CHAPTER EIGHT
Linear Regression

8.1 Scatter Diagram

Example 8.1 A chemical engineer is investigating the effect of process operating temperature (x) on product yield (y). The study results in the following data:

x	100	110	120	130	140	150	160	170	180	190
y	45	51	54	61	66	70	74	78	85	89

(Hines and Montgomery, 1990, p 457) Check if there is any linear relationship between temperature and product yield.

Solution Make a data file with x as Var1 and y as Var2. Then follow the steps in Statistica to get a scatter diagram.

1. Graphs
2. Scatterplot
3. Click Variables (Select Var1 and Var2) / OK
4. Click Advanced (In Graph Type click Regular & in Fit click Off)
5. OK

Since the scatter diagram (See Figure 8.1) between temperature (Var1) and product yield (Var2) shows a linear trend, one recommends estimating the line of best fit.

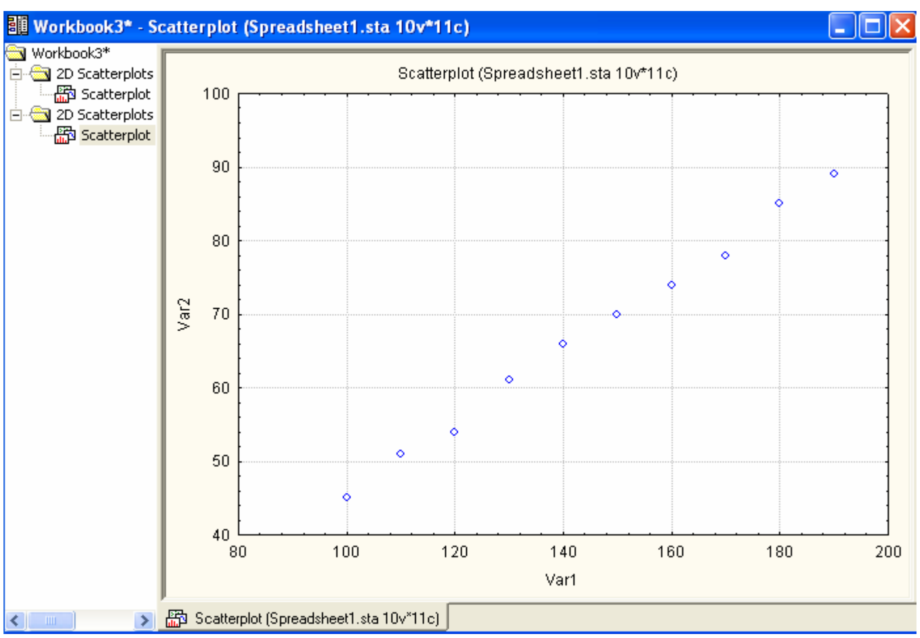


Figure 8.1 Scatterplot

8.2 The Correlation Coefficient

The strength of linear relationship between x and y is measured by the correlation coefficient**,** defined by

$$r = \frac{s_{xy}}{\sqrt{s_{xx}s_{yy}}}$$

where

$$s_{xx} = \sum (x_i - \overline{x})^2 = (n-1)\, s_x^2 \ = \sum x_i^2 - \frac{\left(\sum x_i\right)^2}{n} = \sum x_i^2 - n\overline{x}^2$$

$$s_{yy} = \sum (y_i - \overline{y})^2 = (n-1)\, s_y^2 \ = \sum y_i^2 - \frac{\left(\sum y_i\right)^2}{n} = \sum y_i^2 - n\overline{y}^2$$

$$s_{xy} = \sum (x_i - \overline{x})(y_i - \overline{y}) = \sum x_i y_i - \frac{\left(\sum x_i\right)\left(\sum y_i\right)}{n} = \sum x_i y_i - n\overline{x}\ \overline{y}$$

In Example 8.1, we have $s_{xy} = 3985$, $s_{xx} = 8250$, $s_{yy} = 1932.1$

$$r = \frac{3985}{\sqrt{(8250)(1932.10)}} = 0.99813$$

8.3 Estimating the Line of Best Fit

The simple linear regression model is of the form

$$y = \mu_{y.x} = \beta_0 + \beta_1 x + \varepsilon$$

where $\mu_{y.x}$ is the conditional mean of *y* at *x,*

y and x are respectively the dependent (response) and independent (explanatory) variables,

ε is the random error component,

β_o and β_1 are the $y-$intercept and the slope of the regression line respectively.

The least squares estimators of the regression parameters are given by

$$\hat{\beta}_1 = \frac{s_{xy}}{s_{xx}} \text{ and } \hat{\beta}_0 = \overline{y} - \hat{\beta}_1\ \overline{x}$$

Once the parameters are estimated, the equation $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$ will be called the estimated regression line, the prediction line, the line of best fit or the least squares line. It should be noted that $\hat{y}_0 = \hat{\beta}_0 + \hat{\beta}_1 x_0$ can be used as a point estimate of $\mu_{y.x_0}$ the conditional mean of y at x_0, or a predictor of the response at x_0.

For the data in Example 8.1,

$$\hat{\beta}_1 = 0.4830 \text{ and } \hat{\beta}_0 = -2.7393$$

so that the line of best fit is given by

$$\hat{y} = -2.7393 + 0.4830\,x\,.$$

At a temperature of 140^0 C, we predict the yield to be
$\hat{y}_0 = \hat{\beta}_0 + 140\hat{\beta}_1 = -2.7393 + 140(0.4830) = 64.8848$

Estimating β_o and β_1 Using Statistica

To estimate β_o and β_1 by using ***Basic Statistics and Tables Module***, we can find the estimates of β_o and β_1 by simply plotting a scatter graph of the dependent variable against the independent variable with a ***linear fit***. While you are in ***Basic Statistics and Tables Module*** follow the steps:

1. Enter the values of X in one column, say Var1 and the corresponding Y values in another column, say Var2
2. Graphs / Stats 2D Graphs / Scatterplots
3. Variables / X = Var1 and Y = Var2 / OK
4. In advance, Select Regular for Graph Type and Linear for Fit
5. OK

