# 1 Curve and Surface Evolution

In this chapter we study the evolution of curves and surfaces. We begin with the curve evolution.

## 1.1 Curve Evolution

An evolving curve can be thought of as a family of curves parametrized by time. This means that each curve in the family is a mapping $\gamma : J \times (0,T] \rightarrow \mathbb{R}^2$, that assigns for each space parameter $t \in J$ and each time parameter $\tau \in (0,T]$ a point $\gamma(t,\tau) \in \mathbb{R}^2$. An evolution equation is a differential equation that describes the evolution of $\gamma$ in time. For example, the evolution equation

$$\frac{\partial \gamma}{\partial \tau} = \overrightarrow{N} \tag{1}$$

describes a curve that is evolving with unit speed along its normal direction $\overrightarrow{N}$. Another example is the so called *geometric heat equation*

$$\frac{\partial \gamma}{\partial \tau} = \kappa \overrightarrow{N} \tag{2}$$

which describes a curve evolving along its normal direction with speed equal to its curvature. The name *geometric heat equation* comes from writing the above equation as

$$\frac{\partial \gamma}{\partial \tau} = \frac{\partial^2 \gamma}{\partial t^2},$$

where $\gamma$ is parametrized by arclength.

Consider a general evolution equation is of the form

$$\frac{\partial \gamma}{\partial \tau} = V_1(\gamma(\tau),\tau)\overrightarrow{N} + V_2(\gamma(\tau),\tau)\overrightarrow{T}, \tag{3}$$

where $V_i : \mathbb{R}^2 \times (0,T] \rightarrow \mathbb{R},\ i = 1,2$ is a real valued function. It should be intuitively clear that the tangential component of the evolution equation has no effect on the shape of the curve and affects only its parametrization. This intuition will be made precise by the following lemma.

**Lemma 1** *There is a parametrization $\widetilde{\gamma}$ of the curve $\gamma$ such that the evolution equation (3) reduces to*

$$\frac{\partial \widetilde{\gamma}}{\partial \tau} = V_1 \overrightarrow{N}. \tag{4}$$

**Proof.** Let $\phi : J \times (0,T] \rightarrow I$ be a change of the space parameter, in other words, for each $\tau \in (0,T]$, $\phi(\cdot,\tau)$ is a reparametrization of $\gamma(\cdot,\tau)$ as defined in the previous chapter. Let $s = \phi(t,\tau)$ and $\gamma(t,\tau) = \widetilde{\gamma}(s,\tau)$. Then

$$\frac{\partial \gamma}{\partial \tau} = \frac{\partial \widetilde{\gamma}}{\partial s}\frac{\partial s}{\partial \tau} + \frac{\partial \widetilde{\gamma}}{\partial \tau}.$$

Substituting in (3) we get

$$\begin{aligned}\frac{\partial\widetilde{\gamma}}{\partial s}\frac{\partial s}{\partial\tau}+\frac{\partial\widetilde{\gamma}}{\partial\tau} &= V_1\overrightarrow{N}+V_2\overrightarrow{T}\\ &= V_1\overrightarrow{N}+V_2\frac{\widetilde{\gamma}_s}{\left|\widetilde{\gamma}_s\right|}.\end{aligned}$$

Rearranging, we get

$$\frac{\partial\widetilde{\gamma}}{\partial\tau}=V_1\overrightarrow{N}+\left(\frac{V_2}{\left|\widetilde{\gamma}_s\right|}-\frac{\partial s}{\partial\tau}\right)\frac{\partial\widetilde{\gamma}}{\partial s}.$$

Thus, choosing $s=\phi\left(t,\tau\right)$ to be any particular solution of the partial differential equation

$$V_2-\frac{\partial\phi}{\partial\tau}\left|\widetilde{\gamma}_s\right|=0$$

reduces equation (3) to (4). ■

The above lemma means that it is sufficient to consider only evolutions along the normal direction to the curve $\gamma\left(\cdot,\tau\right)$.

