King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Math 102(12&17) Exam2(B) Spring(042)

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(1) Evaluate each of the following integrals. (45pts)

(i) $\int \frac{dx}{2e^x+1}$.

$u = 2e^x+1$

$du = 2e^x\,dx$

$\frac{du}{2e^x} = dx$

$\frac{du}{u-1} = dx$

$$\int \frac{dx}{2e^x+1} = \int \frac{du}{(u-1)u} = \int \frac{du}{u-1} - \int \frac{1}{u}\,du$$
$$= \ln|u-1| - \ln|u| + C$$
$$= \ln|2e^x| - \ln|2e^x+1| + C$$
$$= \boxed{\ln \frac{2e^x}{2e^x+1} + C}$$

$$\frac{1}{(u-1)u} = \frac{A}{u-1} + \frac{B}{u} = \frac{1}{u-1} + \frac{-1}{u}$$

$1 = Au + B(u-1)$ $\quad u=0$ $\boxed{B=-1}$ $\quad u=1$ $\boxed{A=1}$

(ii) $\int \cos^2\frac{x}{2}\sin^4\frac{x}{2}\,dx$.

$$\int \cos^2\frac{x}{2}\sin^4\frac{x}{2}\,dx = \int \frac{1+\cos x}{2}\cdot\left(\frac{1-\cos x}{2}\right)^2 dx$$
$$= \frac{1}{8}\int (1+\cos x)(1-2\cos x+\cos^2 x)\,dx$$
$$= \frac{1}{8}\int 1-2\cos x+\underline{\cos^2 x}+\cos x-\underline{2\cos^2 x}+\cos^3 x\;dx$$
$$= \frac{1}{8}\int \cos^3 x-\cos^2 x-\cos x+1\;dx$$
$$= \frac{1}{8}\int \cos x(\cos^2 x-1)\,dx - \frac{1}{4}\int \cos x\,dx + \frac{1}{4}\int dx$$
$$= \frac{1}{8}\int \cos x(-\sin^2 x)\,dx - \frac{1}{4}\sin x + \frac{1}{4}x + C$$
$$= -\frac{1}{8}\frac{\sin^3 x}{3} - \frac{1}{8}\sin x + \frac{1}{8}x + C$$
$$= \boxed{\frac{-1}{24}\sin^3 x - \frac{1}{8}\sin x + \frac{1}{8}x + C}$$

(iii) $\int (\cosh x - \sinh x)^{10} dx$

$$\cosh x - \sinh x = \frac{e^{x}+e^{-x}}{2} - \frac{e^{x}-e^{-x}}{2}$$

$$= \frac{\cancel{e^{x}}+e^{-x}-\cancel{e^{x}}+e^{-x}}{2}$$

$$= e^{-x}$$

$$\therefore \int (\cosh x - \sinh x)^{10} dx = \int e^{-10x} dx = \boxed{-\frac{1}{10} e^{-10x} + C}$$

(iv) $\int \frac{dx}{\sqrt{x^2-6x+13}}$

$$\int \frac{dx}{\sqrt{x^2-6x+13}} = \int \frac{dx}{\sqrt{x^2-6x+9+4}} = \int \frac{dx}{\sqrt{(x-3)^2+4}}$$

$u = x-3$

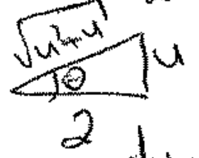
$du = dx$

$$= \int \frac{du}{\sqrt{u^2+4}}$$

$$= \int \frac{\cancel{2}\cancel{\sec^2\theta}\, d\theta}{\cancel{2}\cancel{\sec\theta}}$$

$u = 2\tan\theta$

$\frac{u}{2} = \tan\theta$



$du = 2\sec^2\theta \, d\theta$

$$= \int \sec\theta \, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{u^2+4}}{2} + \frac{u}{2}\right| + C$$

$$= \boxed{\ln\left|\frac{\sqrt{x^2-6x+13} + (x-3)}{2}\right| + C}$$

(iv) $\int (sin^{-1}x)^2 dx$

$$u = (\sin^{-1}x)^2 \qquad dv = dx$$

$$du = 2\sin^{-1}x \cdot \frac{1}{\sqrt{1-x^2}}\,dx \qquad v = x$$

$$\int (\sin^{-1}x)^2\,dx = x(\sin^{-1}x)^2 - \int \frac{2x}{\sqrt{1-x^2}} \sin^{-1}x\,dx$$

