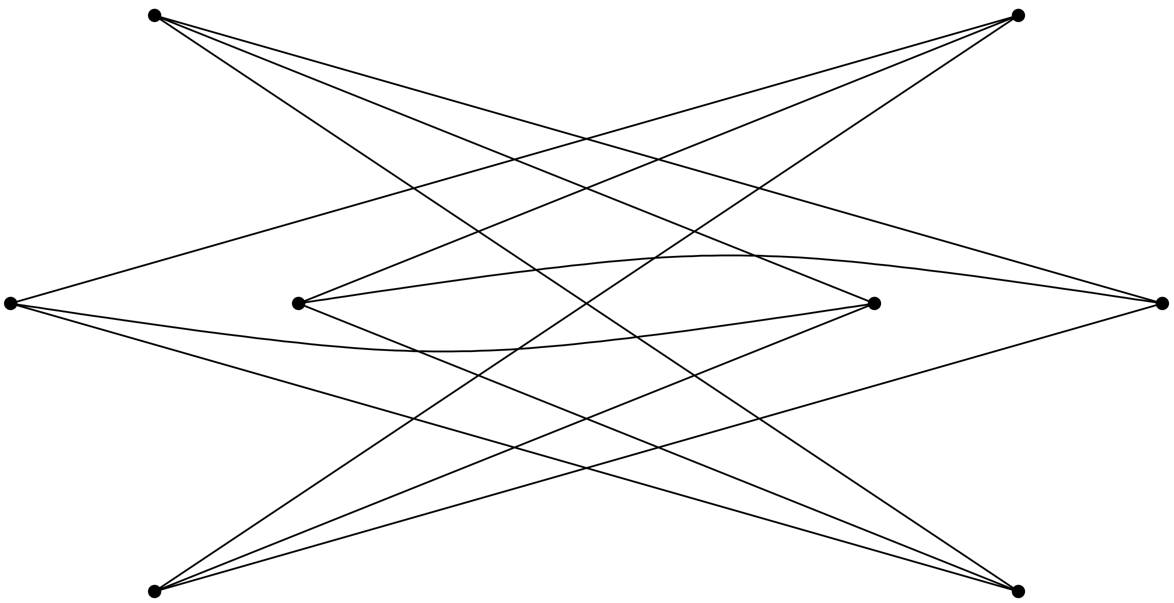
5.4 Generalized $2d$-gons

Here our results for distance-regular covers are presently restricted to antipodal covers of odd diameter, a case covered by Van Bon and Brouwer [12]. Specific cases, for instance the generalized hexagon with parameters $(2, 1)$, can be handled by methods like those of the previous section. It seems likely that for general results we will need to use the full force of the distance-transitive assumption.

Notice that, the finite distance-transitive generalized polygons were classified by Buekenhout and Van Maldeghem [19].

Proposition 5.4.1. *(see Corollary 2.3 [12]) The generalized $2d$-gons with diameter $d \geq 3$ have no antipodal distance-regular covers of odd diameter.*

Bipartite Doubles



Chapter 6

Bipartite Doubles

In this chapter we prove results that, in particular, prove Theorem 4.1 in the case of bipartite imprimitive graphs. For this chapter, the primary references are the book of Brouwer, Cohen, and Neumaier [17] and the two papers of Hemmeter [42,43].

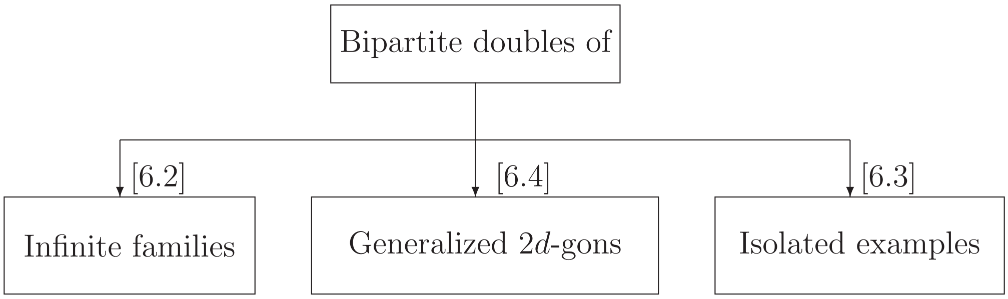


Figure 6.1: bipartite doubles main tree

6.1 Halved Graphs

For an easy reference we will start this section by restating the definition of both the bipartite double and the halved graph.

Suppose G is a bipartite distance-regular graph with diameter d. Then G_2 has two components, and the graphs induced on these components by G_2 are denoted by G^+ and G^- (or also $\frac{1}{2}G$ for an arbitrary one of these) and are known as the halved graphs of G. Although they have the same parameters they need not be isomorphic. They are isomorphic if G is a distance-transitive graph. In this case, we say that G is a bipartite distance-transitive double (or simply a bipartite double) of its halved graph $\frac{1}{2}G$.

Proposition 6.1.1. *(see Proposition 4.2.2 [17]) Let G be a distance-regular graph with intersection array $i(G) = \{b_0, b_1, ..., b_{d-1}; c_1, c_2, ..., c_d\}$ and diameter $d \in \{2m, 2m+1\}$. Then G is bipartite if and only if $a_i = 0$ for $i = 0, ..., d$. In this case, the halved graph $\frac{1}{2}G$ is* *distance-regular of diameter m with intersection array* $i(\frac{1}{2}G) = \{b'_0, b'_1, ..., b'_{d-1}; c'_1, c'_2, ..., c'_d\}$, *where*

$$b'_i = \frac{b_{2i}b_{2i+1}}{c_2} \text{ for } 0 \leq i \leq d-1 \text{ and } c'_j = \frac{c_{2j}c_{2j-1}}{c_2} \text{ for } 1 \leq j \leq d$$

Proof. We first prove that we have a bipartite DRG if and only if all a_i are 0.

Let G be a bipartite distance-regular graph with parameters a_i, b_i and c_i, with $k = b_0$ being the valency. Let $u, v \in G$ with $d(u, v) = i$.

Case(1) i is even. Then u, v and $G_i(u)$ are in the same part of the partition of $V(G)$. While $G(v)$ (the neighbors of v) are in the other part. Hence $a_i = |G_i(u) \cap G(v)| = 0$.

Case(2) i is odd. Then $u, G(v)$ are in one part and $v, G_i(u)$ are in the other part. Hence $a_i = |G_i(u) \cap G(v)| = 0$. Thus $a_i = 0$ for all $i = 0, ..., d$.

Suppose now $a_i \neq 0$ for some i. Then there are adjacent vertices $v, w \in G_i(u)$. Also, there are paths π, ρ of length i from u to v and u to w. Let x be the last vertex where π and ρ meet and let $m = d(x, v) = d(x, w)$. Then we can form a circuit of length $2m + 1$, an odd number. Hence G is not bipartite.

To show that $b'_i = \frac{b_{2i}b_{2i+1}}{c_2}$ for $0 \leq i \leq d-1$, let $u, v \in V(\frac{1}{2}G)$, $u \in G_{2i}(v)$. Count the number of pairs (w, z) with $w \in G(v) \cap G_{2i+1}(u)$ and $z \in G(v) \cap G_{2i+2}(u)$. There are b_{2i} such $w's$, each of which has b_{2i+1} $z's$. Counting $z's$ first, there are b'_i such $z's$, each of which has c_2 $w's$.

To show that $c'_j = \frac{c_{2j}c_{2j-1}}{c_2}$ for $1 \leq j \leq d$, let u, v, be as above. Count the number of pairs (w, z) with $w \in G(v) \cap G_{2j-1}(u)$ and $z \in G(v) \cap G_{2j-2}(u)$. There are c_{2j} such $w's$, each of which has $z's$, each of which has c_2 $w's$. ■

Now let H be a given DTG and suppose that we are interested in the bipartite distance-transitive doubles G whose halved graph $\frac{1}{2}G$ is H. Hemmeter[43] showed that such problem can be reduced to the study of cliques in H.

