Show all your work

1. Find the rectangular coordinates for the points whose polar coordinates are:

   (a) $\left(7, -\frac{\pi}{4}\right)$ (b) $(-1, \pi)$.

   (a) $x = 7\cos\left(-\frac{\pi}{4}\right) = \frac{7}{\sqrt{2}}$, $y = 7\sin\left(-\frac{\pi}{4}\right) = -\frac{7}{\sqrt{2}}$. Point is $\left(\frac{7}{\sqrt{2}}, -\frac{7}{\sqrt{2}}\right)$.

   (b) $x = -\cos(\pi) = 1$, $y = -\sin(\pi) = -1$. Point is $(1, -1)$.

2. Find polar coordinates (r, θ) for the point whose rectangular coordinates is $\left(\sqrt{3}, -1\right)$ such that:

   (a) $r \geq 0$, $0 \leq \theta < 2\pi$, (b) $r \leq 0$, $-\pi \leq \theta < \pi$.

   (a) $r = \sqrt{3+1} = 2$, $\tan\theta = -\frac{1}{\sqrt{3}}$. $\theta = \frac{11\pi}{6}$. Point is $\left(2, \frac{11\pi}{6}\right)$.

   (b) $r = -\sqrt{3+1} = -2$, $\tan\theta = -\frac{1}{\sqrt{3}}$. $\theta = \frac{11\pi}{6} - \pi = \frac{5\pi}{6}$. Point is $\left(-2, \frac{5\pi}{6}\right)$.

3. Change the following polar equations into rectangular coordinates:

   (a) $r^2 \sin 2\theta = 1$ (b) $r = 4\cos\theta + 4\sin\theta$.

   (a) $r^2 \times 2\sin\theta\cos\theta = 1$. Rearranging, $2(r\sin\theta)(r\cos\theta) = 1$. Therefore, $2xy = 1$.

   (b) $r^2 = 4r\cos\theta + 4r\sin\theta$. Therefore, $x^2 + y^2 = 4x + 4y$.

4. Test the following equations for symmetry with respect to the x-axis, the y-axis and the origin:

   (a) $r = \sin 2\theta$, (b) $r = \cos\theta$.

   (a) replace θ by $-\theta$, the equation changes. Replace θ by $\theta + \pi$ and r by $-r$. The equation does not change. Therefore, we have x-symmetry.

   Replace θ by $\pi - \theta$, equation changes. Replace θ by $-\theta$and r by $-r$. The equation does not change. Therefore, we have y-symmetry. It follows that we also have O-symmetry.

   (b) replace θ by $-\theta$, the equation does not change. Therefore, we have x-symmetry.

   Replace θ by $\pi - \theta$, equation changes. Replace θ by $-\theta$and r by $-r$, The equation changes.

   Therefore, we do not have y-symmetry. It follows that we also do not have O-symmetry.

5. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$ for the parametric curve

$$x = t\cos t, \qquad y = t\sin t.$$

$\frac{dy}{dt} = 1 + \cos t$. $\frac{dx}{dt} = 1 - \sin t$. $\frac{dy}{dx} = \frac{1+\cos t}{1-\sin t}$. $\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{1-\sin t+\cos t}{(1-\sin t)^2}$.
$\frac{d^2y}{dx^2} = \frac{1-\sin t+\cos t}{(1-\sin t)^3}$. At $t = \frac{\pi}{6}$,
$\frac{dy}{dx} = \frac{1+\sqrt{3}/2}{1/2} = 2+\sqrt{3}$. $\frac{d^2y}{dx^2} = \frac{1/2+\sqrt{3}/2}{1/8} = 4+4\sqrt{3}$.

6. (a) Show that the two curves $r = 1+\cos\theta$ and $r = 2\sin\theta$ intersect at $\left(\frac{8}{5}, \cos^{-1}\frac{3}{5}\right)$ and $(0,\pi)$.

$$1+\cos\theta = 2\sin\theta$$

Square both sides and simplify to get

$$\begin{aligned} 1+2\cos\theta+\cos^2\theta &= 4\left(1-\cos^2\theta\right), \\ 5\cos^2\theta+2\cos\theta-3 &= 0. \end{aligned}$$

Therefore,

$$\cos\theta = \frac{-2\pm\sqrt{4+60}}{10} = \frac{-1\pm 4}{5} = -1, \frac{3}{5}.$$

(b) Set up an integral (but do not integrate) to compute the area between the two curves in part (a).

Area $= \frac{1}{2}\int_0^{\cos^{-1}\frac{3}{5}} (2\sin\theta)^2\, d\theta + \frac{1}{2}\int_{\cos^{-1}\frac{3}{5}}^{\pi} (1+\cos\theta)^2\, d\theta$.