### 1.1.1 Invariant Curve Evolutions

We discuss here evolution equations that are based on the invariant forms discussed in the previous chapter. For example, the evolution equation for the Euclidean arclength is given by

$$\begin{aligned}\frac{\partial\gamma}{\partial\tau} &= \left\langle\frac{\partial^2\gamma}{\partial s^2},\overrightarrow{N}\right\rangle\overrightarrow{N}\\ &= \kappa\overrightarrow{N}\end{aligned}$$

and for the equi-affine arclength by

$$\begin{aligned}\frac{\partial\gamma}{\partial\tau} &= \left\langle\frac{\partial^2\gamma}{\partial v^2},\overrightarrow{N}\right\rangle\overrightarrow{N}\\ &= \kappa^{1/3}\overrightarrow{N}.\end{aligned}$$

Another invariant form, to be discussed now comes from the *affine transformation with one fixed point* (also known as *the linear affine transformation*). This transformation has the form

$$\mathbf{y}=T\mathbf{x}=A\left(\mathbf{x}-\mathbf{x}_0\right)+\mathbf{x}_0,$$

where $\det A=1$. The point $\mathbf{x}_0$ is fixed by this transformation and all vectors emanating from $\mathbf{x}_0$ have images which also emanate from $\mathbf{x}_0$. We can easily show that the form $\gamma\times\gamma'$ is invariant under this transformation. To obtain the associated invariant arclength, we find a reparametrization $\widetilde{\gamma}\left(w\right)$ such that

$$\left|\widetilde{\gamma}\times\widetilde{\gamma}_w\right|=1.$$

Thus, with the usual change of variable $w = \phi(t)$, we have

$$1 = |\widetilde{\gamma} \times \widetilde{\gamma}_w| = \frac{1}{\phi'} |\gamma \times \gamma_t| .$$

Therefore,

$$w = \phi(t) = \int_a^t |\gamma(\nu) \times \gamma_t(\nu)| \, d\nu.$$

It is easy to show that the above integral is invariant under reparametrization. Thus

$$\begin{aligned} w &= \phi(s) = \int_0^s |\gamma \times \gamma_s| \, d\nu \\ &= \int_0^s \left| \gamma \times \overrightarrow{T} \right| d\nu = \int_0^s \left| \left\langle \gamma, \overrightarrow{N} \right\rangle \right| d\nu. \end{aligned}$$

Writing $\widetilde{\gamma}(w) = \gamma(s)$ we get

$$\begin{aligned} \widetilde{\gamma}_w &= \gamma_s \frac{ds}{dw} \\ &= \frac{1}{\left| \left\langle \gamma, \overrightarrow{N} \right\rangle \right|} \gamma_s = \frac{1}{\left| \left\langle \gamma, \overrightarrow{N} \right\rangle \right|} \overrightarrow{T} \end{aligned}$$

and

$$\begin{aligned} \widetilde{\gamma}_{ww} &= \gamma_{ss} \left( \frac{ds}{dw} \right)^2 + \gamma_s \frac{d^2 s}{dw^2} \\ &= \frac{\kappa}{\left\langle \gamma, \overrightarrow{N} \right\rangle^2} \overrightarrow{N} + \text{Tangential component.} \end{aligned}$$

Therefore, the evolution equation associated with this form is

$$\widetilde{\gamma}_\tau = \frac{\kappa}{\left\langle \gamma, \overrightarrow{N} \right\rangle^2} \overrightarrow{N}.$$

This evolution equation is known as the *linear affine heat equation.* It becomes singular if $\gamma$ is in the direction to the tangent to the curve $\gamma$. We will be interested only in cases where this does not happen.

