King Fahd University of Petroleum & Minerals

جامعة الملك فهد للبترول و المعادن

CALCULUS OF VARIATIONS

MATH 640

Lecture Notes

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Contents

1 Lecture 1

Convex Sets and their separation

Let V be a vector space, $u, v \in V$. Then

- The line segment between u and v is $[u, v] = \{\lambda u + (1-\lambda)v : \lambda \in [0,1]\}$.
- $A \subseteq V$ is convex iff $A = \{\sum_{i=1}^n \lambda_i u_i : \sum_{i=1}^n \lambda_i = 1, \lambda_i \geq 0, u_i \in A\}$.
- ϕ is convex.
- $A \subseteq V$, $co(A) =$ convex hull of $A =$ smallest convex set containing $A = \{\sum_{i=1}^n \lambda_i u_i : \sum_{i=1}^n \lambda_i = 1, \lambda_i \geq 0, u_i \in A\}$
- V' denotes the set of linear functional on V.
- A hyperplane $H \subseteq V$ is defined by $H = \{u \in V : l(u) = \alpha \text{ for some } l \in V', \alpha \in I\!R\}$. If $l(u) = \alpha$ is replaced by $l(u) < \alpha$ or $l(u) > \alpha$ ($l(u) \geq \alpha$ or $l(u) \leq \alpha$), then we have open (closed) half spaces.

Separation of convex sets

Let V be a topological vector space (tvs) over the reals, $u, v \in V, \alpha \in I\!R$. Here, we have $(u, v) \longrightarrow u + v, (u, \alpha) \longrightarrow \alpha v$ are continuous. V is called locally convex space (lcs) [1] if it has a fundamental sysytem of neighborhoods of zero consisiting of convex sets.

- If $A \subseteq V$ is convex, then so are $\overset{\circ}{A}, \overline{A}$.
- If $u \in \overset{\circ}{A}, v \in \overline{A}$, then $[u, v] \subseteq \overset{\circ}{A}$ and $\overline{\overset{\circ}{A}} = \overline{A}$.

  DEFINITION 1
- (Internal Points) A is convex, a point $u \in A$ is called an internal point of A if every line passing through u intersects A in two distinct points u_1 and u_2 such that $u \in (u_1, u_2)$.
  - Every interior point is internal.
  - If $\overset{\circ}{A} \neq \phi$, then every internal point to A is interior.
- $A \subseteq V, \overline{co}(A) =$ closed convex hull of $A =$ intersection of all closed convex sets containing A.
- In a locally convex space (lcs), a hyperplane H is closed iff its representing functional is continuous.

DEFINITION 2
(Separation of sets by hyperplanes) $A, B \subseteq V$. A hyperplane H is said to (strictly) separates A and B if each one of them is contained in one of the (open) half spaces determined by H.

THEOREM 3 (Hahm-Banach theorem) V is a vs, M is an affine set of V, $\phi \neq A \subseteq V$ convex , there exits a hyperplane H such that $M \subseteq H$ and $A \cap H \neq \phi$.

COROLLARY 4
- If $\phi \neq A \subseteq V$ is open and convex, $\phi \neq B \subseteq V$ is convex. Then there exists a hyperplane that separates A and B.

COROLLARY 5
- C(convex) , $B \subseteq V$ (lcs), $C \cap B = \phi, C \neq \phi B \neq \phi$ and B is compact. Then there exists a hyperplane H which strictly separates A and B.

DEFINITION 6
(Supporting hyperplanes) $A \subseteq V, u \in A$. If there exists H such that A lies on one side of H and $u \in H$, then u is called a supporting hyperplane of A at u and u is called the supporting point.

[1] A normed space is lcs

COROLLARY 7

- If $A \subseteq V$ (tvs), $\mathring{A} \neq \phi$ is convex. Then every point in the boundary of A is a supporting point.

COROLLARY 8

- If V (lcs), $M \subseteq V$ is closed and convex. Then M is the intersection of all closed hyperlanes containing it. boundary of A is a supporting point.

$\sigma(V, V')$ is called the weakest topology. V is a T_2 locally convex space in this topology. $\sigma(V, V')$ is the weakest topology in which V is T_2 locally convex. In a locally convex space, every closed convex set is also weakly closed.

2 Lecture 2

Convex Functions:

Definition:(convex function)
Let V be a real vector space, and let $A \subseteq V$ be convex. Then $F : A \to \overline{R} = [-\infty, \infty]$, is said to be convex iff $\forall \lambda \in [0,1]$, $u, v \in A$, $F(\lambda u + (1-\lambda)v) \leq \lambda F(u) + (1-\lambda)F(v)$
whenever the r.h.s. is defined.

Definition:(effective domain)
The effective domain of F is defined as $dom\ F = \{u \in A : F(u) < \infty\}$

Definition:(Indicator function)
If $A \subseteq V$, then $\chi_A(u) = \{^{0 \ \text{ if } u \in A}_{\infty \ \text{ if } u \notin A}$ is called the indicator function of a set A.

