There are two sections to the algorithm: labelling and augmenting.

LABELLING PROCESS

The vertices of the network (directed graph) G can be *labelled*; a labelled vertex can be *scanned*. Start with no vertices labelled. Label the source s as $[-,\infty]$.

All vertices $q \neq s$ with $f(s,q) < c(s,q)$ are labelled $[s^+, \delta(q)]$, where $\delta(q) = c(s,q) - f(s,q)$. Then all unlabelled vertices q with $f(q,s) > 0$ are labelled $[s^-, \delta(q)]$, where $\delta(q) = f(q,s)$. The source s is now said to be scanned.

In general, as soon as one vertex is scanned, we choose a new vertex p that is labelled but not yet scanned. For each vertex q not yet labelled with $f(p,q) < c(p,q)$, we give q the label $[p^+, \delta(q)]$, where $\delta(q) = \min(\delta(p), c(p,q) - f(p,q))$. [NOTE: this is a little different from what was said in class on Monday.] Next, for each unlabelled q with $f(q,p) > 0$, we give q the label $[p^-, \delta(q)]$, where $\delta(q) = \min(\delta(p), f(q,p))$. When this is done, p is said to be scanned.

Terminate the labelling process when either

(*i*) the sink t is labelled, in which case you go to the FLOW AUGMENTING PROCESS below;

(*ii*) the sink t is not labelled, but no more labels can be assigned (that is, all labelled vertices have been scanned).

In case (*ii*), let $S = \{\text{labelled vertices}\}$ and $T = V(G) \backslash S$. Then the present flow f has maximal value $v(f) = v$, and the cut $S \cup T$ is a minimal cut, having capacity equal to the flow value v.

FLOW AUGMENTING PROCESS

Work backwards from the sink t. In the labelling process, t has been labelled either $[q^+, \delta(t)]$ or $[q^-, \delta(t)]$, for some vertex q. Define

$$f^*(q,t) = f(q,t) + \delta(t) \text{ when } t = [q^+, \delta(t)]$$

$$f^*(t,q) = f(t,q) - \delta(t) \text{ when } t = [q^-, \delta(t)]$$

The vertex q referenced in the label of t is itself labelled either $[p^+, \delta(t)]$ or $[p^-, \delta(t)]$, for some vertex p. Define

$$f^*(p,q) = f(p,q) + \delta(t) \text{ when } t = [p^+, \delta(t)]$$
$$f^*(q,p) = f(q,p) - \delta(t) \text{ when } t = [p^-, \delta(t)]$$

Continue back in this fashion (replacing q by p) until $p = s$. If (u,v) is a pair of vertices of G for which $f^*(u,v)$ is not already defined, let $f^*(u,v) = f(u,v)$. (Remember, non-edges of G always get capacity and flow 0.) Then f^* is a new flow in G with value $v(f^*) = v(f) + \delta(t) > v(f)$.

Replace f by f^*, discard all labels, and return to the LABELLING PROCESS.

REMARKS

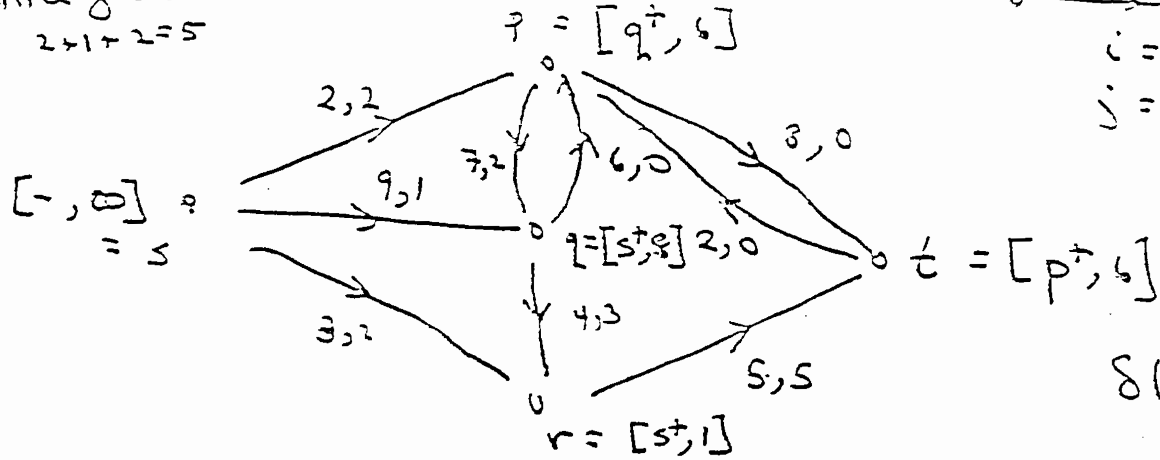
Notice that there is some softness in the algorithm as described – we have not specified the order for scanning the labelled vertices, nor have we specified the order for labelling new vertices while scanning a given, previously labelled vertex. For implementation, these issues can be important. If capacities are irrational, bad choices can lead to a never-ending algorithm. Even for integral capacities, bad choices can unnecessarily prolong the algorithm.