Lemma 6.1.2. *Let G be a bipartite distance-regular graph with bipartition $V(G) = X \cup Y$ and let $\frac{1}{2}G$ be the halved graph of G having X as the set of vertices. Suppose further that $\frac{1}{2}G$ is not a complete graph K_n. Then for every $y \in Y$, $G(y)$ is a maximal clique in $\frac{1}{2}G$. Moreover, if $y_1 \neq y_2$, then $G(y_1) \neq G(y_2)$.*

Proof. Let $y \in Y$. $G(y)$ is clearly a clique of $\frac{1}{2}G$. Suppose it is not maximal. Let $x \in X - G(y)$ and adjacent in $\frac{1}{2}G$ to every vertex of $G(y)$. Then $x \in G_3(y)$ and $c_3 = |G(y) \cap G_2(x)| = |G(y)| = k$. So $b_3 = k - c_3 = 0$. Hence the diameter of G is less than or equal to 3. But this means that $\frac{1}{2}G$ is a complete graph.

To show the last statement of the lemma, let $y_1 \neq y_2 \in Y$. Assume that $G(y_1) \subseteq G(y_2)$. Then $G(y_1) \subseteq G(y_1) \cap G(y_2)$, and so $k \leq c_2$. Thus $k = c_2$, and G must be complete bipartite. But this means that $\frac{1}{2}G$ is a complete graph, contradiction. Hence there exists an element y in $G(y_1)$ that is not in $G(y_2)$ and so, $G(y_1) \neq G(y_2)$. ■

So in order to reconstruct G from $\frac{1}{2}G$ one should find a certain family of cliques F. Then G has the set $V = V(\frac{1}{2}G) \cup \{f | f \in F\}$ as a vertex set. The family F must satisfy certain properties. For example, the cardinality of F must equal to the order of $\frac{1}{2}G$ and each clique in F has the same size k and each vertex of $\frac{1}{2}G$ is contained in exactly k cliques from F where k is the valency of G.

Corollary 6.1.3. *If H is a bipartite distance-regular double of G, then H has a bipartite distance-regular double only if G (and hence H) is a cycle.*

Proof. Suppose that K is the bipartite distance-regular double of H. Since H is a bipartite, its maximal cliques have size 2. Then by the lemma above, the valency of K is 2. So K is a cycle. ■

6.2 Infinite Families

In this section we discuss all bipartite distance-transitive doubles of examples belonging to infinite families (with the exception of the generalized $2d$-gons). Most of these results are due to Hemmeter [42,43].

The following tree gives an overall picture of the current section.

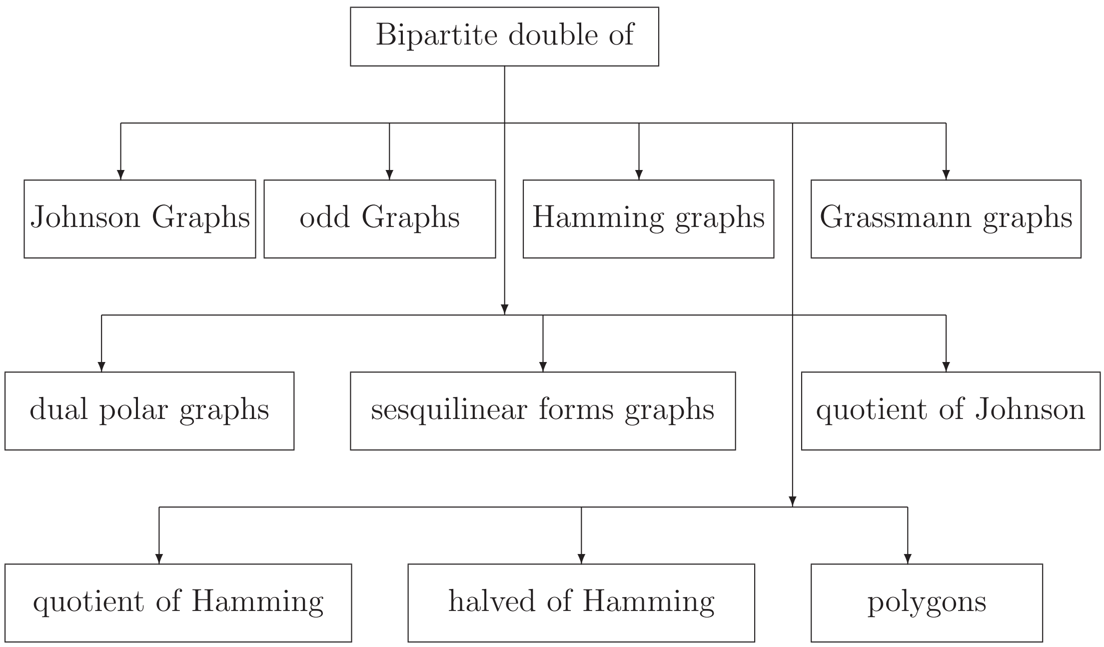


Figure 6.2: bipartite doubles of large diameter DTGs tree

6.2.1 Polygon

Lemma 6.2.1. *(see [44]) The only bipartite distance-regular double of the polygon P_n with $n \geq 6$ is P_{2n}, and it is distance-transitive.*

The ordinary polygon P_n is antipodal when n is even with intersection array $\{2, 1, ..., 1; 1, ..., 1, 2\}$. But the quotient $\overline{P}_{2n} = P_n$, which is already handled above.

6.2.2 Johnson graphs

Theorem 6.2.2. *(see Theorem 1 [43]) For $n \geq 2d$, the only bipartite distance-regular double of the Johnson graph $J(n, d)$ of diameter $d \geq 2$ is the double odd graph $2.O_d$ with $n = 2d+1$, and it is distance-transitive.*

6.2.3 Quotient Johnson graphs $\overline{J}(2k, k)$

Theorem 6.2.3. *(see Theorem 6 [44]) The Johnson quotient graph $\overline{J}(2k, k)$ of diameter $d \geq 2$ has no bipartite distance-regular doubles.*

6.2.4 Odd graphs

Theorem 6.2.4. *(see Theorem 9 [44]) The odd graph O_k of diameter $d \geq 2$ has no bipartite distance-regular doubles.*

6.2.5 Hamming graphs

Theorem 6.2.5. *(see Theorem 2 [43]) Let $n, q \geq 2$. Then the Hamming graph $H(n, q)$ has no bipartite distance-regular doubles, except for $H(2, 2)$ which has the graph H with $V(H) = \{(i, j) : i, j \in \mathbb{F}_2\} \cup \{f_i, s_j : i, j \in \mathbb{F}_2\}$ where (i, j) adjacent to f_i and s_j as the only bipartite distance-transitive double.*

The n-cube $H(n, 2)$ is bipartite, and hence we need to consider its halved graph $\frac{1}{2}H(n, 2)$.

Theorem 6.2.6. *(see Theorem 14[44]) The only bipartite distance-regular double of the halved graph $\frac{1}{2}H(n, 2)$ with $n > 4$ is the n-cube $H(n, 2)$, and it is distance-transitive.*

6.2.6 Quotient Hamming graphs $\overline{H}(n, 2)$

Theorem 6.2.7. *(see Theorems 15 [44]) The quotient Hamming graph $\overline{H}(n, 2)$ with $n \geq 4$ has no bipartite distance-regular doubles.*

Notice that the graphs $\overline{H}(2n, 2)$ is also bipartite. So, we need to consider their halved graphs.

Theorem 6.2.8. *(see Theorem 16 [44]) The only bipartite distance-regular double of the quotient halved Hamming graph $\frac{1}{2}\overline{H}(2n, 2)$ with $n \geq 4$ is $\overline{H}(2n, 2)$, and it is distance-transitive.*

6.2.7 Grassmann graphs

Theorem 6.2.9. *(see Theorem 8 [44]) The only bipartite distance-regular double of the Grassmann graph $J_q(n,d)$ with $d \geq 2$ is the doubled Grassmann graph $2J_q(2d+1,d)$, and it is distance-transitive.*

6.2.8 Dual polar graphs

Theorem 6.2.10. *(see Theorem 11 [44]) A dual polar graph of diameter $d \geq 3$ has no bipartite distance-regular doubles.*

Since $[D_d(q)]$ is bipartite, we need to consider its halved graph.