Definition:(Extension function)
If F is defined on $A \subseteq V \to R$, then $\widetilde{F} : V \to \overline{R}$ which is given by
$\widetilde{F} = \{^{F(u) \ \text{ if } u \in A}_{\infty \ \text{ if } u \notin A}$ is an extension of F on $\overline{R}$

Exercises:
i) Prove that if F is convex, then $S_a = \{u \in A : F(u) \leq a\}$ and $S_{\overline{a}} = \{u \in A : F(u) < \overline{a}\}$ are convex, where $a \in R$.
ii) Prove that if F is convex, then $dom\ F$ is too.
iii) Prove that if F is convex, then $\widetilde{F}$ is too.

iv) Theorem χ_A is convex iff A is convex.

Theorem:
If F is convex and $F(\overline{u}) = -\infty$ for some $\overline{u} \in V$, then on any half line starting from $\overline{u}$, either
$F(v) = -\infty \ \forall\, v \in [\overline{u}, \infty)$ or
$\exists\, v \in (\overline{u}, \infty)$ such that $| F(v) | < \infty$ and
$F(w) = \{^{-\infty \ \ w \in [\overline{u}, v)}_{\infty \ \ w \in (v, \infty)}$

Proof:

Assume $\exists\, v \in (u, \infty)$ such that $| F(v) | < \infty$.
Let $w \in [\overline{u}, v)$, then $w = \lambda v + (1-\lambda)\overline{u}$ where $\lambda \in [0,1)$ and
$F(w) \leq \lambda F(v) + (1-\lambda)F(\overline{u}) = -\infty$
For $w \in (v, \infty)$ we have $v = (1-\lambda)\overline{u} + \lambda w$ $\lambda \in (0,1)$ and
$F(v) \leq \lambda F(w) + (1-\lambda)F(\overline{u})$ assume here that $| F(w) | < \infty$
so $F(w) \geq \frac{1}{\lambda}[F(v) - (1-\lambda)F(\overline{u})] = \infty$ (contradiction!)(Try to consider different cases).

Definition:(proper function)
A function $F : V \to \overline{R}$ is called proper if $-\infty \notin dom\ F$ i.e. $F(u) > -\infty \ \forall\, u \in V$.

Definition:(epigraph of a function)

Let $F : V \to \overline{R}$ be a function. The epigraph of is given by:
$epi\ F = \{(v, a) \in V \times R : F(v) \leq a\}$
Note that the projection of epi onto V is $dom\ F$.
If $(v, a) \in epi\ F$, then $F(v) \in a < \infty$ i.e. $v \in dom\ F$.
If $v \in dom\ F$, then $(v, F(v)) \in epi\ F$.

Proposition:
Let $F : V \to \overline{R}$ be a function. Then
i) F is convex iff $epi\ F$ is convex.

Lecture 2

ii) F is convex, $\lambda > 0 \Rightarrow \lambda F$ is convex.
iii) F, G are convex $\Rightarrow F + G$ are convex (provided that $\infty - \infty = \infty$).
iv) $(F_i)_{i \in I}$,$F(u) = \sup F_i(u) \Rightarrow F$ is convex where each F_i is convex.

Definition:(lower semicontinuous functions)
Let $F : V \to \overline{R}$ be a function. Then it is called l.s.c. if $\underline{lim}_{u \to \overline{u}} F(u) \geq F(\overline{u}) \;\; \forall\, u \in V$.

end of Lec# 2

3 Lecture 3

Recall that a function $F : V \longrightarrow \overline{I\!R}$ is lower semicontinuous if

$$\liminf_{u\to\bar{u}} F(u) \geq F(\bar{u})$$

LEMMA 9
F is lsc iff $S_a = \{u \in V : F(u) \leq a\}$ is closed in V.

Proof. The necessary condition was done in the previous lecture. For sufficient condition, suppose that S_a for all $a \in I\!R$ and let $\bar{u} \in V$ and $a = \liminf_{u\to\bar{u}} F(u)$.
Case 1: If $a = \infty$ then nothing to prove.
Case 2: If a is finite ($\|a\| < \infty$), take a sequence $\{u_n\}$ such that $u_n \to \bar{u}$. For each k, we can find n_k such that

$$F(u_{n_k}) \leq a + \frac{1}{k}$$

these $u_{n_k} \in S_{a+\frac{1}{k}}$ and we have

$$u_{n_k} \in \bigcap_{i=1}^{k} S_{a+\frac{1}{i}}$$

and since $\bigcap_{i=1}^{k} S_{a+\frac{1}{i}}$ is closed and $u_{n_k} \to \bar{u}$ then

$$u \in \bigcap_{i=1}^{\infty} S_{a+\frac{1}{i}} = S_a$$

$\therefore F(\bar{u}) \leq a = \liminf_{u\to\bar{u}} F(u)$
Case 3: If $a = -\infty$ consider $S_n = \{u \in V : F(u) \leq -n\}$. ■

PROPOSITION 10
F is lsc iff $epiF$ is closed.