**Definition 2** *A curve $\gamma$ is said to be a simple closed curve if for $t_1 \neq t_2, \gamma(t_1) = \gamma(t_2)$, if and only if $t_1 = a$ and $t_2 = b$. A simple closed curve $\gamma$ is said to be positively oriented if and only if its unit normal $\overrightarrow{N}$ points into int$(\gamma)$.*

**Proposition 3** *If $\gamma$ is positively oriented simple closed curve then the area inside $\gamma$ is given by*

$$A = \frac{1}{2} \int_a^b \gamma \times \gamma_t d\nu. \tag{5}$$

### 1.1.2 Calculus of Curve Evolution

We study in what follows some differential properties of curves evolving under a general equation

$$\frac{\partial \gamma}{\partial \tau} = V\overrightarrow{N}. \tag{6}$$

For ease of notation we will use $(\cdot)'$ to indicate the partial derivative with tespect to space parameter.

**Lemma 4** *For a general parametiization $\gamma$ we have*

1. $\kappa = \frac{1}{|\gamma'|^2}\left\langle \gamma'', \overrightarrow{N} \right\rangle$
2. $\overrightarrow{T}' = \frac{1}{|\gamma'|^2}\left\langle \gamma'', \overrightarrow{N} \right\rangle \overrightarrow{N} = \kappa\,|\gamma'|\,\overrightarrow{N}$
3. $\overrightarrow{N}' = -\kappa\,|\gamma'|\,\overrightarrow{T}$

**Proof.** (1):

$$\begin{aligned}
\kappa &= \frac{\gamma' \times \gamma''}{|\gamma'|^3} = \frac{1}{|\gamma'|^2}\frac{\gamma'}{|\gamma'|} \times \gamma'' \\
&= \frac{1}{|\gamma'|^2}\overrightarrow{T} \times \gamma'' = \frac{1}{|\gamma'|^2}\left\langle R\left(\frac{\pi}{2}\right)\overrightarrow{T}, \gamma'' \right\rangle \\
&= \frac{1}{|\gamma'|^2}\left\langle \overrightarrow{N}, \gamma'' \right\rangle.
\end{aligned}$$

(2):

$$\begin{aligned}
\overrightarrow{T}' &= \left(\frac{\gamma'}{|\gamma'|}\right)' = \frac{|\gamma'|\,\gamma'' - \left\langle \frac{\gamma'}{|\gamma'|}, \gamma'' \right\rangle \gamma'}{|\gamma'|^2} \\
&= \frac{1}{|\gamma'|}\left(\gamma'' - \left\langle \overrightarrow{T}, \gamma'' \right\rangle \overrightarrow{T}\right) \\
&= \frac{1}{|\gamma'|}\left\langle \gamma'', \overrightarrow{N} \right\rangle \overrightarrow{N} = \kappa\,|\gamma'|\,\overrightarrow{N}.
\end{aligned}$$

(3):

$$\begin{aligned}
\overrightarrow{N}' &= \frac{\partial}{\partial t} R\left(\frac{\pi}{2}\right)\overrightarrow{T} = R\left(\frac{\pi}{2}\right)\overrightarrow{T}' \\
&= \kappa\,|\gamma'|\,R\left(\frac{\pi}{2}\right)\overrightarrow{N} = -\kappa\,|\gamma'|\,\overrightarrow{T}
\end{aligned}$$

■

Next we give the time derivative of the various vectors associated with an evolving curve. As ususal, we evolve the curve along its normal direction.

**Lemma 5** *Suppose $\gamma$ is evolving under the general equation (6). Then*

1. $(|\gamma'|)_\tau = -\kappa\,|\gamma'|\,V$
2. $\overrightarrow{T}_\tau = \frac{1}{|\gamma'|}V'\overrightarrow{N}$
3. $\overrightarrow{N}_\tau = -\frac{1}{|\gamma'|}V'\overrightarrow{T}$
4. $(\gamma'')_\tau = \left(V'' - \kappa^2\,|\gamma'|^2\right)\overrightarrow{N} - \left(\kappa\,|\gamma'|\,V' + (\kappa\,|\gamma'|\,V)'\right)\overrightarrow{T}$.
5. $\kappa_\tau = \frac{\left(V''+\kappa^2|\gamma'|^2V\right)-\frac{V}{|\gamma'|^2}\langle\gamma'',\gamma'\rangle}{|\gamma'|^2}$.