EXAMPLE

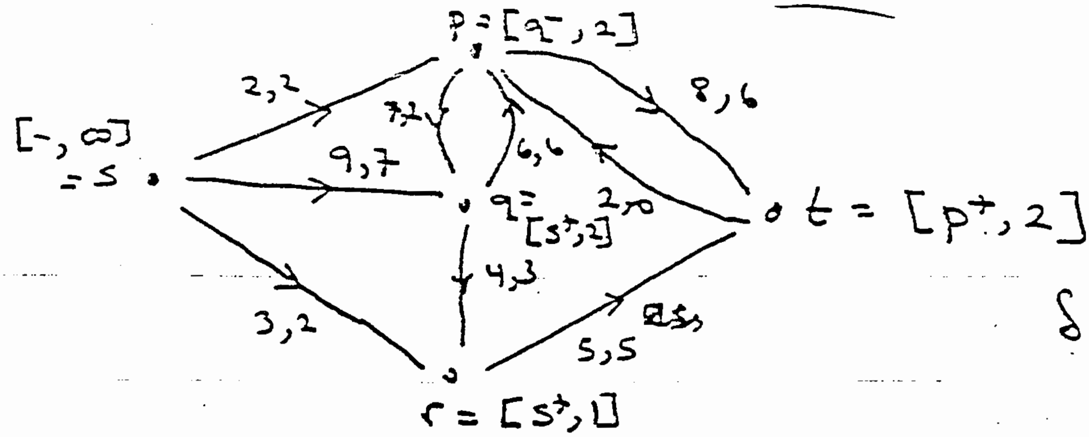
initial flow:
2+1+2=5

$\circ \xrightarrow{i,j} \circ$

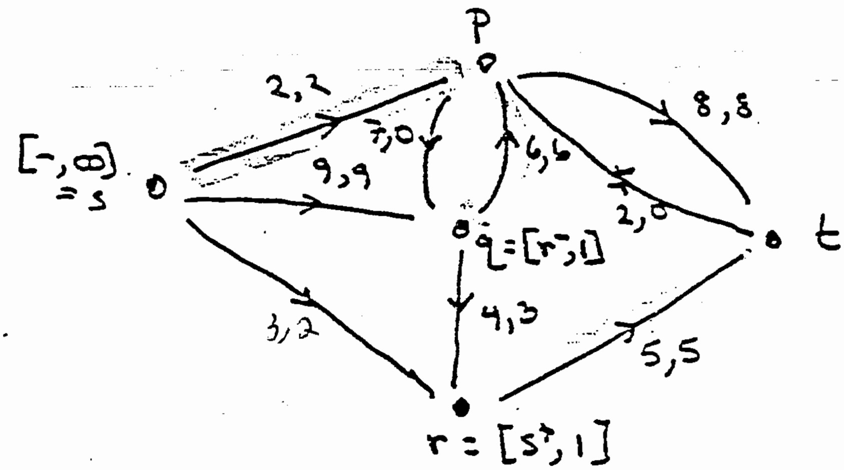
i = capacity c
j = flow f



$\delta(t) = 6$



$\delta(t) = 2$



t unlabelled

flow = 8+0+5=13
[= 5+6+2]

cut =
{labelled} ∪ {unlabelled}
= {s, q, r} ∪ {p, t}

MRF