$x = \sin\theta$

$dx = \cos\theta\,d\theta$

$$= \int \frac{2\sin\theta}{\cancel{\cos\theta}} \cdot \theta \cdot \cancel{\cos\theta}\,d\theta$$

$$= \int 2\theta \sin\theta\,d\theta$$

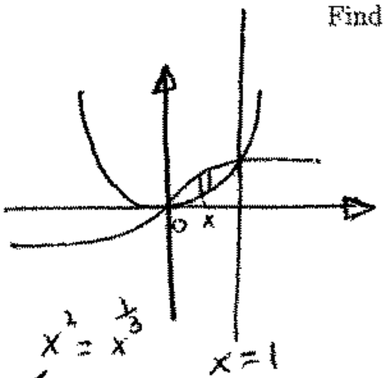
2θ +	$\sin\theta$
2 −	$-\cos\theta$
0	$-\sin\theta$

$$= -2\theta\cos\theta + 2\sin\theta + C$$

$$= -2\sin^{-1}x \cdot \sqrt{1-x^2} + 2x + C$$

$$\boxed{\int (\sin^{-1}x)^2\,dx = x(\sin^{-1}x)^2 + 2\sqrt{1-x^2}\,\sin^{-1}x - 2x + C}$$

(2) The region bounded by the graph of $y = x^2$ and $x = y^3$ is revolved about the line $x = 1$. Find the volume of the resulting solid. (15pts)



$x^2 = x^{\frac{1}{3}}$

$x^6 = x$

$x^6 - x = 0$

$x(x^5 - 1) = 0$

$\boxed{x = 0}$ $\boxed{x = 1}$

by Cylindrical Shells Method

$$V = 2\pi \int_a^b r\,h \quad dx$$

$$= 2\pi \int_0^1 (1-x)\,(x^{\frac{1}{3}} - x^2)\,dx$$

$$= 2\pi \int_0^1 x^{\frac{1}{3}} - x^2 - x^{\frac{4}{3}} + x^3\,dx$$

$$= 2\pi \left[\frac{3}{4}x^{\frac{4}{3}} - \frac{x^3}{3} - \frac{3x^{\frac{7}{3}}}{7} + \frac{x^4}{4}\right]_0^1$$

$$= 2\pi \left[\left(\frac{3}{4} - \frac{1}{3} - \frac{3}{7} + \frac{1}{4}\right) - 0\right]$$

$$= 2\pi \left[\frac{20}{84}\right]$$

$$= \frac{20\pi}{42} = \boxed{\frac{10\pi}{21}}$$

(3) Find the exact arc length of the curve

(10pts)

$y = (\frac{x}{2})^{2/3}$ from $x = 0$ to $x = 2$.

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_0^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^2 \frac{\sqrt{9(x/2)^{2/3}+4}}{3(x/2)^{1/3}}\,dx = \frac{1}{3}\int_0^2 \frac{\sqrt{9(x/2)^{2/3}+1}}{(x/2)^{1/3}}$$

$$\frac{dy}{dx} = \cancel{\tfrac{2}{3}}\left(\frac{x}{2}\right)^{-\frac{1}{3}}\cancel{\tfrac{1}{2}}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{9}\left(\frac{x}{2}\right)^{-\frac{2}{3}}$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{1}{9(x/2)^{2/3}}$$

$$= \boxed{\frac{9(x/2)^{2/3}+1}{9(x/2)^{2/3}}}$$

$$u = 3(x/2)^{\frac{1}{3}}$$

$$\Rightarrow du = (x/2)^{-2/3}\cdot\frac{1}{2}\,dx$$

$$2(x/2)^{2/3}\,du = dx$$

$$2\left(\frac{u}{3}\right)^2 du = dx$$

$$= \frac{2}{3}\int_0^3 \frac{\sqrt{u^2+1}\,\left(\frac{u}{3}\right)^2}{\left(\frac{u}{3}\right)}\,du$$