Lemma 6.2.11. *(see Lemma 12 [44]) Let M be a maximal clique of $\frac{1}{2}[D_d(q)]$. Then either M is the neighborhood in $[D_d(q)]$ of some vertex $y \in [D_d(q)] \setminus \frac{1}{2}[D_d(q)]$, or $|M| \leq 2(q^2+1)(q+1)$.*

Proposition 6.2.12. *The only bipartite distance-transitive double of the halved graph $\frac{1}{2}[D_d(q)]$ with $d = 6$ or 7 is $[D_d(q)]$.*

Proof. Let H be a bipartite distance-regular double of the halved graph $\frac{1}{2}D_d(q)$ with $d = 6$ or 7. Further, let a_i, b_i and c_i denote the parameters of H, with $k = b_0$ being the valency. The corresponding parameters of $\frac{1}{2}[D_d(q)]$ will be a'_i, b'_i, c'_i and k'. Since $c_1 = 1$ and $b_1 + c_1 = k$, we have $b_1 = k - 1$. $k' = \frac{(q^d-1)(q^d-q)}{(q-1)^2(q+1)}$ (see sec. 9.4 [17]). Lemma 6.1.1 with $i = 0$ gives $c_2 = \frac{k(k-1)}{k'}$. Since $c_2 \geq 1$, $k(k-1) \geq k'$. Using this, and assuming that $k \leq 2(q^2+1)(q+1)$, we get $\frac{(q^d-1)(q^d-q)}{(q-1)^2(q+1)} \leq (2q^3+2q^2+2q+2)(2q^3+2q^2+2q+1)$. But for $d = 6$ or $d = 7$ last inequality gives $q = 1$.

So if $d = 6$ or $d = 7$, then the size of the maximal clique which appears as $H(y)$ for some $y \in Y$ (see lemma 6.1.2) must be larger than $2(q^2+1)(q+1)$. By lemma 6.2.11, it must be $[D_d(q)]$ for some vertex $y \in [D_d(q)] \setminus \frac{1}{2}[D_d(q)]$. There are just $|V(\frac{1}{2}[D_d(q)])|$ of these available, so all must be used. Thus $H \cong [D_d(q)]$. ■

Theorem 6.2.13. *(see Theorem 13 [44]) The only bipartite distance-transitive double of the halved graph $\frac{1}{2}D_d(q)$ with $d > 7$ is $D_d(q)$.*

6.2.9 Bilinear forms graphs

Theorem 6.2.14. *(see Theorem 18 [44]) A bilinear forms graph of diameter $d \geq 2$ has no bipartite distance-regular doubles.*

6.2.10 Alternating forms graphs

Theorem 6.2.15. *(see Theorem 20 [44]) The alternating forms graph $Alt(n,q)$ on $\mathbb{F}_q$ with $n > 3$ has no bipartite distance-regular doubles.*

Notice that for $n \leq 3$, the alternating forms graphs $Alt(n,q)$ over $\mathbb{F}_q$ is just the complete graphs.

6.2.11 Hermitian forms graphs

Theorem 6.2.16. *(see Theorem 21 [44]) The Hermitian forms graph of diameter $d \geq 2$ has no bipartite distance-regular doubles.*

6.3 Isolated Examples

In this section we will discuss the bipartite distance-transitive doubles of all known primitive distance-transitive graphs with diameter $d > 2$ that are not covered in the previous section. Our conclusions are based on two steps. First, we list all feasible intersection arrays of possible doubles. Then we look for the corresponding bipartite distance-transitive doubles if any exist, using the known list of all distance-transitive graphs with small diameter or prove the nonexistence of such doubles using the regularity conditions.

6.3.1 Affine sporadic graphs

In this subsection, we discuss in detail the bipartite distance-transitive doubles of the affine sporadic graphs given in Table 3.1.

A. Extended ternary Golay graph

Proposition 6.3.1. *The extended ternary Golay graph G with 729 vertices and intersection array $\{24, 22, 20; 1, 2, 12\}$ has no bipartite distance-regular doubles.*

Proof. Let H be a bipartite distance-regular double of G with diameter D. Then by 6.1.1, $|V(H)| = 1458$ and $D = 6$ or 7. However, none of the feasible intersection arrays of the bipartite distance-regular graphs with diameter 6 or 7 and having at most 4096 vertices (given in [17]) has 1458 vertices. Hence no such doubles exist. ■

B. Truncated binary Golay graph

Proposition 6.3.2. *The truncated Golay graph G with 1024 vertices and intersection array $\{22, 21, 20; 1, 2, 6\}$ has no bipartite distance-regular doubles.*

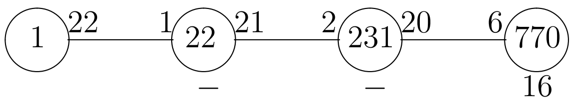
Proof. Let H be a bipartite distance-regular double of G with parameters a_i, b_i, c_i. Then (6.1.1) with $j = 2$, gives $c_4 = 2c_2/c_3$. In view of 2.2.1(3) then, $c_4 \leq 2$. Thus, the only possibilities are: $c_2 = c_3 = \omega$ for $\omega = 1, 2$ and $c_4 = 2$. By 2.2.4(2), $\omega = 1$. Now (6.1.1) with $i = 0, 1$ gives $22 = b_0 b_1$ and $21 = b_2 b_3$. But $b_0 \geq b_1 \geq b_2 \geq b_3$, a contradiction. Hence no such a double H exists. ■

C. Distance two graph of truncated binary Golay graph

Lemma 6.3.3. *If M is a maximal clique of distance two graph of truncated binary Golay graph G, then M is the neighborhood in the double coset graph H of truncated binary Golay code of some vertex $y \in V(H) \setminus V(G)$.*

Proof. Let a_i, b_i, c_i be the parameters of the double coset graph H of truncated binary Golay code with valency $k = b_0$. H has distance two graph of truncated binary Golay graph G as its halved graph. As in lemma 6.1.2, the vertex set of H has bipartition $G \cup Y$ with $|G| = |Y|$ and each $G(y)$, for $y \in Y$, a maximal clique of $G \simeq \frac{1}{2}H$. Then by lemma 6.1.1, $c_2 = \frac{k(k-1)}{231}$. Since $c_2 \geq 1$, $k(k-1) \geq 231$. Thus $k \geq 16$. Since the valency k must be equal to the size of the maximal clique (lemma 6.1.2), then $|M| \geq 16$. In the other hand, since G has smallest eigenvalue $\lambda_d = -9$, then the size of a clique M in G is bounded by $|M| \leq 1 - \frac{k}{\lambda_d} = 26.7$ (see Proposition 4.4.6 [17]). So, we have $16 \leq |M| \leq 26$. But if $k \in \{16, 17, 18, 19, 20, 21, 23, 24, 25, 26\}$, then $c_2 = \frac{k(k-1)}{231}$ is not an integer. Hence $|M| = 22$.