Proof. Let $\phi : V \times I\!R \longrightarrow \overline{I\!R}$ be defined by

$$\phi(u, a) = F(u) - a$$

Now, let $(u_n, a_n) \to (u, a)$; that is $u_n \to u$ and $a_n \to a$. Then

$$\liminf \phi(u_n, a_n) = \liminf F(u_n) - a_n \geq F(u) - a = \phi(u, a)$$

So $\phi(u, a)$ is lsc, then by previous lemma the set $\{(u, a) : \phi(u, a) \leq \alpha\}$ is closed for each $\alpha \in \overline{I\!R}$. In particular, if $\alpha = 0$, the set

$$\{(u, a) : \phi(u, a) \leq 0\} = \{(u, a) : F(u) \leq a\}$$

is closed; which is the epigraph of f.
Now suppose that $epiF$ is closed. Then the set

$$\{(u, a) \in V \times I\!R : \phi(u, a) \leq r\} = \{(u, a) \in V \times I\!R : F(u) \leq a + r\} = \{(u, a) \in V \times I\!R : F(u) \leq a\} - \{(0, r) : r \in I\!R\}$$

is closed. Therefore, ϕ is lsc. It remains to show that F is lsc if ϕ is. For this let $\{u_n\}$ be a sequence such that $u_n \to \bar{u}$ and consider

$$\liminf_{u_n\to\bar{u}} F(u_n) - \quad a = \liminf F(u_n) - \liminf a = \liminf (F(u_n) - a) = \liminf \phi(u_n, a) \geq \phi(u, a) = F(u) - a$$

Therefore F is lsc. ■

LEMMA 11
If $(F_i)_{i\in I}$ is a family of lsc functions, then $F(u) = \sup_{i\in I} F_i(u)$ is as well lsc.

Proof. Claim: $epiF = \bigcap epiF_i$. To show this,

$$\begin{aligned} \text{let } (u,a) \in epiF &\Leftrightarrow F(u) \le a \\ &\Leftrightarrow \sup F_i(u) \le a \\ &\Leftrightarrow F_i(u) \le a \qquad \forall i \\ &\Leftrightarrow (u,a) \in epiF_i \qquad \forall i \\ &\Leftrightarrow (u,a) \in \bigcap epiF_i \end{aligned}$$

So F is lsc. ■

DEFINITION 12
A function $\bar{F}$ is called the lsc regularization of F if it is the greatest lsc minorant of F (i.e. $\bar{F}(u) \le F(u)$ for all $u \in V$).

THEOREM 13 If $F : V \longrightarrow \overline{I\!R}$. Then

(a) $epi\bar{F} = \overline{epiF}$.

(b) $\bar{F}(u) = \liminf F(u)$.

Proof.

(a) Read the book.

(b) Let $(u,a) \in epi\bar{F}$, then $(u,a) \in \overline{epiF}$ and there exists a sequence (u_n, a_n) in $epiF$ such that $(u_n, a_n) \leftarrow (u,a)$. Now for each n we have $\bar{F}(u_n) \le F(u_n) \le a_n$ and

$$\bar{F}(u) \le \liminf \bar{F}(u_n) \le \liminf F(u_n) \le \liminf a_n = a = \bar{F}(u)$$

Therefore $\bar{F}(u) = \liminf F(u)$ as desired. ■

COROLLARY 14
The function $F : V \longrightarrow \overline{I\!R}$ is lsc and convex iff F is weakly lsc and convex.

Proof.
F is lsc and convex$\Leftrightarrow epiF$ is convex and closed
$\Leftrightarrow epiF$ is the intersection of all half spaces containing it.
$\Leftrightarrow epiF$ is weakly lsc and convex.
$\Leftrightarrow F$ is weakly lsc and convex.
which concludes the proof. ■

PROPOSITION 15
If $F : V \longrightarrow \overline{I\!R}$ is lsc and convex and $F(\bar{u}) = -\infty$ for some $\bar{u} \in V$, then F can not take any finite value.

Proof. Assume $|F(u)| < \infty$. Let $u_n = \alpha_n \bar{u} + (1-\alpha_n)u, \alpha_n \leftarrow 0$ then

$$F(u_n) = F(\alpha_n \bar{u} + (1-\alpha_n)u) \le \alpha_n F(\bar{u}) + (1-\alpha_n)F(u) = -\infty$$

which is a contradiction. ■

4 Lecture 4

Continuity of Convex Function

PROPOSITION 16
$F : V \longrightarrow \bar{R}$,If F is Convex and bounded above in a nbhd of a point $u \in V$, then F is continuous at u.

Proof. Assume $u = 0$, and $F(0) = 0$, let W be nbhd of 0 and F is bounded by $a < \infty$ on W. Let $W_1 = W \cap -W$, let $\epsilon > 0$ be given, let $v \in \epsilon W_1$.

$$F(v) = F(\frac{\epsilon v}{\epsilon}) \leq \epsilon F(\frac{v}{\epsilon}) \leq \epsilon a,$$

also,$-v \in W_1$

$$0 = \frac{1}{2}v - \frac{1}{2}v$$

$$0 = F(0) \leq \frac{1}{2}F(v) + \frac{1}{2}F(-v) \Longrightarrow$$

$$-F(v) \leq F(-v) = F(\epsilon\frac{-v}{\epsilon}) \leq \epsilon F(\frac{-v}{\epsilon}) \leq \epsilon a,$$

then

$$|F(v)| \leq \epsilon a \quad \Longrightarrow F \text{ is continuous at } 0.$$

■

PROPOSITION 17
Let $F : V \to \bar{R}$ be a convex function, TFAE

(i) $\exists$ an open, non-empty $O \subseteq V$, s.t. F is bounded above (by $a < \infty$) on V and $F(O) \neq \{-\infty\}$.