For the data in Example 8.1 you will get $\text{Var2} = -2.7394 + 0.483 * x$ (Figure 8.2) which means that the estimates of β_o and β_1 are given by $\hat{\beta}_o = -2.7394$ and $\hat{\beta}_1 = 0.483$ respectively. Thus, the predicted simple linear regression model is given by

$$\hat{y} = -2.7394 + 0.483x.$$

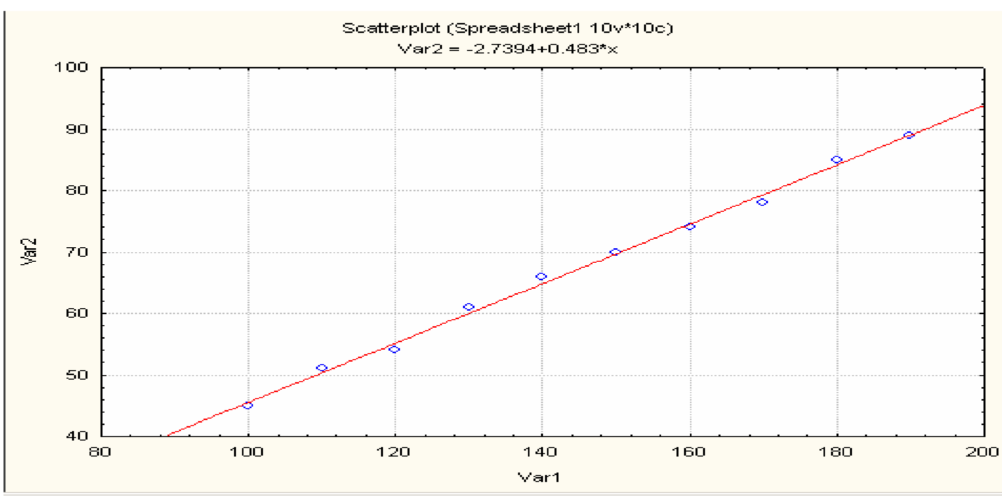


Figure 8.2 A Graph of Y versus X

8.4 Sources of Variation

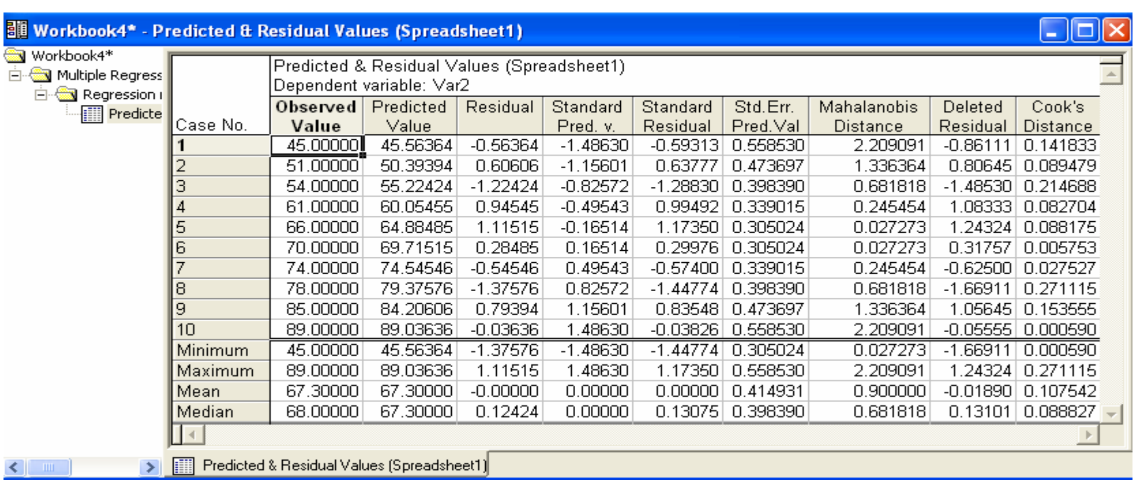
The variation in the dependent variable say $TSS = s_{yy}$ is attributed partly to that in the independent variable. The rest is attributed to what is called the Sums of Squares Due to Errors defined by $SSE = \sum (y_i - \hat{y})^2$ and can be calculated by the following table:

x	y	$\hat{y}$	$y - \hat{y} = e$	e^2
100	45	45.5636	−0.5636	0.3176
110	51	50.3939	0.6061	0.3674
120	54	55.2242	−1.2242	1.4987
130	61	60.0545	0.9455	0.8940
140	66	64.8848	1.1152	1.2437
150	70	69.7151	0.2849	0.0812
160	74	74.5454	−0.5454	0.2975
170	78	79.3757	−1.3757	1.8926
180	85	84.2060	0.7940	0.6304
190	89	89.0363	−0.0363	0.0013
			$1(10)^{-8}$	7.2244

The SSE = 7.2244, here compare these errors (or residuals) with that obtained by Statistica.

Predicted and Residual Values Using Statistica
Follow the steps:

1. Statistics / Multiple regression
2. Variables (select the dependent and independent variables) / OK / OK
3. Click Residual / assumption / prediction
4. Click perform residual analysis
5. In advanced click Summary: Residual & predicted (get Figure 8.3)



Case No.	Observed Value	Predicted Value	Residual	Standard Pred. v.	Standard Residual	Std.Err. Pred.Val	Mahalanobis Distance	Deleted Residual	Cook's Distance
1	45.00000	45.56364	-0.56364	-1.48630	-0.59313	0.558530	2.209091	-0.86111	0.141833
2	51.00000	50.39394	0.60606	-1.15601	0.63777	0.473697	1.336364	0.80645	0.089479
3	54.00000	55.22424	-1.22424	-0.82572	-1.28830	0.398390	0.681818	-1.48530	0.214688
4	61.00000	60.05455	0.94545	-0.49543	0.99492	0.339015	0.245454	1.08333	0.082704
5	66.00000	64.88485	1.11515	-0.16514	1.17350	0.305024	0.027273	1.24324	0.088175
6	70.00000	69.71515	0.28485	0.16514	0.29976	0.305024	0.027273	0.31757	0.005753
7	74.00000	74.54546	-0.54546	0.49543	-0.57400	0.339015	0.245454	-0.62500	0.027527
8	78.00000	79.37576	-1.37576	0.82572	-1.44774	0.398390	0.681818	-1.66911	0.271115
9	85.00000	84.20606	0.79394	1.15601	0.83548	0.473697	1.336364	1.05645	0.153555
10	89.00000	89.03636	-0.03636	1.48630	-0.03826	0.558530	2.209091	-0.05555	0.000590
Minimum	45.00000	45.56364	-1.37576	-1.48630	-1.44774	0.305024	0.027273	-1.66911	0.000590
Maximum	89.00000	89.03636	1.11515	1.48630	1.17350	0.558530	2.209091	1.24324	0.271115
Mean	67.30000	67.30000	-0.00000	0.00000	0.00000	0.414931	0.900000	-0.01890	0.107542
Median	68.00000	67.30000	0.12424	0.00000	0.13075	0.398390	0.681818	0.13101	0.088827