G is the distance two graph of the coset graph H^* of the truncated binary Golay code C. Using the following distance distribution diagram of H^*



$v = 1024$ (see 11.3F [17])

and the properties of the truncated binary Golay code C, we will give a full description of the vertices of H^* as follow. Let M be a maximal clique of G with $x := C + \underline{0} \in M$ where $\underline{0}$ is the 22-tuple with 0 in all positions. Let e_i $(1 \leq i \leq 22)$ be the 22-tuple with 1 in the i^{th} place and 0 in all other positions. We denote by x_i $(1 \leq i \leq 22)$ the 22 adjacent vertices of x where $x_i := C + e_i$. Now, let e_{ij} be the 22-tuple with 1 in the i^{th} and j^{th} positions and 0 elsewhere. We denote by $x_{i,j}(= x_{j,i})$ $(i \neq j, 1 \leq i, j \leq 22)$ the 231=$\binom{22}{2}$ vertices that are at distance two from x where $x_{i,j} := C + e_{ij}$. It is clear that, $x_{i,j}$ is only connected to x_i and x_j. Similarly, let e_{ijk} be the 22-tuple with 1 in the i^{th}, j^{th}, and k^{th} positions and 0 elsewhere. We denote by $x^{a,b,c}_{i,j,k}$ the 770=$\frac{1}{2}\binom{22}{3}$ vertices that are at distance three from x where $x^{a,b,c}_{i,j,k} := C + e_{ijk} = C + e_{abc}$. Since the words of weight 6 in the truncated binary Golay code C form the Steiner system $S(3, 6, 22)$, then each triple a, b, c lies in exactly one word of weight 6. It is clear that, $x^{a,b,c}_{i,j,k}$ is only connected to $x_{i,j}$, $x_{i,k}$, $x_{j,k}$, $x_{a,b}$, $x_{a,c}$, and $x_{b,c}$.

Case (1) Suppose that M has two vertices $x_{i,j}$ and $x_{a,b}$ with no common neighbor that at distance 1 from x. Clearly, a, b, i, j are distinct. Since $d(x_{i,j}, x_{a,b}) = 2$ in H^*, then $|H^*(x_{i,j}) \cap H^*(x_{a,b})| = c_2^* = 2$. Thus there are exactly two vertices $x^{a,b,c}_{i,j,k}$, $x^{a,b,k}_{i,j,c}$ that are adjacent to both $x_{i,j}$ and $x_{a,b}$ and at distance three from x.
Subcase (i) If all the other elements of M are adjacent to $x^{i,j,k}_{a,b,c}$ or to $x^{a,b,k}_{i,j,c}$ (or to both), then $|M| \leq 1 + 10 = 11$, contradiction.
Subcase (ii) If there is $x_{s,t} \in M$ that has no common neighbor to both $x_{a,b}$ and $x_{i,j}$. Then we have three possibilities of s, t.

1. $s, t \in \{a, b, i, j\}$
2. s or t (not both) in $\{a, b, i, j\}$
3. $s, t \notin \{a, b, i, j\}$.

(1). Without loss of generality, we can assume that $s = a$ and $t = i$. Since $d(x_{a,b}, x_{a,i}) = 2$, then $|H^*(x_{a,b}) \cap H^*(x_{a,i})| = c_2^* = 2$. We already know one common neighbor, namely x_a. Hence there is only one vertex that is adjacent to both $x_{a,b}$ and $x_{a,i}$ and at distance three from x. But a, b, i is located in only one word of weight 6, namely $abcijk$. Hence the only possibility is $x_{c,j,k}^{a,b,i}$. Since $d(x_{i,j}, x_{a,i}) = 2$, then $|H^*(x_{i,j}) \cap H^*(x_{a,i})| = c_2^* = 2$. We already know one common neighbor, namely x_i. Hence there is only one vertex that is adjacent to both $x_{i,j}$ and $x_{a,i}$ and at distance three from x. But a, i, j is located in only one word of weight 6, namely $abcijk$. Hence the only possibility is $x_{b,c,k}^{a,i,j}$. Now, let $x_{g,h}$ be any new element of M and let x_{g,h,n_2}^{i,j,n_1}, x_{g,h,n_1}^{i,j,n_2}, x_{g,h,n_4}^{a,b,n_3}, x_{g,h,n_3}^{a,b,n_4}, x_{g,h,n_6}^{a,i,n_5}, and x_{g,h,n_6}^{a,i,n_5} be its common neighbors with $x_{i,j}$, $x_{a,b}$, and $x_{a,i}$, respectively. Then we must have $n_1 = a$, $n_2 = b$, $n_3 = i$, $n_4 = j$, $n_5 = j$, and $n_6 = b$. Hence the only possibility of g, h is $g = c$ and $h = k$ (or $g = k$ and $h = c$). But then $|M| \leq 1 + 14 = 15$, contradiction.

(2). Without loss of generality, we can assume that $s = a$. Since $d(x_{i,j}, x_{a,t}) = 2$, then $|H^*(x_{i,j}) \cap H^*(x_{a,t})| = c_2^* = 2$. Let $x_{a,t,v}^{i,j,u}$, $x_{a,t,u}^{i,j,v}$ denote its common neighbors with $x_{i,j}$. But i, j, a is located in only one word of weight 6, namely $abcijk$. Hence $t \in \{b, c, k\}$.

- If $t = b$, then $(s, t) \in \{a, b, i, j\}$, contradiction.
- If $t = c$, then $(s, t) = (a, c)$. But $x_{a,c}$ has $x_{i,j,k}^{a,b,c}$ as a common neighbor to both $x_{i,j}$ and $x_{a,b}$, contradiction.
- If $t = k$, then $(s, t) = (a, k)$. But $x_{a,k}$ has $x_{i,j,c}^{a,b,k}$ as a common neighbor to both $x_{i,j}$ and $x_{a,b}$, contradiction.

(3). Since $d(x_{s,t}, x_{a,b}) = d(x_{s,t}, x_{i,j}) = 2$, then $|H^*(x_{s,t}) \cap H^*(x_{a,b})| = |H^*(x_{s,t}) \cap H^*(x_{i,j})| = c_2^* = 2$. Thus there are two common neighbors $x_{s,t,v}^{a,b,u}$ and $x_{s,t,u}^{a,b,v}$ of $x_{s,t}$ with $x_{a,b}$ and another two neighbors $x_{s,t,y}^{i,j,w}$ and $x_{s,t,w}^{i,j,y}$ with $x_{i,j}$.

1. $s, t, u, v, y, w \notin \{a, b, c, i, j, k\}$. Now, suppose that $x_{a,u} \in M$. Since $d(x_{i,j}, x_{a,u}) = 2$, then $|H^*(x_{i,j}) \cap H^*(x_{a,u})| = c_2^* = 2$. Thus there are two common neighbors x_{a,u,f_2}^{i,j,f_1} and x_{a,u,f_2}^{i,j,f_2} of $x_{i,j}$ with $x_{a,u}$. But a, i, j is located in only one word of weight 6, namely $abcijk$. Hence $u \in \{b, c, k\}$, contradiction. Hence $x_{a,u} \notin M$. Likewise, $x_{b,v}, x_{a,v}, x_{b,v}, x_{i,y}, x_{j,y}, x_{i,w}, x_{j,w} \notin M$. Now, suppose $x_{s,v} \in M$. Since $d(x_{i,j}, x_{s,v}) = 2$, then $|H^*(x_{i,j}) \cap H^*(x_{s,v})| = c_2^* = 2$. Thus there are two common neighbors x_{a,u,f_4}^{i,j,f_3} and x_{a,u,f_3}^{i,j,f_4} of $x_{i,j}$ with $x_{s,v}$. But i, j, s is located in only one word of weight 6, namely $ijwsty$. Hence $v \in \{y, w\}$. Similarly, $x_{s,u} \in M$ gives $u \in \{y, w\}$. Without loss of generality, we can assume that $v = y$ and $u = w$. Now, if $x_{g,h}$ is any other element of M, then $(g, h) = (a, b)$. Hence $|M| \leq 1 + 10 + 5 = 16$, contradiction.
2. If any of these elements is in $\{a, b, c, i, j, k\}$, then all the others are in the same set. Hence $|M| \leq 1 + 10 = 11$, contradiction.

Case (2) Every two vertices in M have common neighbor that is at distance 1 from x. Now, suppose $x_{i,j}, x_{i,b} \in M$. Let $x_{s,t}$ be any new vertex of M. Then (s,t) has the following possibilities:

1. $(s,t) = (i,t)$ (or (i,s)). In this case, all vertices are adjacent to a common neighbor that is at distance 1 from x. Using the distance diagram of H^* given above, there are 1024 cliques of this kind.