**Proof.** Using Lemma 4 we have

(1):

$$\begin{aligned}
(|\gamma'|)_\tau &= \left\langle \frac{\gamma'}{|\gamma'|}, (\gamma')_\tau \right\rangle = \left\langle \overrightarrow{T}, (\gamma_\tau)' \right\rangle \\
&= \left\langle \overrightarrow{T}, \left(V\overrightarrow{N}\right)' \right\rangle = \left\langle \overrightarrow{T}, V'\overrightarrow{N} + V\overrightarrow{N}' \right\rangle \\
&= \left\langle \overrightarrow{T}, V\overrightarrow{N}' \right\rangle = \left\langle \overrightarrow{T}, -\kappa\,|\gamma'|\,V\overrightarrow{T} \right\rangle = -\kappa\,|\gamma'|\,V.
\end{aligned}$$

(2):

$$\begin{aligned}
\overrightarrow{T}_\tau &= \left(\frac{\gamma'}{|\gamma'|}\right)_\tau = \frac{|\gamma'|\,(\gamma')_\tau - (|\gamma'|)_\tau\,\gamma'}{|\gamma'|^2} \\
&= \frac{|\gamma'|\left(V\overrightarrow{N}\right)' + \kappa\,|\gamma'|\,V\gamma'}{|\gamma'|^2} = \frac{V'\overrightarrow{N} + V\overrightarrow{N}' + \kappa V\gamma'}{|\gamma'|} \\
&= \frac{V'\overrightarrow{N} - \kappa\,|\gamma'|\,V\overrightarrow{T} + \kappa V\gamma'}{|\gamma'|} = \frac{1}{|\gamma'|}V'\overrightarrow{N}.
\end{aligned}$$

(3):

$$\begin{aligned}
\overrightarrow{N}_\tau &= \left(R\left(\frac{\pi}{2}\right)\overrightarrow{T}\right)_\tau \\
&= R\left(\frac{\pi}{2}\right)\overrightarrow{T}_\tau \\
&= \frac{1}{|\gamma'|}V'R\left(\frac{\pi}{2}\right)\overrightarrow{N} \\
&= -\frac{1}{|\gamma'|}V'\overrightarrow{T}
\end{aligned}$$

(4):

$$
\begin{aligned}
(\gamma'')_\tau &= (\gamma_\tau)'' = \left(V\overrightarrow{N}\right)'' = \left(V'\overrightarrow{N} + V\overrightarrow{N}'\right)' \\
&= \left(V'\overrightarrow{N} - \kappa\left|\gamma'\right| V\overrightarrow{T}\right)' = V''\overrightarrow{N} + V'\overrightarrow{N}' - \left(\kappa\left|\gamma'\right| V\right)'\overrightarrow{T} - \kappa\left|\gamma'\right| V\overrightarrow{T}' \\
&= V''\overrightarrow{N} - \kappa\left|\gamma'\right| V'\overrightarrow{T} - \left(\kappa\left|\gamma'\right| V\right)'\overrightarrow{T} - \kappa^2\left|\gamma'\right|^2 V\overrightarrow{N} \\
&= \left(V'' - \kappa^2\left|\gamma'\right|^2 V\right)\overrightarrow{N} - \left(\kappa\left|\gamma'\right| V' + \left(\kappa\left|\gamma'\right| V\right)'\right)\overrightarrow{T}.
\end{aligned}
$$