Figure 8.3 Predicted and Residual Values

Decomposition of the Sum of Squares

It can be proved that $TSS = SSR + SSE$ where TSS is the Total Sum of Squares, SSR is the Sum of Squares due to Regression and SSE is the Sum of Squares of Errors, also known as the residual sum of squares.

$$TSS = s_{yy}, \quad SSR = \hat{\beta}_1 \; s_{xy} = \frac{s_{xy}^2}{s_{xx}}$$

The coefficient of determination is defined by

$$R^2 = \frac{SSR}{TSS}$$

In Example 8.1, $TSS = 1932.1$, $SSR = 1924.8757$ so that $SSE = TSS - SSR = 7.2243$. Note that the expression $SSR = \hat{\beta}_1^2 \, s_{xx}$ may not be computationally efficient.

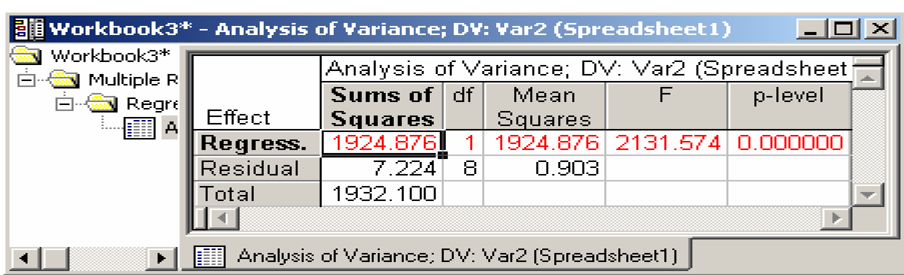
The coefficient of determination $R^2 = 0.9963$

Calculation of Sums of Squares Using Statistica

To calculate the sum of squares using Statistica follow the steps:

1. Enter the values of x in one column, say Var1 and the corresponding y values in another column, say Var2.
2. Statistics / Multiple Regression
3. Variables (select the dependent and independent variables) / OK /OK
4. In Advanced click ANOVA (Overall Goodness of Fit)

The resulting spreadsheet of result shows the Total Sum of Squares (TSS), the sum of squares due to regression (SSR) and the sum of squares due to errors (SSE), the mean squares and the F value.
For the data in Example 8.1 above, we have $TSS = 1932.1$, $SSR = 1924.876$ and $SSE = 7.224$ (Figure 8.4).



Analysis of Variance; DV: Var2 (Spreadsheet

Effect	Sums of Squares	df	Mean Squares	F	p-level
Regress.	1924.876	1	1924.876	2131.574	0.000000
Residual	7.224	8	0.903		
Total	1932.100				

Figure 8.4 Analysis of Variance

8.5 Confidence Interval Estimation of Regression Parameters

Confidence Interval (CI) for the Slope Parameter β_1

A $100(1-\alpha)\%$ CI for β_1 is given by $\hat{\beta}_1 \mp t_{\alpha/2} \sqrt{\dfrac{MSE}{s_{xx}}}$

where $t_{\alpha/2}$ is the $100(1-\alpha/2)$th percentile of the *t*-distribution with $df = n-2$, and

$$MSE = \frac{SSE}{n-2}, \text{ the estimate for } \sigma^2$$

For the data in Example 8.1, $\hat{\sigma}^2 = 0.9030$, and thus a 95% CI for β_1 is given by

$$0.4830 \mp (2.306) \sqrt{\frac{0.9030}{8250}}$$

In other words,

$$0.458877405 \le \beta_1 \le 0.507156201.$$

Confidence Interval for the Conditional Mean $\mu_{y.x}$

A $100(1-\alpha)\%$ CI for the conditional mean at x_0 is given by

$$(\hat{\beta}_0 + \hat{\beta}_1 x_0) \mp t_{\alpha/2} \sqrt{\left(\frac{1}{n} + \frac{(x_0 - \bar{x})^2}{s_{xx}}\right) MSE}$$

In example 8.1, a 95% CI for the conditional mean at 140 ^0C is given by

$$64.8848 \mp (2.306) \sqrt{(0.103)(0.9030)} = 64.8848 \mp 0.7033$$

i.e. $64.1814 \le \mu \le 65.5882$

The above problem can be solved using Statistica following the steps:

1. Statistics
2. Multiple Regression
3. Click variables (Choose Dependent and Independent Variable say Var2 and Var1)
4. OK/OK
5. Click Residuals/Assumptions/Prediction
6. Click Compute Confidence Limits (Checked by default)
7. Click Predict Dependent Variable (Under Common value put 140) Click Apply, then OK

We find that 95% confidence interval for $\mu_{y.140} = \beta_0 + 140\beta_1$ is given by [64.18146, 65.58823] (See Figure 8.5).