(ii) $\overset{o}{\widehat{domF}} \neq \phi$, F is continuous and proper on $\overset{o}{\widehat{domF}}$.

Proof. Clearly (ii)$\Longrightarrow$(i). Conversely for (i)$\Longrightarrow$(ii), $\overset{o}{\widehat{domF}} \neq \phi$ since $O \subseteq domF$.Let $u \in \overset{o}{\widehat{domF}}$ and choose $v \in O$ s.t. $|F(v)| < \infty$. since u is a internal point of the convex set $\overset{o}{\widehat{domF}}$, there exists a $w_1 \in \overset{o}{\widehat{domF}}$ s.t. $u \in (w_1, v)$

$$u = \alpha w_1 + (1-\alpha)v \in \alpha w_1 + (1-\alpha)O,$$

let $z \in \alpha w_1 + (1-\alpha)O$

$$z = \alpha w_1 + (1-\alpha)z_2 \text{ where } z_2 \in O,$$

$$F(z) \leq \alpha F(w_1) + (1-\alpha)F(z_2) \leq \alpha F(w_1) + (1-\alpha)a$$

therefore F is bounded above on the open nbhd $\alpha w_1 + (1-\alpha)O$ of u. Then F is continuous at u. ■

COROLLARY 18
$F : V \to R$ convex, V is finite dimension, then F is continuous on $\overset{o}{\widehat{domF}}$.

Proof. If $\overset{o}{\widehat{domF}} \neq \phi$, then $\overset{o}{\widehat{domF}}$ contains an interior point. $\overset{o}{\widehat{domF}}$ contains $(n+1)$ affinely independent vectors $(u_1, u_2, .., u_{n+1})$. For $u \in \overset{o}{\widehat{domF}}$, there exists an open set of the form $I_1 \times I_2 \times ...I_n$, u can be written as $u = \sum_{i=1}^{n+1} \lambda_i u_i$ s.t. $0 \leq \lambda_i \leq 1$ and $\sum_{i=1}^{n+1} \lambda_i = 1$, then

$$F(u) \leq \sum_{i=1}^{n+1} \lambda_i F(u_i) \leq \sum_{i=1}^{n+1} F(u_i),$$

therefor F is bounded above on a nbhd of u. ■

COROLLARY 19
Let V be a normed space, $F : V \to \bar{R}$ is a a proper convex function. TFAE:

(i) $\exists$ an open set $O \subseteq V$ on which F is bounded in O.

(ii) $\overset{o}{\widehat{domF}} \neq \phi$, and F is locally Lipschitz on $\overset{o}{\widehat{domF}}$.

5 Lecture 5

Theorem:
Let V be a real vector space and let $F : V \to \overline{R}$.Then the following are equivalent:

(1) $\exists\, \emptyset \neq O \subseteq V$ such that F is bounded above in O.
(2)$\overset{\circ}{\widehat{dom}}\, F \neq \emptyset$, F is locally Lipschitz on $\overset{\circ}{\widehat{dom}}\, F$

Proof:
$(1) \Rightarrow (2)$
Let $u \in \overset{\circ}{\widehat{dom}}\, F$. Then, F is continuous at u. So F is absolutely bouned (by a) in a ball $\overline{B(u,r)}$, $r > 0$. Let $v \in B(u,r)$. Write $v = (1-\lambda)u + \lambda w_1$

$$
\begin{aligned}
\Rightarrow \quad & v - u = \lambda(w_1 - u) \\
\Rightarrow \quad & \|v - u\| = \lambda r \\
\Rightarrow \quad & F(v) - F(u) = F((1-\lambda)u + \lambda w_1) - F(u) \\
& \leq (1-\lambda)F(u) + \lambda F(w_1) - F(u) \\
& = \lambda(F(w_1) - F(u)) < 2a\tfrac{\|u-v\|}{r}
\end{aligned}
$$

Now if $u = (1-\overline{\lambda})v + \overline{\lambda} w_2$

$$
\begin{aligned}
& \Rightarrow \quad u - v = \overline{\lambda}(w_2 - v) \quad \Rightarrow \overline{\lambda} = \tfrac{\|u-v\|}{r+\|u-v\|} \\
& F(u) - F(v) \leq 2a\overline{\lambda} = 2a\tfrac{\|u-v\|}{r+\|u-v\|} \leq \tfrac{2a}{r}\|u-v\| \quad \Rightarrow \\
& \mid F(u) - F(v) \mid \leq \tfrac{2a}{r}\|u-v\|
\end{aligned}
$$

For any $v \in \overset{\circ}{\widehat{dom}}\, F$ cover $[u,v]$ by a finite set $B(u_i, r_i)$, $i = 1, 2, ..., n$ for which
$u_1 = u, \quad u_n = v$ and $u_{i+1} \in B(u_i, r_i)$. Then,

$$
\begin{aligned}
\mid F(u) - F(v) \mid & \leq \sum_{i=1}^{n-1} \mid F(u_{i+1}) - F(u_i) \mid \\
& \leq \sum_{i=1}^{n-1} \tfrac{2a_i}{r_i}\|u_{i+1} - u_i\| \\
& \leq \sum_{i=1}^{n-1} \tfrac{2a_i}{r_i} c_i \|u - v\|
\end{aligned}
$$

where $c_i = \frac{\|u_{i+1} - u_i\|}{\|u-v\|}$

Definition:(Cafs)
A caf is the pointwise (pw) supermum of a continuous affine fanuctionals.