2. $(s,t) = (j,b)$. In this case, there is no other possibility to add as a new vertex. Hence $|M| = 1 + 3 = 4$, contradiction.

Hence there are only 1024 maximal cliques in G each of size 22. On the other hand, since $G \cong \frac{1}{2}H$, then for each $y \in Y$, $H(y)$ is a maximal clique in G. Thus a maximal clique M of G must be of the form $H(y)$ for some $y \in V(H) \setminus V(G)$. ■

Proposition 6.3.4. *The only bipartite distance-transitive double of the distance two graph of truncated Golay graph G with* 1024 *vertices and intersection array* $\{231, 160, 6; 1, 48, 210\}$ *is the double coset graph H of truncated binary Golay code.*

Proof. Let H be a bipartite distance-regular double of G. The maximal clique of G is the neighborhood in the double coset graph of truncated binary Golay code of some vertex y not in G (see lemma 6.3.3). There are just $|V(G)|$ of these available, so all must be used. That is, H must be isomorphic to the double coset graph of truncated binary Golay code. This graph is distance-transitive (see 11.3F [17]). Moreover, since the distance two truncated Golay graph G with 1024 vertices and intersection array $\{231, 160, 6; 1, 48, 210\}$ is uniquely determined by its parameters as a distance-transitive graph (see theorem 3.2.7), then H is uniquely determined by its parameter as a distance-transitive graph. ■

D. Perfect Golay graph

Proposition 6.3.5. *The perfect Golay graph G with* 2048 *vertices and intersection array* $\{23, 22, 21; 1, 2, 3\}$ *has no bipartite distance-regular doubles.*

Proof. Let H be a bipartite distance-regular double of G with parameters a_i, b_i, c_i. Then (6.1.1), with $j = 2$, gives $c_4 = 2c_2/c_3$. In view of 2.2.1(3) then, $c_4 \leq 2$. Thus, the only possibilities are: $c_2 = c_3 = \omega$ for $\omega = 1, 2$ and $c_4 = 2$. By 2.2.4(2), $\omega = 1$. (6.1.1) again, with $j = 3$ gives $3 = c_5c_6$. But $c_6 \geq c_5 \geq c_4 = 2$, a contradiction. Hence no such double exist. ■

E. Distance two graph of perfect Golay graph

Proposition 6.3.6. *The only bipartite distance-transitive double of distance two graph G of perfect Golay graph with* 2048 *vertices and intersection array* $\{253, 210, 3; 1, 30, 231\}$ *is the double coset graph of the binary Golay code.*

Proof. Let H be a bipartite distance-regular double of G with diameter D. Then by 6.1.1, $|V(H)| = 4096$ and $D = 6$ or 7. The only feasible intersection array of bipartite DRGs with 4096 vertices and diameter $D = 6$ or 7 is $\{23, 22, 21, 20, 3, 2, 1; 1, 2, 3, 20, 21, 22, 23\}$ (see ch.14 [17]). This is the parameters of the double coset graph H of the binary Golay code. It is distance-transitive and uniquely determined by its intersection array (see 11.3E [17]). Moreover, H has G as its halved graph. Thus H is the only bipartite distance-transitive double of G. ■

6.3.2 Simple socle graphs of Lie type

In this subsection, we discuss in detail the bipartite distance-transitive doubles of the simple socle graphs of Lie type given in Table 3.2 and Table 3.3 (with the exception of the generalized $2d$-gons).

A. Coxeter graph

Proposition 6.3.7. *The Coxeter graph G with 28 vertices and intersection array $\{3, 2, 2, 1; 1, 1, 1, 2\}$ has no bipartite distance-regular doubles.*

Proof. Let H be a bipartite distance-regular double of G with diameter D. Then by 6.1.1, $|V(H)| = 56$ and $D = 8$ or 9. However, none of the feasible intersection arrays of the bipartite distance-regular graphs with diameter 8 or 9 with at most 4096 vertices (given in [17]) has 56 vertices. Hence no such doubles exist. ■

B. Sylvester graph

Proposition 6.3.8. *The Sylvester graph G with 36 vertices and intersection array $\{5, 4, 2; 1, 1, 4\}$ has no bipartite distance-regular doubles.*

Proof. Let H be a bipartite distance-regular double of G with diameter D. Then by 6.1.1, $|V(H)| = 72$ and $D = 6$ or 7. However, none of the feasible intersection arrays of the bipartite distance-regular graphs with diameter 8 or 9 with at most 4096 vertices (given in [17]) has 72 vertices. Hence no such doubles exist. ■

C. Doro graph

Proposition 6.3.9. *The Doro graph G with 68 vertices and intersection array $\{12, 10, 3; 1, 3, 8\}$ has no bipartite distance-regular doubles.*

Proof. Let H be a bipartite distance-regular double of G with diameter D. Then by 6.1.1, $|V(H)| = 136$ and $D = 6$ or 7. However, none of the feasible intersection arrays of the bipartite distance-regular graphs with diameter 6 or 7 and at most 4096 vertices (given in [17]) has 136 vertices. Hence no such doubles exist. ■

D. Biggs-Smith graph

Proposition 6.3.10. *The Biggs-Smith graph G with 102 vertices and intersection array $\{3, 2, 2, 2, 1, 1, 1; 1, 1, 1, 1, 1, 1, 3\}$ has no bipartite distance-regular doubles.*

Proof. Let H be a bipartite distance-regular double of G with diameter D. Then by 6.1.1, $|V(H)| = 204$ and $D = 14$ or 15. However, none of the feasible intersection arrays of bipartite distance-regular graphs with diameter 14 or 15 and at most 4096 vertices (given in [17]) has 204 vertices exists. Hence no such doubles exist. ■

E. Perkel graph

Proposition 6.3.11. *The Perkel graph G with 57 vertices and intersection array $\{6, 5, 2; 1, 1, 3\}$ has no bipartite distance-regular doubles.*

Proof. Let H be a bipartite distance-regular double of G with diameter D. Then by 6.1.1, $|V(H)| = 114$ and $D = 6$ or 7. However, none of the feasible intersection arrays of the bipartite distance-regular graphs with diameter 6 or 7 and at most 4096 vertices (given in [17]) has 114 vertices. Hence no such doubles exist. ■

F. Locally Petersen graph

Proposition 6.3.12. *The Locally Petersen graph G with 65 vertices and intersection array $\{10, 6, 4; 1, 2, 5\}$ has no bipartite distance-regular doubles.*

Proof. Let H be a bipartite distance-regular double of G with diameter D. Then by 6.1.1, $|V(H)| = 130$ and $D = 6$ or 7. However, none of the feasible intersection arrays of the bipartite distance-regular graphs with diameter 6 or 7 (given in [17]) has 130 vertices. Hence no such doubles exist. ■

G. The distance three graph of the Hermitian graph $Her(3, 4)$

Proposition 6.3.13. *The distance three graph $G = (Her(3, 4))_3$ of the Hermitian graph $Her(3, 4)$ with 280 vertices and intersection array $i(G) = \{9, 8, 6, 3; 1, 1, 3, 8\}$ has no bipartite distance-regular doubles.*

Proof. Let H be a bipartite distance-regular double of G with diameter D. Then by 6.1.1, $|V(H)| = 560$ and $D = 8$ or 9. However, none of the feasible intersection arrays of the bipartite distance-regular graphs with diameter 8 or 9 and at most 4096 vertices (given in [17]) has 560 vertices. Hence no such doubles exist. ■

H. The Johnson graph $J(8, 3)$

$J(8, 3)$ has no bipartite distance-regular doubles (See sec. 6.2.2).