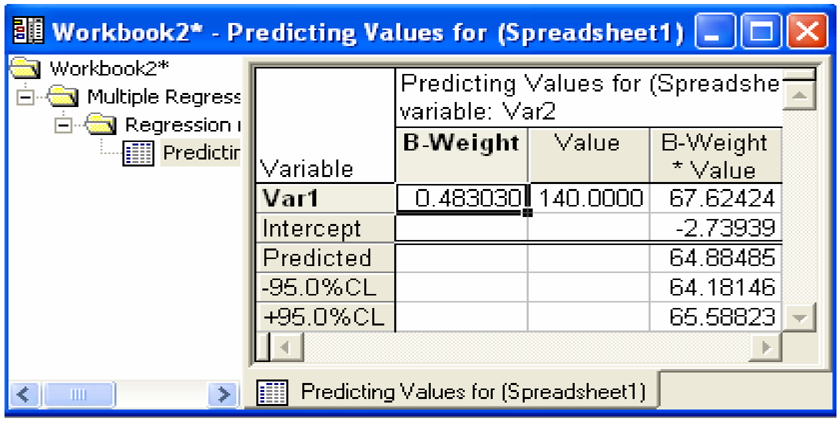


Figure 8.5 Confidence Interval for Mean

8.6 Prediction Interval (PI) for a Future Observation y_0

A $(1-\alpha)100\%$ PI for a future observation y_0 at x_0 is given by

$$(\hat{\beta}_0 + \hat{\beta}_1\, x_0) \;\mp\; t_{\alpha/2}\; \sqrt{\left(1+\frac{1}{n}+\frac{(x_0-\overline{x})^2}{s_{xx}}\right) MSE}$$

For the data in Example 8.1 a 95% prediction interval for the yield at 140 ^0C is given by

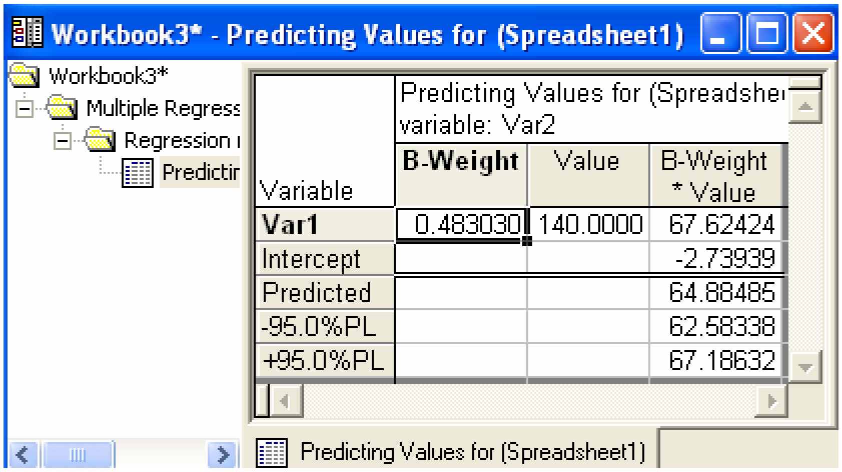
$$64.8848 \;\mp\; (2.306)\; \sqrt{(1+0.103)(0.90303)} = 64.8848 \;\mp\; 2.3014$$

i.e, $62.5834 \le \hat{y}_0 \le 67.1862$

The problem can be solved using Statistica following the steps:

1. Statistics
2. Multiple Regression
3. Click variables (Choose Dependent and Independent Variable say Var2 and Var1)
4. OK / OK
5. Click Residuals/Assumptions/Prediction
6. Click Compute Prediction Limits
7. Click Predict Dependent Variable to enter fixed x say 140
8. Click Apply, then OK

It is predicted with 95% confidence that product yield will be in the interval [62.58338, 67.18632] (See Figure 8.6).



Variable	Predicting Values for (Spreadshe variable: Var2 B-Weight	Value	B-Weight * Value
Var1	0.483030	140.0000	67.62424
Intercept			-2.73939
Predicted			64.88485
-95.0%PL			62.58338
+95.0%PL			67.18632

Figure: 8.6 Prediction Interval for Product Yield

8.7 Testing the Slope of the Regression Line

The following table has a list of possible null hypotheses involving the slope β_1, the critical region and the p-value in each case.

Hypotheses about β_1 and their respective rejection regions and p-values

H_o vs H_a	Rejection Region	*p*-value
$\beta_1 = \beta_{10}$ vs $\beta_1 \neq \beta_{10}$	$\{t : t < -t_{\alpha/2} \quad or \quad t > t_{\alpha/2}\}$	$2P(\lvert t \rvert > t_{\alpha/2})$
$\beta_1 = \beta_{10}$ vs $\beta_1 < \beta_{10}$	$\{t : t < -t_{\alpha}\}$	$P(t < -t_{\alpha})$
$\beta_1 = \beta_{10}$ vs $\beta_1 > \beta_{10}$	$\{t : t > t_{\alpha}\}$	$P(t > t_{\alpha})$

The test Statistic for these hypotheses is

$$t = \frac{\hat{\beta}_1 - \beta_{10}}{\sqrt{\dfrac{MSE}{s_{xx}}}}$$

The hypothesis $H_0 : \beta_1 = 0$ vs. $H_a : \beta_1 \neq 0$ is known as the hypothesis of the significance of the regression.

In example 8.1, test the hypothesis $H_0 : \beta_1 = 0$ vs. $H_a : \beta_1 \neq 0$ at significance level $\alpha = 0.05$.

The value of the test statistics is $t = \dfrac{0.4830 - 0}{\sqrt{\dfrac{0.90303}{8250}}} = 46.1689$

Since $t = 46.1689 > t_{0.025} = 2.306$, we reject H_0 in favor of the alternative hypothesis H_a, and conclude that the regression is significant.

8.8 Testing the Significance of the Regression by Analysis of Variance

In order to test the hypothesis $H_o: \ \beta_1 = 0 \quad vs \quad H_a: \ \beta_1 \neq 0$ at the 5% significant level, using an *F* test, we reproduce here the ANOVA table of example 8.1 shown in figure 8.4

SV	SS	DF	MS	f
Regression	1924.875758	1	1924.8757	2131.5738
Error	7.22424243	8	0.90303	
Total	1932.10	9		

The test statistic for the above hypothesis is

$$\frac{MSR}{MSE} = \frac{SSR/1}{SSE/(n-2)}.$$

The observed value of the test statistic is $f = 2131.5738$. Since $f = 2131.5738 > f_{0.05} = 5.32$, the critical value from the F distribution with 1 and *n – 2* degrees of freedom, we reject $H_0: \beta_1 = 0$ in favor of the alternative hypothesis H_a at 5% level of significance.