(5):

$$
\begin{aligned}
\kappa_\tau &= \left(\frac{\left\langle \gamma'', \overrightarrow{N}\right\rangle}{\left|\gamma'\right|^2}\right)_\tau \\
&= \frac{\left|\gamma'\right|^2\left(\left\langle \gamma''_\tau, \overrightarrow{N}\right\rangle + \left\langle \gamma'', \overrightarrow{N}_\tau\right\rangle\right) - \left(\left\langle \gamma', \gamma'\right\rangle\right)_\tau \left\langle \gamma'', \overrightarrow{N}\right\rangle}{\left|\gamma'\right|^4} \\
&= \frac{\left|\gamma'\right|^2\left(\left(V'' - \kappa^2\left|\gamma'\right|^2 V\right) + \left\langle \gamma'', -\frac{1}{\left|\gamma'\right|}V'\overrightarrow{T}\right\rangle\right) - 2\left\langle \gamma', \gamma'_\tau\right\rangle \left\langle \gamma'', \overrightarrow{N}\right\rangle}{\left|\gamma'\right|^4} \\
&= \frac{\left|\gamma'\right|^2\left(\left(V'' - \kappa^2\left|\gamma'\right|^2 V\right) + \left\langle \gamma'', -\frac{1}{\left|\gamma'\right|}V'\overrightarrow{T}\right\rangle\right) - 2\left\langle \gamma', V'\overrightarrow{N} + V\overrightarrow{N}'\right\rangle \left\langle \gamma'', \overrightarrow{N}\right\rangle}{\left|\gamma'\right|^4} \\
&= \frac{\left|\gamma'\right|^2\left(\left(V'' - \kappa^2\left|\gamma'\right|^2 V\right) + \left\langle \gamma'', -\frac{1}{\left|\gamma'\right|}V'\overrightarrow{T}\right\rangle\right) - 2\left\langle \gamma', -\kappa\left|\gamma'\right| V\overrightarrow{T}\right\rangle \left\langle \gamma'', \overrightarrow{N}\right\rangle}{\left|\gamma'\right|^4} \\
&= \frac{\left|\gamma'\right|^2\left(\left(V'' - \kappa^2\left|\gamma'\right|^2 V\right) + \left\langle \gamma'', -\frac{1}{\left|\gamma'\right|}V'\overrightarrow{T}\right\rangle\right) + 2\kappa^2\left|\gamma'\right|^4 V}{\left|\gamma'\right|^4} \\
&= \frac{\left(V'' + \kappa^2\left|\gamma'\right|^2 V\right) - \frac{V'}{\left|\gamma'\right|^2}\left\langle \gamma'', \gamma'\right\rangle}{\left|\gamma'\right|^2}
\end{aligned}
$$

■

**Corollary 6** *If $\gamma$ is parametrized by arclength, then*

1. $\overrightarrow{T}_\tau = V'\overrightarrow{N}$
2. $\overrightarrow{N}_\tau = -V'\overrightarrow{T}$
3. $\kappa_\tau = V'' + \kappa^2 V.$

We also recall that the Euclidean arclength is given by

$$L(\tau) = \int_a^b \left|\gamma'\right| d\nu$$

**Lemma 7** *Suppose $\gamma$ is a positively oriented simple closed curve evolving under the general equation (6) then*

1. $L_\tau = -\int_0^L \kappa V ds$
2. $A_\tau = -\int_0^L V ds,$
   *where $L = L(\tau)$ denotes the length of $\gamma$ at time $\tau$.*

**Proof.** (1):

$$\begin{aligned} L_\tau &= \int_a^b |\gamma'|_\tau \, d\nu = -\int_a^b \kappa V |\gamma'| \, d\nu \\ &= -\int_0^L \kappa V ds, \end{aligned}$$

where we used a change of variable to parametrization by arclength.