I. Unitary nonisotropics graph on 208 points

Proposition 6.3.14. *The unitary nonisotropics graph G with 208 vertices and intersection array $i(G) = \{12, 10, 5; 1, 1, 8\}$ has no bipartite distance-regular doubles.*

Proof. Let H be a bipartite distance-regular double of G with diameter D. Then by 6.1.1, $|V(H)| = 416$ and $D = 6$ or 7. However, none of the feasible intersection arrays of the bipartite distance-regular graphs with diameter 6 or 7 and at most 4096 vertices (given in [17]) has 416 vertices. Hence no such doubles exist. ■

J. Line graph of the Hoffman-Singleton graph

Proposition 6.3.15. *The line graph of the Hoffman-Singleton graph G with 175 vertices and intersection array $i(G) = \{12, 6, 5; 1, 1, 4\}$ has no bipartite distance-regular doubles.*

Proof. Let H be a bipartite distance-regular double of G with diameter D. Then by 6.1.1, $|V(H)| = 350$ and $D = 6$ or 7. However, none of the feasible intersection arrays of the bipartite distance-regular graphs with diameter 6 or 7 and at most 4096 vertices (given in [17]) has 350 vertices. Hence no such doubles exist. ■

K. E_7 Graphs

Proposition 6.3.16. *The collinearity graph of the points in a finite building of type E_7 defined over $\mathbb{F}_q$ with intersection array $\{q(q^8+q^4+1)\frac{q^9-1}{q-1}, q^9(q^4+1)\frac{q^5-1}{q-1}, q^{17}; 1, (q^4+1)\frac{q^5-1}{q-1}, (q^8+q^4+1)\frac{q^9-1}{q-1}\}$ has no bipartite distance-regular doubles.*

Proof. Let H be a bipartite distance-regular double of the collinearity graph G of the points in a finite building of type E_7 over $\mathbb{F}_q$. Further, let a_i, b_i and c_i denote the parameters of H, with $k = b_0$ being the valency. The corresponding parameters of G will be a'_i, b'_i, c'_i and k'. The maximal cliques (maximal singular subspaces) of G are projective spaces. Moreover, maximal singular subspaces should have rank equal 5 or 6. Hence lemma 6.1.2, gives $k = \frac{q^{n+1}-1}{q-1}$ (the size of the maximal cliques) with $n \in \{5, 6\}$. Since $c_1 = 1$ and $b_1+c_1 = k$, we have $b_1 = k(k-1)$. Since $k' = q(q^8+q^4+1)\frac{q^9-1}{q-1}$, lemma 6.1.1 with $i = 0$ gives $c_2 = \frac{k(k-1)(q-1)}{q(q^8+q^4+1)(q^9-1)}$ which is not an integer unless $q = 0$. Hence no such a double exists. ■

L. The affine E_6 Graph

Proposition 6.3.17. *The affine E_6 graph defined over $\mathbb{F}_q$ with intersection array $\{\frac{(q^{12}-1)(q^9-1)}{q^4-1}, q^8(q^4+1)(q^5-1), q^{16}(q-1); 1, q^4(q^4+1), q^8\frac{q^{12}-1}{q^4-1}\}$ has no bipartite distance-regular doubles.*

Proof. Let H be a bipartite distance-regular double of the affine E_6 graph G defined over $\mathbb{F}_q$. Further, let a_i, b_i and c_i denote the parameters of H, with $k = b_0$ being the valency. The corresponding parameters of G will be a'_i, b'_i, c'_i and k'. The maximal cliques (maximal singular subspaces) of the parapolar space $E_{6,1}(q)$ are projective spaces of rank equal to that of the maximal subdiagrams of type A_n. Thus, of rank 4 and 5. Further, the subgraph of the affine $E_6(q)$ graph induced on $(E_6(q))(x)$ (the neighbors of x) is a $(q-1)$-clique extension of the strongly regular graph of Lie type $E_{6,1}(q)$ (see remark after theorem 10.8.1 [17]). Hence the maximal cliques of the affine $E_6(q)$ graph are affine spaces of order q and rank $n = 4$ or 5. Lemma 6.1.2, gives $k = q^n$ (the size of the maximal cliques) with $n \in \{4, 5\}$. Since $c_1 = 1$

and $b_1 + c_1 = k$, we have $b_1 = q^n - 1$. Since $k' = \frac{(q^{12}-1)(q^9-1)}{q^4-1}$, lemma 6.1.1 with $i = 0$ gives $c_2 = \frac{q^n(q^n-1)(q^4-1)}{(q^{12}-1)(q^9-1)}$ which is not an integer unless $q = 0$. Hence no such a double exists. ■

6.3.3 Sporadic simple socle graphs

In this subsection we discuss in detail the bipartite distance-transitive doubles of the sporadic simple socle examples given in Table 3.4.

A. Livingstone graph

Proposition 6.3.18. *The Livingstone graph G with 266 vertices and intersection array $\{11, 10, 6, 1; 1, 1, 5, 11\}$ has no bipartite distance-regular doubles.*

Proof. Let H be a bipartite distance-regular double of G with diameter D. Then by 6.1.1, $|V(H)| = 532$ and $D = 8$ or 9. However, none of the feasible intersection arrays of the bipartite distance-regular graphs with diameter 8 or 9 and at most 4096 vertices (given in [17]) has 532 vertices. Hence no such doubles exist. ■

B. Hall-Janko near octagon

Proposition 6.3.19. *The Hall-Janko near octagon graph G with 315 vertices and intersection array $\{10, 8, 8, 2; 1, 1, 4, 5\}$ has no bipartite distance-regular doubles.*

Proof. Let H be a bipartite distance-regular double of G with diameter D. Then by 6.1.1, $|V(H)| = 630$ and $D = 8$ or 9. However, none of the feasible intersection arrays of the bipartite distance-regular graphs with diameter 8 or 9 (given in [17]) has 630 vertices. Hence no such doubles exist. ■

C. Witt graph

Proposition 6.3.20. *The large Witt graph with 759 vertices and intersection array $\{30, 28, 24; 1, 3, 15\}$ has no bipartite distance-regular doubles.*

Proof. Let H be a bipartite distance-regular double of G with diameter D. Then by 6.1.1, $|V(H)| = 1518$ and $D = 6$ or 7. However, none of the feasible intersection arrays of the bipartite distance-regular graphs with diameter 6 or 7 (given in [17]) has 1518 vertices. Hence no such doubles exist. ■

D. Truncated from Witt graph

Proposition 6.3.21. *The truncated Witt graph with 506 vertices and intersection array $\{15, 14, 12; 1, 1, 9\}$ has no bipartite distance-regular doubles.*

Proof Let H be a bipartite distance-regular double of G with diameter D. Then by 6.1.1, $|V(H)| = 1012$ and $D = 6$ or 7. However, none of the feasible intersection arrays of the bipartite distance-regular graphs with diameter 6 or 7 and at most 4096 vertices (given in [17]) has 1012 vertices. Hence no such doubles exist. ■

E. Doubly truncated from Witt graph

Proposition 6.3.22. *The doubly truncated Witt graph with* 330 *vertices and intersection array* $\{7, 6, 4, 4; 1, 1, 1, 6\}$ *has no bipartite distance-regular doubles.*

Proof. Let H be a bipartite distance-regular double of G with diameter D. Then by 6.1.1, $|V(H)| = 660$ and $D = 8$ or 9. However, none of the feasible intersection arrays of the bipartite distance-regular graphs with diameter 8 or 9 and at most 4096 vertices (given in [17]) has 660 vertices. Hence no such doubles exist. ■

F. Patterson graph of Suz type

Proposition 6.3.23. *The Patterson graph G of Suz type with* 22880 *vertices and intersection array* $\{280, 243, 144, 10; 1, 8, 90, 280\}$ *has no bipartite distance-regular doubles.*

Proof. Let H be a bipartite distance-regular double of G with parameters a_i, b_i, c_i. Then (6.1.1) with $j = 2$, gives $c_4 = 8c_2/c_3$. In view of 2.2.1(3) then, $c_4 \leq 8$. Thus, we have the following possibilities:

1. $c_2 = c_3 = \omega$ for $\omega = 1, 2, 3, 4, 5, 6, 7, 8$ and $c_4 = 8$.