Testing the Significance of the Regression Using Statistica

Using a *t*-test we follow the steps:

1. Enter the values of X in one column, say Var1 and the corresponding Y values in another column, say Var2.
2. Statistics / Multiple Regression, to get Figure 8.7,click Advanced

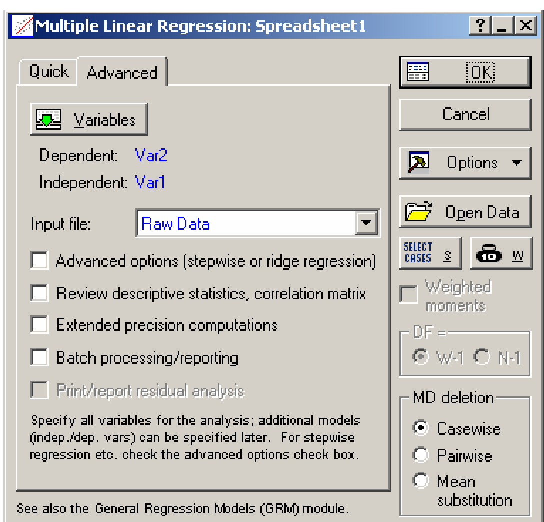


Figure 8.7 Multiple Regression Setup

3. Variables (select the dependent and independent variables) / OK
4. OK (to get Figure 8.8)

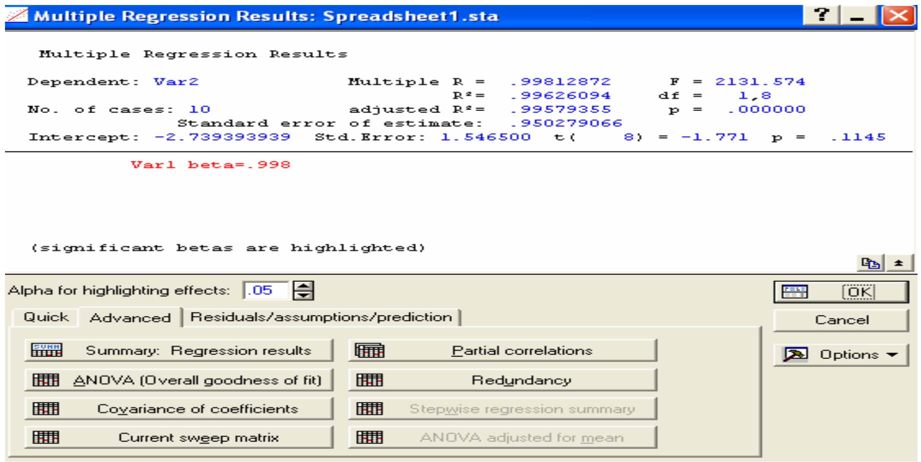
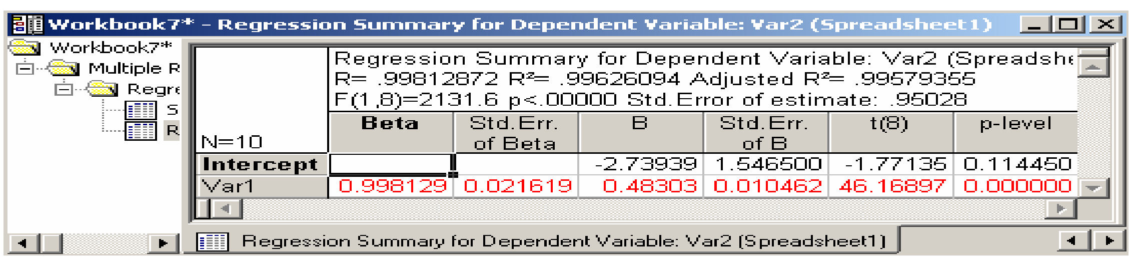


Figure 8.8 Regression Results

5. In Advanced, click Summary: Regression results.You will get Figure 8.9



Regression Summary for Dependent Variable: Var2 (Spreadshe
R= .99812872 R²= .99626094 Adjusted R²= .99579355
F(1,8)=2131.6 p<.00000 Std.Error of estimate: .95028

N=10	Beta	Std.Err. of Beta	B	Std.Err. of B	t(8)	p-level
Intercept			-2.73939	1.546500	-1.77135	0.114450
Var1	0.998129	0.021619	0.48303	0.010462	46.16897	0.000000

Figure 8.9 Regression Summary

The regression parameters are given by $\hat{\beta}_0 = -2.7393$ and $\hat{\beta}_1 = 0.4830$ (See Column B of Figure 8.9).

Since the *p-value* for testing the null hypothesis $H_0 : \beta_1 = 0$ against the alternative hypothesis that $H_a : \beta_1 \neq 0$ is given by 0 (See the column labeled by p-level), we reject the null hypothesis at any $\alpha > 0$ in favor of the alternative hypothesis.

Using an F test, follow the steps:

1. Enter the values of x in one column, say Var1 and the corresponding Y values in another column, say Var2.
2. Statistics / Multiple Regression, to get Figure 8.7, click Advanced
3. Variables (select the dependent and independent variables) / OK /OK
4. In Advanced click ANOVA (Overall Goodness of Fit)

For the data in Example 8.1, we have p-value $= P(F > 2131.574) = 0 < \alpha$ (See Figure 8.4) so that we reject null hypothesis at any $\alpha > 0$ and accept the alternative hypothesis , indicating that the regression of y on x is significant whether we are testing at 1% or 5% level of significance.

8.9 Checking Model Assumptions

We now discuss how to verify the assumptions that the random errors are normally distributed and that they have a constant variance.

Checking the Assumption of Normality
To check the assumption that the errors follow a normal distribution, a normal probability plot of residuals is drawn. If the plot is approximately linear, then the assumption is justified, otherwise, the assumption is not justified.
In Statistica, to get the normal probability plot of residuals, we follow the steps below assuming that we have the data of Example 8.1 in the Multiple Regression Module.

1. Statistics / Multiple Regression
2. Variable / Select the dependent and independent variables / *OK / OK*
3. In Residual / assumption / prediction, click Perform residual analysis
4. In Quick, click Normal plot of residuals

Since the normal probability plot of residuals (See Figure 8.10) for the data in Example 8.1 exhibits a linear trend. Thus, the normality assumption is valid.