(2): Using equation (5),

$$\begin{aligned} A_\tau &= \frac{1}{2}\int_a^b (\gamma_\tau \times \gamma' + \gamma \times \gamma'_\tau) \, d\nu \\ &= \frac{1}{2}\int_a^b \gamma_\tau \times \gamma' d\nu + \frac{1}{2}\int_a^b \gamma \times \gamma'_\tau d\nu \\ &= \frac{1}{2}\int_a^b \gamma_\tau \times \gamma' d\nu + \frac{1}{2}\left[\gamma \times \gamma\tau|_a^b - \int_a^b \gamma' \times \gamma_\tau d\nu\right] \\ &= \int_a^b \gamma_\tau \times \gamma' d\nu - \frac{1}{2}\gamma \times \gamma_\tau|_a^b \end{aligned}$$

Now

$$\begin{aligned} \gamma \times \gamma_\tau|_a^b &= \gamma \times V\overrightarrow{N}\Big|_a^b \\ &= \gamma(b,\tau) \times V(\gamma(b,\tau),\tau)\overrightarrow{N}(\gamma(b,\tau)) - \gamma(a,\tau) \times V(\gamma(a,\tau),\tau)\overrightarrow{N}(\gamma(a,\tau)). \end{aligned}$$

Since $\gamma$ is a simple closed curve, $\gamma(b,\tau) = \gamma(b,\tau)$. Hence, $\gamma \times \gamma_\tau|_a^b = 0$. Therefore,

$$\begin{aligned} A_\tau &= \int_a^b \gamma_\tau \times \gamma' d\nu = \int_a^b V\overrightarrow{N} \times |\gamma'| \overrightarrow{T} d\nu \\ &= -\int_a^b V |\gamma'| \, d\nu = -\int_0^L V ds. \end{aligned}$$

■

**Example** Suppose a curve $\gamma$ is evolving under the constant evolution equation (1) (i.e., $V = 1$) such that at $\tau = 0$, $\gamma(s,0)$ is a circle of radius $r$ . Note that the parametrization of $\gamma(s,0)$ by arclength $\gamma : [0, 2\pi r] \to \mathbb{R}^2$ takes the form

$$\gamma(s,0) = \left(r\cos\left(\frac{s}{r}\right), r\sin\left(\frac{s}{r}\right)\right).$$

From this we obtain

$$\begin{aligned}
\overrightarrow{T}(s,0) &= \left(-\sin\left(\frac{s}{r}\right), \cos\left(\frac{s}{r}\right)\right), \\
\overrightarrow{N}(s,0) &= R\left(\frac{\pi}{2}\right)\overrightarrow{T}(s,0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -\sin\left(\frac{s}{r}\right) \\ \cos\left(\frac{s}{r}\right) \end{bmatrix} \\
&= \begin{bmatrix} -\cos\left(\frac{s}{r}\right) \\ -\sin\left(\frac{s}{r}\right) \end{bmatrix} = -\frac{1}{r}\gamma(s,0), \\
\kappa(s,0) &= \gamma_s \times \gamma_{ss} = \begin{bmatrix} -\sin\left(\frac{s}{r}\right) \\ \cos\left(\frac{s}{r}\right) \end{bmatrix} \times \begin{bmatrix} -\frac{1}{r}\cos\left(\frac{s}{r}\right) \\ -\frac{1}{r}\sin\left(\frac{s}{r}\right) \end{bmatrix} = \frac{1}{r}.
\end{aligned}$$

The evolution of the normal vector is governed by the equation

$$\overrightarrow{N}_\tau = -V'\overrightarrow{T} = 0.$$

Therefore,

$$\overrightarrow{N}(s,\tau) = C(s).$$

The constant is evaluated by putting $\tau = 0$, which gives

$$C(s) = \overrightarrow{N}(s,0) = -\frac{1}{r}\gamma(s,0).$$

The evolution of the curve $\gamma$ is governed by the equation

$$\gamma_\tau = \overrightarrow{N}(s,\tau) = \overrightarrow{N}(s,0) = -\frac{1}{r}\gamma(s,0).$$