Definition:($\Gamma(V)$)
$\Gamma(V)$ is the set of funtions $F : V \to \overline{R}$ which are the pw superma of families of cafs.

Note:
(1) ∞ and $-\infty \in \Gamma(V)$
(2) $\Gamma_\circ(V) = \Gamma(V) \backslash \{-\infty, \infty\}$.
(3) $F \in \Gamma(V) \Rightarrow F$ is convex and l.s.c.

Proposition:
The following are equivalent:

(i) $F \in \Gamma(V)$
(ii) F is convex and l.s.c. and if F assumes the value of $-\infty$, then $F \equiv -\infty$

Proof:
(ii)⇒(i)
Suppose that F is convex and l.s.c. If $F \equiv -\infty,\ F \in \Gamma(V)$ and if $F \equiv \infty,\ F \in \Gamma(V)$.
If F is proper and (F is not $\equiv \infty$). Let $u \in V$. Then we have two cases:

Case(1): $F(u) < \infty$
Let $\overline{a} < F(u)$. Then $\exists$ a hyperplane $H : L(v) + \alpha a + \beta = 0 \ \ \forall\, v \in V$ that strictly separate $epi\ F$ and $(u, \overline{a})$. i.e.

$$L(v) + \alpha a + \beta > 0 \ \ \forall \ (v, a) \in epi\ F \quad \text{and}$$
$$L(v) + \alpha\overline{a} + \beta < 0$$

Claim that $\alpha > 0$.

For $(u, F(u)) \in epi\ F$ we have $L(u) + \alpha F(u) + \beta > 0$
and $-L(u) - \alpha\overline{a} - \beta > 0$
$\Rightarrow \alpha(F(u) - \overline{a}) > 0 \quad \Rightarrow \alpha > 0$

So, $F(v) > -\frac{1}{\alpha}(L(v) + \beta) \ \ \forall\, v \in V$
$\Rightarrow \quad \overline{a} < -\frac{1}{\alpha}(L(u) + \beta) < F(u)$

Case (2): $F(u) = \infty$
This means that $\exists$ a hyperplane $H : L(u) + \alpha a + \beta = 0$ that strictly separate $epi\ F$ and $(u, \overline{a})$.If $\alpha \neq 0$, we are back to case (1).
If $\alpha = 0$, then $H : L(u) + \beta = 0$
and $L(u) + \beta < 0$ if we substitute with $(u, \overline{a})$.
From case(1) we can find a caf minorant $m(v) + \gamma$

$$F(v) \geq m(v) + \gamma \quad \forall\, v \in V$$
$$\therefore \ F(v) \geq m(v) + \gamma - c(L(v) + \beta) \quad \forall\, c \geq 0$$

We want to choose c such that

$$m(u) + \gamma - c(L(u) + \beta) > \overline{a}$$
$$c > \frac{\overline{a} - m(u) - \gamma}{-(L(u) + \beta)}$$
$$\Rightarrow F \in \Gamma(V)$$

end of Lec# 5

6 Lecture 6

$\Gamma-$reqularization

Definition: Let $F : V \longrightarrow \bar{\mathbb{R}}$, a function $G \in \Gamma(V)$ is called the Γregularizer of F if G is the pointwise supremum of all caf minorant of F.
Remark:s
* If G is the $\Gamma - reg\ F$, then G is lower semicontinuous and convex.
* note that $G = \Gamma - reg\ F$ iff G is the greatest minorant in $\Gamma(V)$ of F.
* G is a minorant and if $\widetilde{G} \in \Gamma(V)$ is a minorant of F, then $G \geqslant \widetilde{G}$. on the other hand, suppose that G is the greatest minorant of F in
$\Gamma(V)$.Let G_1 be the $\Gamma - reg\ F \Longrightarrow G_1 \geqslant G$, but by hypothesis $G \geqslant G_1 \Longrightarrow G = G_1$.

Proposition: Let $F : V \longrightarrow \bar{\mathbb{R}}$, F has a caf minorant, $G = \Gamma - reg\ F \Longrightarrow epiG = \overline{co}\ epiF$
Proof:
$epiG \supseteq \overline{co}\ epiF$. on the other hand, assume $\left(\overline{\overline{v}},\ \overline{a}\ \right) \notin \overline{co}\ epiF$ $\Longrightarrow$there exists a caf $l(u) + \alpha a + \beta$ strictly separating $\left(\overline{\overline{v}},\ \overline{a}\ \right)$ and $\overline{co}\ epiF$.