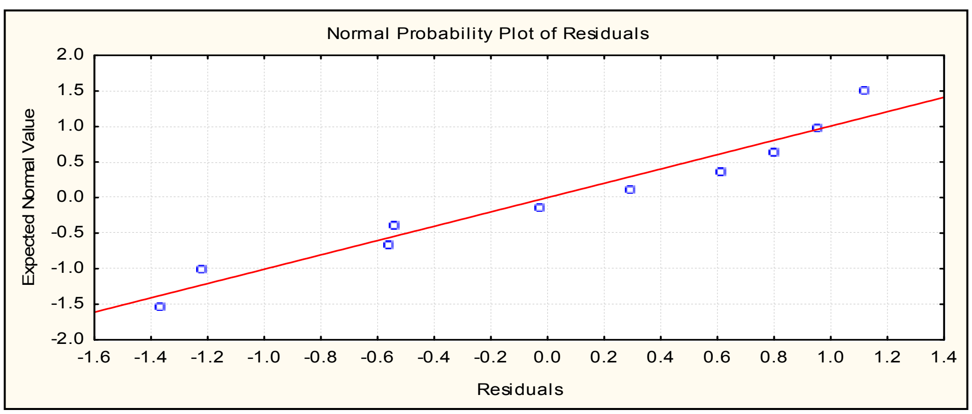


Figure 8.10 Normal probability Plot

Checking the Assumption of Constancy of Variance
To check the assumption that the errors have a constant variance, a graph of the residuals versus the independent variable is plotted. If the graph shows no pattern, then the assumption is justified. Otherwise, the assumption is not justified.

In Statistica, to plot the graph of the residuals versus the independent variable, we proceed as follows assuming that we have the data in multiple regression module:

1. Statistics / Multiple Regression
2. Variable / Select the dependent and independent variables / *OK*
3. In Residuals / assumption / prediction, click Perform residual analysis
4. under Type of residual click Raw residuals
5. In Residuals, click Residuals vs. independent Var, "select the independent variable"

The graph in Figure 8.11 is the residual plot for example 8.1. Since it does not exhibit any pattern, we conclude that the constant variance assumption is justified. The value $r = 0.63E\text{-}6$ states that raw residuals (e) and temperatures (x) are almost uncorrelated.

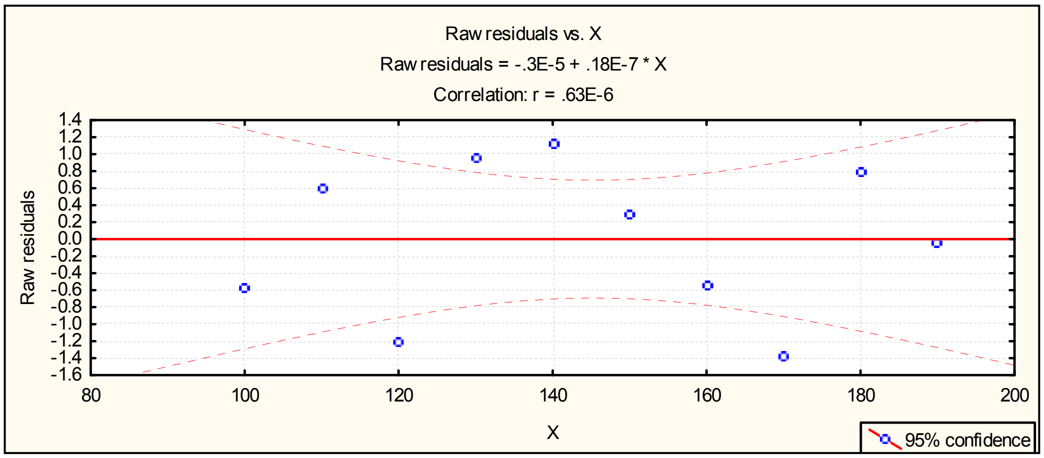


Figure 8.11 Graph of Residuals vs Independent Variable

8.10 Multiple Linear Regression

The multiple regression model is a mathematical model which explains the relationship between the dependent variable and two or more independent variables. For example, a manufacturer wants to model the quality of a product (Y) as a function of temperature (x_1) and pressure (x_2) at which it is produced.

The multiple linear regression model with 2 independent variables x_1 and x_2 is given by

$$Y = \beta_o + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$$

where β_o and ε are the intercept and the random error term respectively. We shall refer to the $\beta' s$ in the model as the regression parameters.

Example 8.2 Consider the problem of predicting gasoline mileage y (in miles per gallon), where the independent variables are fuel octane rating x_1 and average speed (mile per hour) x_2. The sample data obtained from 20 test runs with cars at various speeds are as follows:

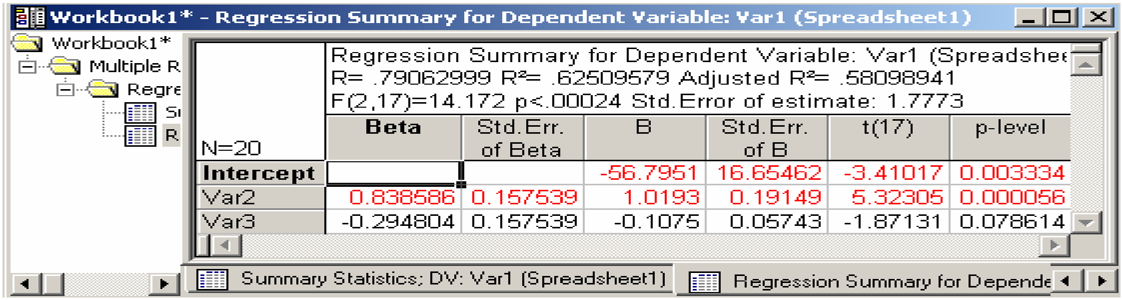
y	x_1	x_2
24.8	88	52
30.6	93	60
31.1	91	28
28.2	90	52
31.6	90	55
29.9	89	46
31.5	92	58
27.2	87	46
33.3	94	55
32.6	95	62
30.6	88	47
28.1	89	58
25.2	90	63
35.0	93	54
29.2	91	53
31.9	92	52
27.7	89	52
31.7	94	53
34.2	93	54
30.1	91	58

Estimate the linear regression model and interpret your results.

Solution To solve this problem by using Statistica, you must be in ***Multiple Regression*** module and follow the steps:

1. Enter the values of each variable in a separate column (or variable)
2. Statistics / Multiple Regression
3. Variables (select the dependent variable from the list on the left)
4. Hold down the Ctrl key and select the independent variables from the list on the right.
5. OK
6. In Advanced Click Summary: Regression Results

For the data in Example 8.2, one can read the estimates of the regression parameters as : $\hat{\beta}_o = -56.7951,\ \hat{\beta}_1 = 1.0193,\ \hat{\beta}_2 = -0.1075$ (See column labeled B in Figure 8.12).