2. $c_2 = 1$, $c_3 = 2$ and $c_4 = 4$.

3. $c_2 = 2$, $c_3 = 4$ and $c_4 = 4$.

4. $c_2 = 3$, $c_3 = 4$ and $c_4 = 6$.

Case (1): By (2.2.4(2)), $\omega = 1$. Now (6.1.1) with $i = 0, 1$ gives $280 = b_0 b_1$ and $243 = b_2 b_3$. But $b_0 \geq b_1 \geq b_2 \geq b_3$, a contradiction.

Case(2) (6.1.1) with $i = 0, 1$ gives $280 = b_0 b_1$ and $243 = b_2 b_3$. But $b_0 \geq b_1 \geq b_2 \geq b_3$, a contradiction.

Case(3): (6.1.1) with $i = 0, 1$ gives $560 = b_0 b_1$ and $486 = b_2 b_3$. But $b_0 \geq b_1 \geq b_2 \geq b_3$, a contradiction.

Case(4): (6.1.1) with $i = 0, 1$ gives $840 = b_0 b_1$ and $729 = b_2 b_3$. But $b_0 \geq b_1 \geq b_2 \geq b_3$, a contradiction.

Hence no such double H exists. ■

6.4 Generalized $2d$ gons

Here are our results are complete.

Theorem 6.4.1. *Let G be the collinearity graph of a finite generalized $2d$-gon with diameter $d \geq 2$ and parameters (s,t). Then there is a bipartite, distance-regular graph H with halved graph G if and only if $s = t$. In that case, H is uniquely determined as the incidence graph of G.*

Proof. Let H be a bipartite distance-regular double of the generalized $2d$-gon G of order (s,t) and diameter $d \geq 2$. Further, let a_i, b_i and c_i denote the parameters of H, with $k = b_0$ being the valency. The corresponding parameters of G will be a'_i, b'_i, c'_i and k'. Since the maximal cliques of the generalized $2d$-gons (s,t) are K_{s+1}, lemma 6.1.2 gives $k = s+1$ (the size of the maximal cliques). Since $c_1 = 1$ and $b_1 + c_1 = k$, we have $b_1 = s$. Since $k' = s(t+1)$, lemma 6.1.1 with $i = 0$ gives $c_2 = \frac{(s+1)}{(t+1)}$ (1). Since $c'_2 = 1$, lemma 6.1.1 with $j = 2$ gives $c_4 = 1\frac{c_2}{c_3}$. In view of 2.2.1(3) then, $c_4 \leq 1$. Thus the only possibility is: $c_2 = c_3 = c_4 = 1$ (2). (1)&(2) implies that $s = t$. As in lemma 6.1.2, the point set of H has bipartition $G \cup Y$ with $|G| = |Y|$ and each $G(y)$, for $y \in Y$, a maximal unique of $G \cong \frac{1}{2}H$. This can only be a line of the generalized polygon G. As $s = t$, G has the same number of points and lines. So for every line l there is a unique y in Y with $l = G(y)$. Thus H is the incidence graph of G, as claimed. ■

Corollary 6.4.2. *Let G be the collinearity graph of a finite distance-transitive generalized $2d$-gon with $d \geq 2$. Then there is a bipartite, distance-transitive graph H with halved graph G if and only if G is a generalized 4-gon of type $Sp_4(q)$ with q a power of 2 or G is a generalized hexagon of type $G_2(q)$ with q a power of 3. In both cases, H is the incidence graph of G and so is a generalized octagon (dodecagon) respectively.*

Proof. The generalized octagon of order $(1,q)$ and intersection array $\{q+1, q, q, q; 1, 1, 1, q+1\}$ is distance-transitive precisely when q is a power of 2 (see sec. 6.5 [17]). The generalized dodecagon of order $(1,q)$ and intersection array $\{q+1, q, q, q, q, q; 1, 1, 1, 1, 1, q+1\}$ is distance-transitive precisely when q is a power of 3 (see sec. 6.5 [17]). ■

Bibliography

[1] Aldred, R.E.L. and C.D. Godsil, Distance-regular antipodal covering graphs, J. Combin. Th. B 45 (1988) 127-134.

[2] Araya, M., A. Hiraki, and A. Jurisic, Distance-regular graphs with $b_2 = 1$ and antipodal covers, Europ. J. Combin. 18 (1997) 243-248.

[3] Bannai, E., On distance-regular graphs with fixed valency, III, J. Alg. 107 (1987) 43-52.

[4] Bannai, E. and T. Ito, On distance-regular graphs with fixed valency, Graphs and Combinatorics 3 (1987) 95-109.

[5] Bannai, E. and T. Ito, On distance-regular graphs with fixed valency, II, Graphs and combinatorics 4 (1988) 219-228.

[6] Bannai, E. and T. Ito, On distance-regular graphs with fixed valency, IV, Europ. J. Combin. 10 (1989) 137-148.

[7] Biggs, N.L., A.G. Boshier, and J. Shawe-Taylor, Cubic distance-regular graphs, J. London Math. Soc.(2) 33 (1986) 385-394.

[8] Biggs, N.L. and D.H. Smith, On trivalent graphs, Bull. London Math. Soc. 3 (1971) 155-158.

[9] Blokhuis, A. and A.E. Brouwer, Locally 4-by-4 grid graphs, J. Graph Theory, 13 (1989) (2) 229-244.

[10] Bon, J. van, Affine distance-transitive graphs, Thesis, Univ. Utrecht, 1990.

[11] Bon, J. van, Affine distance-transitive graphs with quadratic forms, Math. Proc. Cambridge Phil. Soc., 112 (1992) 507-517.

[12] Bon, J.T.M. van and A.E. Brouwer, The distance-regular antipodal covers of classical distance-regular graphs, pp. 141-166 in: Colloq. Math. Soc. Janos Bolyai, Proc. Eger 1987, 1988.

[13] Bon, J. van and A.M. Cohen, Linear groups and distance-transitive graphs, Europ. J. Combin. 10 (1989) 399-411.

[14] Bon, J. van and A.M. Cohen, Affine distance-transitive graphs and exceptional Chevalley groups, Proc. London Math. Soc. (3) 83 (2001) 51-70.

[15] Bon, J. van, A.M. Cohen and H. Cuypers, Affine distance-transitive graphs and classical groups, March 31, 2003.

[16] Bon, J. van, A. A. Ivanov and J. Saxl, Affine distance-transitive graphs with sporadic stabilizer, Europ. J. Comb. 20 (2) (1999) 163-177.

[17] Brouwer, A. E, A.M. Cohen and A. Neumaier, Distance-regular Graphs, Springer, Berlin, 1989.

[18] Brouwer, A. E, Uniqueness and nonexistence of some graphs related to M_{22}, Graphs and Combinatorics, 2 (1986) 21-29.

[19] Buekenhout, F. and H. van Maldeghem, Finite distance-transitive generalized polygons, Geometriae Dedicata 52 (1994) 41-51.

[20] Cameron, P.J., Dual polar spaces, Geometriac Dedicata 12 (1982) 75-85.

[21] Cameron, P.J., There are only finitely many finite distance-transitive graphs of given valency greater than two, Combinatorica 2 (1) (1982) 9-13.

[22] Cameron, P.J., Covers of graphs and EGQs, Discrete Math. 97 (1991) 83-92.

[23] Cameron, Peter J., Automorphism groups of graphs, in: Selected Topics in Graph Theory II, L.W. Beineke and R.J. Wilson (eds). Academic Press, London, 1983, pp. 89-127.

[24] Cameron, P.J., C.E. Praeger, J. Saxl, and G.M. Seitz, On the Sims conjecture and distance-transitive graphs, Bull. London Math. Soc. 15 (1983) 499-506.

[25] Cohen, A.M., Distance-transitive graphs, Eindhoven Univ. of Technology (March 2001).