Therefore,

$$\gamma(s,\tau) = -\frac{\tau}{r}\gamma(s,0) + C(s).$$

At $\tau = 0$,

$$\gamma(s,0) = C(s).$$

Thus

$$\begin{aligned}
\gamma(s,\tau) &= \gamma(s,0)\left(1-\frac{\tau}{r}\right) \\
&= \left(r\cos\left(\frac{s}{r}\right), r\sin\left(\frac{s}{r}\right)\right)\left(1-\frac{\tau}{r}\right) \\
&= .\left(r\left(1-\frac{\tau}{r}\right)\cos\left(\frac{s}{r}\right), r\left(1-\frac{\tau}{r}\right)\sin\left(\frac{s}{r}\right)\right)
\end{aligned}$$

Thus, $\gamma(s,\tau)$ is a circle of radius $r\left(1-\frac{\tau}{r}\right) = (r-\tau)$. This means that the circle collapses to a point at $\tau = r$.

The evolution of the curvature is governed by the equaiton

$$\begin{aligned}
\kappa_\tau &= V'' + \kappa^2 V \\
&= \kappa^2.
\end{aligned}$$

This equation has the solution

$$\begin{aligned}\kappa(s,\tau) &= \frac{\kappa(s,0)}{1-\tau\kappa(s,0)} \\ &= \frac{1/r}{1-\tau/r} = \frac{1}{r-\tau}.\end{aligned}$$

As $\tau \to r$, $\kappa \to \infty$ (the curvature of a point).

The arclength evolution is governed by the differential equation

$$\begin{aligned}L_\tau &= -\int_0^L \kappa V ds \\ &= -\frac{1}{r-\tau}\int_0^L ds \\ &= -\frac{1}{r-\tau}L,\end{aligned}$$

which has the solution

$$L(\tau) = 2\pi(r-\tau).$$

Clearly $L(r) = 0$. The evolution of the area is governed by the differential equation

$$\begin{aligned}A_\tau &= -\int_0^L V ds \\ &= -L = -2\pi(r-\tau),\end{aligned}$$

which has the solution

$$A(\tau) = \pi(r-\tau)^2.$$

Again $A(r) = 0$.

**Example** Suppose $\gamma$ is a simple closed curve evolving under the curvature flow (2) (i.e., $V = \kappa$). The cahnge of curvature, length and area are goverened by the equations

$$\begin{aligned}\kappa_\tau &= \kappa_{ss} + \kappa^3, \\ L_\tau &= -\int_0^L \kappa^2 ds, \\ A_\tau &= -\int_0^L \kappa ds,\end{aligned}$$

respectively. The change in area can be simplified as follows

$$A_\tau = -\int_0^L \frac{d\theta}{ds} ds = -[\theta(L) - \theta(0)] = -2\pi.$$

Hence,

$$A(\tau) = A(0) - 2\pi\tau,$$

which means that the area will vanish after finite time $\tau = A(0)/(2\pi)$.

More information about curvature flows is provided in the following theorem.

**Theorem 8** *(Gage-Hamilton) The curvature flow shrinks a convex curve* $\gamma$ *to a point.* $\gamma$ *becomes circular as it evolves in the sense that*

1. *the ratio of the radii of the inscribed circle to the circomscribed circle becomes,*
2. *the ration of the maximum curvature to the minimum curvature approaches 1, and*
3. *the higher order of the derivatives of the curvature converge uniformly to 0.*

## 1.2 Surface Evolution

An example of surface evolution is

$$\frac{\partial \sigma}{\partial \tau} = H\overrightarrow{N},$$

where $H$ is the mean curvatues of the surface. This flow is called the *mean curvature flow.* The steady state equation for this flow is

$$H\overrightarrow{N} = 0,$$

which is the Euler-Lagrange equation for the surface area

$$A(\sigma) = \int |\sigma_u \times \sigma_v| \, dV_2(\mathbf{u}).$$

The vanishing of the "first derivative" means that the corresponding surface $\sigma$ has minimum area. Therefore, the mean curvature flow aims at minimizing the surface area.