Regression Summary for Dependent Variable: Var1 (Spreadshee
R= .79062999 R²= .62509579 Adjusted R²= .58098941
F(2,17)=14.172 p<.00024 Std.Error of estimate: 1.7773

N=20	Beta	Std.Err. of Beta	B	Std.Err. of B	t(17)	p-level
Intercept			-56.7951	16.65462	-3.41017	0.003334
Var2	0.838586	0.157539	1.0193	0.19149	5.32305	0.000056
Var3	-0.294804	0.157539	-0.1075	0.05743	-1.87131	0.078614

Figure 8.12 Regression Summary

Thus, the predicted multiple linear regression model for the given data is

$$\hat{y} = -56.7951 + 1.0193x_1 - 0.1075x_2$$

If 'average speed' (x_2 or Var3) is held fixed, it is estimated that a 1-unit increase in octane (x_1 or Var2) would result in 1.0193 unit increase in the expected 'gasoline mileage'. Similarly if 'octane' (x_1 or Var2) is held fixed, it is estimated that a 1-unit increase in average speed (x_2 or Var3) would result in, 0.1075 unit decrease in the expected 'gasoline mileage'.

Exercises

8.1 (cf. Devore, J. L., 2000, 510). The following data represent the burner area liberation rate $(= x)$ and emission rate (Nox) $(= y)$:

x	100	125	125	150	150	200	200	250	250	300	300	350	400
y	150	140	180	210	190	320	280	400	430	440	390	600	610

(a) Assuming that the simple linear regression model is valid, obtain the least square estimate of the true regression line.
(b) What is the estimate of the expected Nox emission rate when burner area liberation rate equals 225?
(c) Estimate the amount by which you expect Nox emission rate to change when burner area liberation rate is decreased by 50.

8.2 (cf. Devore, J. L., 2000, 510). The following data represent the wet deposition (NO3) $(= x)$ and lichen N (% dry weight) $(= y)$:

x	0.05	0.10	0.11	0.12	0.31	0.42	0.58	0.68	0.68	0.73	0.85
y	0.48	0.55	0.48	0.50	0.58	0.52	0.86	1.0	0.86	0.88	1.04

(a) What are the least square estimates of β_0 *and* β_1 ?
(b) Predict lichen N for an NO3 deposition value of 0 .5.
(c) Test the significance of regression at 5% level of significance.

8.3 (Devore, J. L., 2000, 510). The following data represent x = available travel space in feet, and y = separation distance:

x	12.8	12.9	12.9	13.6	14.5	14.6	15.1	17.5	19.5	20.8
y	5.5	6.2	6.3	7.0	7.8	8.3	7.1	10.0	10.8	11.0

(a) Derive the equation of the estimated line.
(b) What separation distance would you predict if available travel space value is 15.0?

8.4 (Devore, J. L., 2000, 511). Consider the following data set in which the variable of interest are x = commuting distance and y = commuting time:

x	15	16	17	18	19	20	5	10	15	20	25	50	5	10	15	20	25	50
y	42	45	35	42	49	46	16	32	44	45	63	115	8	16	22	23	31	60

Obtain the least square estimate of the regression model.

8.5 (cf. Devore, J. L., 2000, 584). Soil and sediment adsorption, the extent to which chemicals collect in a condensed form on the surface, is an important characteristic influencing the effectiveness of pesticides and various agricultural chemicals, The article “Adsorption of Phosphate, Arsenate, Methancearsonate, and Cacodylate by Lake and Stream sediments: Comparisons with Soils” (J. of

Environ. Qual., 1984: 499-504) gives the accompanying data on y = phosphate adsorption index, x_1 = amount of extractable iron, and x_2 = amount of extractable aluminum.

x_1	61	175	111	124	130	173	169	169	160	244	257	333	199
x_2	13	21	24	23	64	38	33	61	39	71	112	88	54
y	4	18	14	18	26	26	21	30	28	36	65	62	40

(a) Find the least square estimates of the parameters and write the equation of the estimated model.
(b) Make a prediction of Adsorption index resulting from an extractable iron = 250 and extractable aluminum = 55.
(c) Test the null hypothesis that $\beta_2 = 0$ against the alternative hypothesis that $\beta_2 \neq 0$ at 5% level of significance.

8.6 (Johnson, R. A., 2000, 345). The following table shows how many weeks a sample of 6 persons have worked at an automobile inspection station and the number of cars each one inspected between noon and 2 P.M. on a given day:

Number of weeks employed (x)	2	7	9	1	5	12
Number of cars inspected (y)	13	21	23	14	15	21

(a) Find the equation of the least squares line, which will enable us to predict y in terms of x .
(b) Use the result of part (a) to estimate how many cars someone who has been working at the inspection station for 8 weeks can be expected to inspect during the given 2-hour period.

8.7 (cf. Devore, J. L., 2000, 590). An investigation of die casting process resulted in the accompanying data on x_1= on furnace temperature, x_2= die close time and y = temperature difference on the die surface (A Multiple Objective Decision Making Approach for Assessing Simultaneous Improvement in Die Life and Casting Quality in a Die Casting Process," Quality Engineering, 1994: 371-383).

x_1	1250	1300	1350	1250	1300	1250	1300	1350	1350
x_2	6	7	6	7	6	8	8	7	8
y	80	95	101	85	92	87	96	106	108

(a) Write the equation of the estimated model.
(b) Test the null hypothesis that $\beta_1 = 0$ against the alternative hypothesis that $\beta_1 \neq 0$ at 5% level of significance.

8.8 (cf. Johnson, R. A., 2000, 334). The following are measurements of the air velocity and evaporation coefficient of burning fuel droplets in an impulse engine:

Air Velocity (cm/sec) x	20	60	100	140	180	220	160	300	340	380
Evaporation coefficient mm2/sec) y	0.18	0.37	0.35	0.78	0.56	0.75	1.18	1.36	1.17	1.65

(a) Fit a straight line to the data by the method of least square and use it to estimate the evaporation coefficient of a droplet when the air velocity is 190 cm/s.
(b) Test the null hypothesis that $\beta_1 = 0$ against the alternative hypothesis $\beta_1 \neq 0$ at the 0.05 level of significance.