$\therefore l(\overline{\overline{v}})+\alpha\overline{a}+\beta < 0andl(v)+\alpha a+\beta > 0 \forall (v,a) \in \overline{co}epiF.\left(\overline{\overline{v}},\ F(\ \overline{\overline{v}})\right) \in epiF \subseteq \overline{co}epiF \Longrightarrow l(\overline{\overline{v}})+\alpha F(\ \overline{\overline{v}})+\beta > 0and-l(\overline{\overline{v}})-\alpha\overline{a}-\beta >$

adding these inequalities

$$\Longrightarrow \alpha(F(\ \overline{\overline{v}}) - \overline{a}) > 0 \Longrightarrow \alpha > 0$$

for

$(v, F(v) \in epiF \Longrightarrow l(v)+\alpha F(v)+\beta > 0 \Longrightarrow F(v) > \dfrac{-1}{\alpha}(l(v)+\beta) \Longrightarrow G(v) > \dfrac{-1}{\alpha}(l(v)+\beta)\ \forall\ v \in domF \Longrightarrow G(\overline{v}) > \dfrac{-1}{\alpha}(l(\overline{v})+\beta$

remark:
$F : V \longrightarrow \bar{\mathbb{R}}$, $\overline{F} = lsc\ reqF$ and $G = \Gamma - reg\ F \Longrightarrow$
1) $G \leq \overline{F} \leq F$
2) if F is convex, with one caf minorant, then $G = \overline{F}$. indeed;

$$F \text{ is convex} \Longrightarrow \overline{F} \text{ convex },\ epi\ \overline{F} = \overline{epiF},\ \overline{F} \in \Gamma(V) \Longrightarrow \overline{F} \leq G \Longrightarrow G = \overline{F}$$

1.4 polar Functions

Let $F : V \longrightarrow \bar{\mathbb{R}}$,suppose $\left\langle u, \overset{*}{u} \right\rangle - \alpha$ is a minorant of F. the polar function $\overset{*}{F} : \overset{*}{V} \longrightarrow \bar{\mathbb{R}}$ is defined by

$$\overset{*}{F}(\overset{*}{u}) = \sup_{u \in V} \left\langle u, \overset{*}{u} \right\rangle - F(u)$$

7 Lecture 7

Polar Functions

Let $F : V \longrightarrow \bar{\mathbb{R}}$,suppose $\left\langle u, \overset{*}{u} \right\rangle - \alpha$ is a minorant of F. the polar function $\overset{*}{F} : \overset{*}{V} \longrightarrow \bar{\mathbb{R}}$ is defined by

$$\overset{*}{F}(\overset{*}{u}) = \sup_{u \in V} \left\langle u, \overset{*}{u} \right\rangle - F(u)$$

Note that for any caf minorant $\left\langle u, \overset{*}{u} \right\rangle - \alpha$ of F we have

$$\left\langle u, \overset{*}{u} \right\rangle - \alpha \leq \left\langle u, \overset{*}{u} \right\rangle - \overset{*}{F}(\overset{*}{u})$$

Properties of the polar function
1) $\overset{*}{F}(0) = -\inf_{u \in V} F(u)$
2) $F \leq G \Longrightarrow \overset{*}{G} \leq \overset{*}{F}$
3) $\left(\inf_{i \in I} F_i \right)^* = \sup_{i \in I} \overset{*}{F_i}$
4) $\left(\sup_{i \in I} F_i \right)^* \leq \inf_{i \in I} \overset{*}{F_i}$
5) $(\lambda F)^* (u^*) = \lambda \, F^* \left(\frac{u^*}{\lambda} \right), \lambda > 0$
6) $(F + a)^* = F^* - a$
7) $(F_a)^* (u^*) = \overset{*}{F}(\overset{*}{u}) + \langle a, u^* \rangle$ where $F_a(u) = F(u - a)$

*Bipolar Function
The bipolar function is defined by

$$\overset{**}{F}(u) = \sup_{\overset{*}{u} \in \overset{*}{V}} \left\langle u, \overset{*}{u} \right\rangle - \overset{*}{F}(\overset{*}{u})$$

Remarks:

1) $\overset{**}{F} \in \Gamma(V)$
2) $\overset{**}{F} = \Gamma - reg \; F$

Proof of (2)
Step 1: we show it is a minorant of F

$$\overset{*}{F}(\overset{*}{u}) \geqslant \left\langle u, \overset{*}{u} \right\rangle - F(u) \Longrightarrow \left\langle u, \overset{*}{u} \right\rangle - \overset{*}{F}(\overset{*}{u}) \leq F(u) \Longrightarrow \sup_{\overset{*}{u} \in \overset{*}{V}} \left\langle u, \overset{*}{u} \right\rangle - \overset{*}{F}(\overset{*}{u}) \leq F(u)$$
$$\Longrightarrow \overset{**}{F}(u) \leq F(u) \Longrightarrow \overset{**}{F} \text{ is a minorant of } F \;\therefore \overset{**}{F} \leq \Gamma - regF$$

on the other hand

$$\sup_{\overset{*}{u} \in \overset{*}{V}} \sup_{\alpha} \left\langle u, \overset{*}{u} \right\rangle - \alpha \leq \left\langle u, \overset{*}{u} \right\rangle - \overset{*}{F}(\overset{*}{u}) \Longrightarrow \sup_{\overset{*}{u} \in \overset{*}{V}} \sup_{\alpha} \left\langle u, \overset{*}{u} \right\rangle - \alpha \leq \left\langle u, \overset{*}{u} \right\rangle - \overset{*}{F}(\overset{*}{u}) \Longrightarrow \Gamma - regF \leq \overset{**}{F}$$

hence, $\overset{**}{F} = \Gamma - regF$.