[26] Cohen, A.M., Ivanov, and Alexander A., Affine distance-transitive groups of dimension one, Europ. J. Combin. 21 (2000) 191-195.

[27] Cohen, A.M., K. Magaard, and S. Shpectorov, Affine distance-transitive graphs: the cross characteristic case, Europ. J. Combin. 20 (1999) 351-373.

[28] Chuvaeva, I.V and D.V. Pasechnik, Distance-transitive graphs of type $q.K_{q,q}$ and projective planes, Europ. J. Combin. 11 (1990) 341-346.

[29] Faradjev, I.A, A.A. Ivanov, and A.V. Ivanov, Distance-transitive graphs of valency 5,6 and 7, Europ. J. Combin. 7 (1986) 303-319.

[30] Faradzev, I.A. and A.A. Ivanov, Distance-transitive representations of groups G with $PSL_2(q) \trianglelefteq G \trianglelefteq P\Gamma L_2(q)$, Europ. J. Combin. 11 (1990) 347-356.

[31] Gardiner, A., Antipodal Covering Graphs, J. Combin. Th. (B) 16 (1974) 255-273.

[32] Gardiner, A., Imprimitive distance-regular graphs and projective planes, J. Combin. Th. (B) 16 (1974) 274-281.

[33] Gardiner, A., On trivalent graphs, J. London Math. Soc. (2), 10 (1975), 507-512.

[34] Gardiner, A., When is an array relaised by a distance-regular graph?, pp. 209-219 in: Algebraic Methods in Graph Theory, Szeged, 1978, Vol. 1, Colloq. Math. Soc. J*á*nos Bolyai 25

[35] Gardiner, A., Classifying distance-transitive graphs, pp. 67-88 in: Combinatorial Math IX, Brisbane 1981, Lecture Notes in Math. 952 (E.J. Billington, S. Oates-Williams & A.P. Street, eds.), Springer Verlag, 1982.

[36] Gardiner, A., An elementary classification of distance-transitive graphs of valency four, Ars Combin. 19A (1985) 129-141.

[37] Gardiner, A. and Ch.E. Praeger, Distance-transitive graphs of valency five, Proc. Edinburgh Math Soc. (2) 30 (1987) 73-81.

[38] Gardiner, A. and C.E. Praeger, Distance-transitive graphs of valency six, Ars Combin. 21A (1986) 195-210.

[39] Gardiner, A. and C.E. Praeger, A geometrical approach to imprimitive graphs, Proc. London Math. Soc. (3) 71 (1995) 524-546.

[40] Godsil, C.D., and A.D. Hensel, Distance regular covers of the complete graph, J. Combin. Th. B 56 (1992) 205-238.

[41] Godsil, C.D., R.A. Liebler, and C.E. Praeger, Antipodal distance transitive covers of complete graphs, Europ. J. Combin. 19 (1998) 455-478.

[42] Hall, J. I., Locally Petersen graphs, J. Graph Th., 4 (1980) 173-187.

[43] Hemmeter, J., Halved graphs, Johnson and Hamming graphs, Utilitas Math. 25 (1984) 115-118.

[44] Hemmeter, J., Distance-regular graphs and halved graphs, Europ. J. Combin. 7 (1986) 119-129.

[45] Hemmeter, J., The large cliques in the graph of quadratic forms, Europ. J. Combinatorics (1988) 9, 395-410.

[46] Ito, T., Bipartite distance-regular graphs of valency three, Lin. Alg. Appl. 46 (1982) 195-213.

[47] Ivanov, A.A., Distance-transitive graphs and their classification, in Investigations in the Algebraic Theory of Combinatorial Objects (I. A. Faradžev, A. A. Ivanov, M. H. Klin and A. J. Woldar, eds.), Kluwer, Dordrecht, 1994, pp. 283-378.

[48] Ivanov, A.A and A.V. Ivanov, Distance-transitive graphs of valency k, $8 \leq k \leq 13$, pp. 112-145 in: Algebraic, Extremal and Metric Combinatorics, 1986, Cambridge Univ. Press, Cambridge, 1988.

[49] Ivanov, A.A., R. A. Liebler, T. Penttila, and C.E. Praeger, Antipodal distance-transitive covers of complete bipartite graphs, Europ. J. Combin. 18(1997) 11-33.

[50] Ivanov, A.A., S. A. Linton, K. Lux, J. Saxl and L. H. Soicher, Distance-transitive representations of the sporadic groups, Comm. Algebra 23 (1995) 3379-3472.

[51] Ivanov, A.A., S. V. Shpectorov, The P-geometry for M_{23} has no nontrivial 2-coverings, Europ. J. Comb. 11 (1990), 373-379.

[52] Jurisic, A., Distance regular antipodal covers of strongly regular graphs, Master's Essay, Univ. of Waterloo (April 1990).

[53] Jurisic, A., Antipodal covers, Ph. D's Thesis, Univ. of Waterloo, (January 1995).

[54] Liebeck, M.W, The affine permutation groups of rank three, Proc. London Math. Soc. (3) 54 (1987), no.3, 477-516.

[55] Liebeck, M.W and Ch.E. Praeger, Affine distance-transitive groups with alternating or symmetric point stabilizer, Europ. J. Comb. 13 (1992) 489-501.

[56] Liebeck, M.W, Ch.E. Praeger, and J. Saxl, Distance transitive graphs with symmetric or alternating automorphism group, Bull. Austral. Math. Soc. 35 (1987) 1-25.

[57] Praeger, C.E., Finite transitive permutation groups and finite vertex-transitive graphs, in Graph Symmetry (G. Hahn and G. Sabidussi, eds.), Kluwer, Dordrecht, 1997, pp. 277-318.

[58] Praeger, C.E., Symmetric graphs and the classification of the finite simple groups, pp. 99-110 in: Groups, Proc. Conf. on combinatorial group theory, Kyoungju, Korea, 1983, Lect. Notes Math. 1098 (A.C. Kim & B.H. Neumann, eds.), 1984.

[59] Praeger, C.E., Imprimitive symmetric graphs, Ars Combin. 19A (1985) 149-163.

[60] Praeger, C.E., J. Saxl, and K. Yokoyama, Distance transitive graphs and finite simple groups, Proc. London Math. Soc. (3) 55 (1987) 1-21.

[61] Shult, E. and A. Yanushka, Near n-gons and line systems, Geometriae Dedicata, 9 (1980), 1-72.

[62] Smith, D.H., Primitive and imprimitive graphs, Quart. J. Math. Oxford (2) (1971) 551-557.

[63] Smith, D.H., Bounding the diameter of a distance-transitive graph, J. Combin. Th. (B) 16, (1973) 139-144.

[64] Smith, D.H., On tetravalent graphs, J. London Math. Soc.(2) 6 (1973) 659-662.

[65] Smith, A.H., Distance-transitive graphs of valency four, J. London Math. Soc.(2) 8 (1974) 377-384.

[66] Smith, A.H., On bipartite tetravalent graphs, Discrete Math. 10 (1974) 167-172.

[67] Taylor, D.E., Two-graphs and doubly transitive groups, J. Combin. Th. A 61 (1992) 113-122.

[68] Terwilliger, P., P- and Q-polynomial association schemes and their antipodal P-polynomial covers, Europ. J. Combin. 14 (1993) 355-358.

[69] Vijayan, K.S., On a class of distance transitive graphs, J. Combin. Th. (B) 25 (1978) 125-129.

[70] Weiss, R., Distance-transitive graphs and generalized polygons, Arch. Math. 45 (1985) 186-192.

[71] Weiss, R., On distance-transitive graphs, Bull. London Math. Soc. 17 (1985) 253-256. For a given $k \geq 3$, there are, up to isomorphism, only finitely many finite distance-transitive graphs of valency k.

[72] Yokoyama, Z., On distance transitive graphs in which the stabilizer of a point contains an alternating group, Europ. J. Combin. 9 (1988) 379-390.