8.9 (Johnson, R. A., 2000, 344). A chemical company, wishing to study the effect of extraction time on the efficiency on an extraction operation, obtained the data shown in the following table;

Extraction time (minutes) (x)	27	45	41	19	35	39	19	49	15	31
Extraction efficiency (%) (y)	57	64	80	46	62	72	52	77	57	68

(a) Draw a scattergram to verify that a straight line will provide a good fit to the data.
(b) Draw a straight line to predict the extraction efficiency one can expect when the extraction time is 35 minutes.

8.11 (cf. Johnson, R. A., 2000, 347). The cost of manufacturing a lot of certain product depends on the lot size, as shown by the following sample data:

Cost (Dollars)	30	70	140	270	530	1010	2500	5020
Lot Size	1	5	10	25	50	100	250	500

(a) Draw a scattergram to verify the assumption that the relationship is linear, letting lot size be x and cost y .
(b) Fit a straight line to these data by the method of least squares, using lot size as the independent variable, and draw its graph on the diagram obtained in part (a).

8.12 (Johnson, R. A., 2000, 345). The following table, x is the tensile force applied to a steel specimen in thousands of pounds, and y is the resulting elongation thousands of an inch:

x	1	2	3	4	5	6
y	14	33	40	63	76	85

(a) Graph the data to verify that it is reasonable to assume that the regression of Y on x is linear.
(b) Find the equation of the least square line, and use it to predict the elongation when the tensile force is 3.5 thousand pounds.

8.13 (Johnson, R. A., 2000, 385). The following are the data on the number of twists required to break a certain kind of forged alloy bar and the percentage of two alloying elements present in the metal;

No. of twists (y)	41	49	69	65	40	50	58	57	31	36	44	57	19	31	33	43
%age Element A (x_1)	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
%age Element B (x_2)	5	5	5	5	10	10	10	10	15	15	15	15	20	20	20	20

Fit a least square regression plane and use its equation to estimate the number of twists required to break one of the bars when $x_1 = 2.5$ and $x_2 = 12$.

8.14 (Johnson, R. A., 2000, 263). Twelve specimens of cold-reduced sheet steel, having different copper contents and annealing temperatures, are measured for hardness with the following results:

Hardness	78.9	65.1	55.2	56.4	80.9	69.7	57.4	55.4	85.3	71.8	60.7	58.9
Copper content	0.02	0.02	0.02	0.02	0.10	0.10	0.10	0.10	0.18	0.18	0.18	0.18
Annealing Temp.	1000	1100	1200	1300	1000	1100	1200	1300	1000	1100	1200	1300

Fit an equation of the form $y = \beta_o + \beta_1 x_1 + \beta_2 x_2$, where x_1 represents the copper content, x_2 represents the annealing temperature, and y represents the hardness.

8.15 Suppose the following data gives the mass of adults in kilograms sampled from three villages A, B, C

A		B		C	
71.5	68.0	62.0	65.0	60.5	62.8
63.3	73.5	71.3	73.1	58.4	58.7
73.6	74.1	64.7	72.5	65.5	58.1
78.1	73.6	66.0	73.0	50.6	58.5
66.5	76.5	72.6	66.8	62.1	52.6
74.3	73.3	71.1	71.9	64.5	58.8
76.3	76.0	65.4	65.0	60.6	63.7
62.0	64.0	59.1	69.9	62.9	61.6
62.8	69.6	69.7	69.8	60.2	67.2
72.9	69.2	77.1	78.5	63.4	58.2

(a) Assuming that these samples are independent, run *t*-tests to determine which of the villages have identical mean mass of adults stating clearly the hypotheses you are testing. State your conclusions based on the p-value as well as the *t*-value.

(b) State the assumption under which your tests are valid.

8.16 (cf. Dougherty, 1990, 595) When smoothing a surface with an abrasive, the roughness of the finished surface decreases as the abrasive grain becomes finer. The following data give measurements of surface roughness (in micrometers) in terms of the grit numbers of the grains, finer grains possessing larger grit numbers.

x	24	30	36	46	54	60
y	0.34	0.30	0.28	0.22	0.19	0.18

(a) Draw a scatter diagram. Do you recommend fitting a linear regression model?

(b) How strong is the linear correlation between the two variables?
(c) Do you think that there is strong nonlinear correlation between the two variables?

8.17 (cf. Johnson, R. A., 2000, 578). The article " How to optimize and Control the Wire Bonding Process: Part II" (Solid State Technology, Jan. 1991: 67-72) described on experiment carried out to assess the impact of the variable x_1 = force (gm), x_2 = power (mw), x_3 = temperature ($^\circ C$), and x_4 = time (ms) on y = ball bond share strength (gm). The following data generated to be consistent with the information given in the article:

Observations	Force	Power	Temperature	Time	Strength
1	30	60	175	15	26.2
2	40	60	175	15	26.3
3	30	90	175	15	39.8
4	40	90	175	15	39.7
5	30	60	225	15	38.6
6	40	60	225	15	35.5
7	30	90	225	15	48.8
8	40	90	225	15	37.8
9	30	60	175	25	26.6
10	40	60	175	25	23.4
11	30	90	175	25	38.6
12	40	90	175	25	52.1
13	30	60	225	25	39.5
14	40	60	225	25	32.3
15	30	90	225	25	43.0
16	40	90	225	25	56.0
17	25	75	200	20	35.2
18	45	75	200	20	46.9
19	35	45	200	20	22.7
20	35	105	200	20	58.7
21	35	75	150	20	34.5
22	35	75	250	20	44.0
23	35	75	200	10	35.7
24	35	75	200	30	41.8
25	35	75	200	20	36.5
26	35	75	200	20	37.6
27	35	75	200	20	40.3
28	35	75	200	20	46.0
29	35	75	200	20	27.8
30	35	75	200	20	40.3

(a) Find the least square estimates of the parameters and write the equation of the estimated model.
(b) Make a prediction of strength resulting from a force of 35 gm, power of 75 mw, temperature of 200 degrees and time of 20 ms.
(c) Test the null hypothesis that $\beta_3 = 0$ against the alternative hypothesis that $\beta_3 \neq 0$ at 5% level of significance.