Cor 1: If $F \in \Gamma(V) \Longrightarrow \overset{**}{F} = F$
Cor2: $\overset{***}{F} = \overset{*}{F}$

EFS(1): copute $(\ln x)^*$

answer: $(\ln x)^* = \infty$
EFS(2): If $F(x) = \left|x^2 - 1\right|$,compute $\overset{**}{F}$.
answer: $\overset{**}{F}(x) = \begin{cases} x^2 - 1 & \text{if} \quad |x| \geqslant 1 \\ 0 & \text{if} \quad |x| \leq 1 \end{cases}$
Remark: the mapping $F \longrightarrow \overset{*}{F}$ is a bijection between $\Gamma(V)$ and $\Gamma(V^*)$indeed;
Define $T : \Gamma(V) \longrightarrow \Gamma(V^*)$ by $TF = \overset{*}{F}$
1)T is one-to-one: assume $TF = TG$ for $F, G \in \Gamma(V) \Longrightarrow \overset{*}{F} = \overset{*}{G} \Leftrightarrow \overset{**}{F} = \overset{**}{G} \Leftrightarrow F = G$
2) T is on to.indeed; suppose $G \in \Gamma(\overset{*}{V}) \Longrightarrow \overset{*}{G} \in \Gamma(V)$ and $T\overset{*}{G} = \overset{**}{G} = G$

Dual Functions
two functions $F \in \Gamma(V)$ and $G \in \Gamma(\overset{*}{V})$ are called induality if $\overset{*}{F} = G$ and $\overset{*}{G} = F$

* $\infty \in \Gamma(V)$ is dual with $-\infty \in \Gamma(V^*)$.
** $\pm\infty$ are dual with $\mp\infty$
*** the mapping $F \longmapsto \overset{*}{F}$ is a bijection between $\Gamma_0(V)$ and $\Gamma_0(\overset{*}{V})$
Examples:
1) $\chi_A(u) = \begin{cases} 0 & \text{if } u \in A \\ \infty & \text{if } u \notin A \end{cases}$

$$\overset{*}{\chi}_A(\overset{*}{u}) = \sup_{u \in V} \left\langle u, \overset{*}{u} \right\rangle - \chi_A(u) = \sup_{u \in A} \left\langle u, \overset{*}{u} \right\rangle$$

*$\overset{*}{\chi}_A(\overset{*}{u})$ is lwoer semicontinuous, convex and positive homogeneous
** $\overset{*}{\chi}_A$is called the support function of A

EFS(3): show that $\overset{**}{\chi}_A = \chi_{\overline{A}}$

2) let $\Phi \in \Gamma_0(\mathbb{R})$ be an even function and let $\overset{*}{\Phi} \in \Gamma_0(\mathbb{R})$ be the polar function of Φ. Let V be a normed space, Define $F \in \Gamma(V)$ and $G \in \Gamma(\overset{*}{V})$ by

$$F(u) = \Phi\left(\|u\|\right) \text{ and } G(u) = \overset{*}{\Phi}\left(\left\|\overset{*}{u}\right\|\right) \text{, then } F \text{ and } G \text{ are dual.}$$

indeed;

$$\begin{aligned} \overset{*}{F}(\overset{*}{u}) &= \sup_{u \in V} \left\langle u, \overset{*}{u} \right\rangle - \Phi\left(\|u\|\right) = \sup_{t \in [0,\infty)} \left\langle u, \overset{*}{u} \right\rangle - \Phi\left(t\right) = \sup_{t \in [0,\infty)} \sup_{\|v\|=1} t \left\langle v, \overset{*}{u} \right\rangle - \Phi\left(t\right) \\ &= \sup_{t \in [0,\infty)} t \left\|\overset{*}{u}\right\| - \Phi\left(t\right) = \sup_{t \in \mathbb{R}} t \left\|\overset{*}{u}\right\| - \Phi\left(t\right) = \overset{*}{\Phi}\left(\left\|\overset{*}{u}\right\|\right) = G(u) \Longrightarrow G \in \Gamma(\overset{*}{V}) \end{aligned}$$

Similarly, $\overset{*}{G}(u) = F(u)$.

3) let $\Phi(x) = \frac{1}{p}|x|^p$ and $\overset{*}{\Phi}(t) = \frac{1}{q}|x|^q$ where $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

$$\overset{*}{\Phi}(t) = \sup_{x \in \mathbb{R}} tx - \frac{1}{p}|x|^p$$

let $f(x) = x - \frac{1}{p}|x|^p$.
Case1: $x = 0 \Longrightarrow f_m = 0$

Case2: $x \neq 0$

Lecture 7

$$f^{/}(x) = t - |x|^{p-1}\,\frac{x}{|x|} = 0 \Longrightarrow t = x\,|x|^{p-2} \Longrightarrow f_m = x^2\,|x|^{p-2} - \frac{1}{p}\,|x|^p = |x|^p\left(1 - \frac{1}{p}\right) = \frac{1}{q}\,|x|^p$$

$$|t| = |x|^{p-1} \Longrightarrow |t|^{\frac{p}{p-1}} = |x|^p \Longrightarrow |t|^q = |x|^p \Longrightarrow f_m = \frac{1}{q}\,|t|^q$$

8 Lecture 8

Subdifferentiability

The function $F : V \longrightarrow \overline{I\!R}$ is called subdifferentiable at $u \in V$ if there exists a $u^* \in V^*$ such that $\forall v \in V, < u - v, u^* > +F(u)$ is a caf minorant of F. The set of all subgradients (may be empty) at u is denoted by $\partial F(u)$.