$$= \frac{2}{9}\int_0^3 u\sqrt{u^2+1}\,du$$

$$= \frac{1}{9}\int_0^3 2u\sqrt{u^2+1}\,du$$

$$= \frac{1}{9}\cdot\frac{2}{3}\left(u^2+1\right)^{3/2}\Big|_0^3$$

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$$= \boxed{\frac{2}{27}\left[10^{3/2}-1\right]}$$

(4) Find the area of the surface formed by revolving about the x-axis the parametric curve

$$x = \sin^2 t,\ y = \sin t \cos t,\ 0 \le t \le \tfrac{\pi}{2}. \qquad \text{(10pts)}$$

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\frac{dx}{dt} = 2\sin t \cos t$$

$$\left(\frac{dx}{dt}\right)^2 = 4\sin^2 t \cos^2 t$$

$$\frac{dy}{dt} = \cos^2 t - \sin^2 t$$

$$\left(\frac{dy}{dt}\right)^2 = \cos^4 t - 2\cos^2 t \sin^2 t + \sin^4 t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\cos^2 t + \sin^2 t\right)^2$$

$$= 1$$

$$\therefore\ S = \int_0^{\frac{\pi}{2}} 2\pi \sin t \cos t \, dt$$

$$= 2\pi \left.\frac{\sin^2 t}{2}\right|_0^{\frac{\pi}{2}}$$

$$= \pi \left.\sin^2 t\right|_0^{\frac{\pi}{2}} = \boxed{\pi}$$

(5) Set up the partial fraction decomposition of $\frac{1}{(x^3+1)(x^4-1)^2}$.

(DO NOT CALCULATE THE CONSTANTS) (10pts)

$$\frac{1}{(x^3+1)(x^4-1)^2} = \frac{1}{(x+1)(x^2-x+1)\ (x^2-1)^2(x^2+1)^2}$$

$$= \frac{1}{(x+1)\ (x^2-x+1)\ (x-1)^2(x+1)^2\ (x^2+1)^2}$$

$$= \frac{1}{(x+1)^3\ (x^2-x+1)(x-1)^2\ (x^2+1)^2}$$

$$= \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{x-1} + \frac{E}{(x-1)^2} + \frac{fx+h}{x^2+1}$$

$$+ \frac{gx+k}{(x^2+1)^2} + \frac{mx+n}{x^2-x+1}$$

(6) Complete the blanks. (Show your work) (10pts)

(i) $\coth(\ln x)$ is equal to $\boxed{\dfrac{x^2+1}{x^2-1}}$

$$\coth x = \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$$

$$\coth(\ln x) = \frac{e^{\ln x}+e^{-\ln x}}{e^{\ln x}-e^{-\ln x}} = \frac{x+\frac{1}{x}}{x-\frac{1}{x}} = \frac{x^2+1}{x^2-1}$$

(ii) For all real x, $\frac{\operatorname{csch}^2 x}{1+\coth^2 x}$ is equal to $\boxed{\operatorname{sech} 2x}$

$$\frac{\frac{1}{\sinh^2 x}}{1+\frac{\cosh^2 x}{\sinh^2 x}} = \frac{\frac{1}{\cancel{\sinh^2 x}}}{\frac{\sinh^2 x+\cosh^2 x}{\cancel{\sinh^2 x}}} = \frac{1}{\sinh^2 x+\cosh^2 x}$$

$$= \frac{1}{\cosh 2x}$$

$$= \operatorname{sech} 2x$$

(iii) If $y = (\operatorname{sech} x)(\operatorname{csch} x)$ then y' is equal to ....

$$y' = (-\operatorname{sech} x \tanh x)\operatorname{csch} x + (-\operatorname{csch} x \coth x)\operatorname{sech} x$$

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