Another example of surface evolution is the equiaffine invariant flow

$$\frac{\partial \sigma}{\partial \tau} = K^{1/4}\overrightarrow{N},$$

which is obtained in a similar way to the affine invariant flow for curves.

### 1.2.1 Surfaces that are Graphs of Functions

The graph of a function $I : U \subset \mathbb{R}^2 \to \mathbb{R}$ of the two variables $(x, y) \in U$ is the subset of $\mathbb{R}^3$ defined by

$$\sigma = \{(x, y, I(x, y)) : (x, y) \in U\}.$$

If the function $I$ is smooth, then $\sigma$ can be considered as a surface in $\mathbb{R}^3$. A special case of such functions is the intensity function of an image. In this case $I(x, y)$ is the gray level of the image corresponding to the point $(x, y)$. We are going to consider a mean curvature flow for such surfaces of the form

$$\sigma_\tau = \frac{H}{\left\langle \overrightarrow{N}, \overrightarrow{k} \right\rangle} \overrightarrow{k}, \tag{7}$$

where $H$ is the mean curvature at any poin on the surface, $\overrightarrow{N}$ is the unit normal to the surface and $\overrightarrow{k}$ is the unit vector in the positive $z$-direction: $\overrightarrow{k} = (0, 0, 1)$. The unit normal $\overrightarrow{N}$ is computed as follows

$$\overrightarrow{N} = \frac{\sigma_x \times \sigma_y}{|\sigma_x \times \sigma_y|} = \frac{1}{|\sigma_x \times \sigma_y|} \begin{bmatrix} 1 \\ 0 \\ I_x \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ I_y \end{bmatrix} = \frac{1}{\sqrt{1 + I_x^2 + I_y^2}} \begin{bmatrix} -I_x \\ -I_y \\ 1 \end{bmatrix}.$$

Therefore,

$$\left\langle \overrightarrow{N}, \overrightarrow{k} \right\rangle = \frac{1}{\sqrt{1 + I_x^2 + I_y^2}}.$$

On the other hand

$$\sigma_\tau = \begin{bmatrix} 0 \\ 0 \\ I_\tau \end{bmatrix}.$$

Thus, equation (7) simplifies to

$$I_\tau = H\sqrt{1 + I_x^2 + I_y^2}.$$

Furthermore,

$$H = \frac{LG - 2MF + NE}{2\left(EG - F^2\right)},$$

where $E, F, G, L, M, N$ are the coeffficients of the first and second fundamental forms. Observe that the above formula for $H$ can be found by computing the trace of the matrix $\mathcal{F}_I^{-1}\mathcal{F}_{II}$. For our case, $E = |\sigma_x|^2 = (1 + I_x^2)$, $F = \langle \sigma_x, \sigma_y \rangle = I_x I_y$, $G = |\sigma_y|^2 = \left(1 + I_y^2\right)$, $L = \left\langle \sigma_{xx}, \overrightarrow{N} \right\rangle = RI_{xx}$, $M = \left\langle \sigma_{xy}, \overrightarrow{N} \right\rangle = RI_{xy}$ and $N = \left\langle \sigma_{yy}, \overrightarrow{N} \right\rangle = RI_{yy}$, where$R = \frac{1}{\sqrt{1+I_x^2+I_y^2}}$ Subsituting these values in the expression for $H$ yeilds

$$H = \frac{I_{xx}\left(1 + I_x^2\right) - 2I_{xy}I_xI_y + I_{yy}\left(1 + I_y^2\right)}{\left(1 + I_x^2 + I_y^2\right)^{3/2}}$$

and the expression for $I_\tau$ becomes

$$I_\tau = \frac{I_{xx}\left(1 + I_x^2\right) - 2I_{xy}I_xI_y + I_{yy}\left(1 + I_y^2\right)}{1 + I_x^2 + I_y^2}$$

with the function itself as the initial condition $I(x, y, 0) = I(x, y)$.