PROPOSITION 20
$u^* \in \partial F(u)$ iff $F(u) + F^*(u^*) = \langle u, u^* \rangle$.

Proof. If $u^* \in \partial F(u)$

$$\begin{aligned} \langle v - u, u^* \rangle &\leq F(v) \\ -\langle u, u^* \rangle + F(u) &\leq -\langle v, u^* \rangle + F(v) \\ \langle u, u^* \rangle - F(u) &\geq \langle v, u^* \rangle - F(v) \end{aligned}$$

Taking the supremum over all $v \in V$, we get

$$F^*(u^*) \geq \langle u, u^* \rangle - F(u) \geq \sup_{v \in V} \langle v, u^* \rangle - F(v) \geq F^*(u^*)$$

This shows that $F(u) + F^*(u^*) = \langle u, u^* \rangle$.
Now if $F(u) + F^*(u^*) = \langle u, u^* \rangle$, then $F(u) + \langle v, u^* \rangle - F(v) \leq F(u) + F^*(u^*) = \langle u, u^* \rangle$. Hence

$$F(v) \geq \langle v - u, u^* \rangle + F(u)$$

Which implies that $u^* \in \partial F(u)$. ■

PROPOSITION 21
$u^* \in \partial F(u)$, then $F^{**}(u) = F(u)$ and $u^* \in \partial F^{**}(u)$.

Proof. If $u^* \in \partial F(u)$, then $\langle v - u, u^* \rangle + F(u) \leq F^{**}(u) \leq F(v)$ (because from previous proposition, we have $F(u) + F^*(u^*) = \langle u, u^* \rangle$ also $\langle v - u, u^* \rangle + F(u) \leq F(v)$. Now

$$\langle v - u, u^* \rangle + F(u) = \langle v, u^* \rangle - \langle u, u^* \rangle + F(u) = \langle v, u^* \rangle + F^*(u^*) \leq F^{**}(u)$$

So $\langle v - u, u^* \rangle + F(u) \leq F^{**}(u) \leq F(v))$
From this we conclude that $u^* \in \partial F^{**}(u)$. Furthermore, at $v = u$, we have

$$F(u) \leq F^{**}(u) \leq F(u) \Rightarrow F^{**}(u) = F(u)$$

Now if $F^{**}(u) = F(u)$, then $\partial F(u) = \partial F^{**}(u)$.

$$\begin{aligned} \partial F(u) &= \{u^* \in V^* : F(u) + F^*(u) =< u, u^* >\} \\ &= \{u^* \in V^* : F(u) + F^*(u) \leq< u, u^* >\} \\ &= \{u^* \in V^* : F^*(u) - < u, u^* > \leq F(u)\} \end{aligned}$$

Since $F^* \in \Gamma(V^*)$ (hence F^* is lsc and convex), then $\partial F(u)$ is closed and convex and $\sigma(V^*, V)$ closed. ■

THEOREM 22 If $F : V \longrightarrow \overline{I\!R}$ is convex, continuous and finite at $u \in V$, then $\partial F(v) \neq \phi$ for all $v \in \overbrace{\text{dom} F}^{\circ}$. In particular $\partial F(u) \neq \phi$.

Proof.

1. $\overbrace{\text{dom} F}^{\circ} \neq \phi$, F is continuous on $\overbrace{\text{dom} F}^{\circ}$ and is proper over V.

2. $\overbrace{\text{epi} F}^{\circ} \neq \phi$ ($u \in \overbrace{\text{dom} F}^{\circ}$, F is bounded in a neighbourhood $\mathcal{O}_u$ i.e. $F(v) \leq m$ for all $v \in \mathcal{O}_u$ that means $\mathcal{O}_u \times (m + \epsilon, \infty) \in \text{epi} F$).

3. The set of points $(u, F(u)) \forall\, u \mathrm{dom} F$ are boundary points of epi$F$. epi$F = \overbrace{\mathrm{epi}F}^{\circ} + \mathrm{bd}(\mathrm{epi}F)$.
4. $(u, F(u))$ is a support point for epi$F$ for each $u \in \mathrm{dom}F$.
5. Let $v \in \overbrace{\mathrm{dom}F}$. Since $(v, F(v))$ is a support point for epi$F$, then there is a hyperplane H:

$$\langle w, u^* \rangle + \alpha a + \beta = 0$$

such that $(v, F(v)) \in H$ and $\langle w, u^* \rangle + \alpha a + \beta \geq 0$ for all $(w, a) \in \mathrm{epi}F$.

$$(v, F(v) \in H \Rightarrow \beta = -\langle v, u^* \rangle - \alpha F(v)$$

So H is

$$\langle w - v, u^* \rangle + \alpha(a - F(v)) = 0$$

α must be positive; take $\bar{a}$ sufficiently large then $(v, \bar{a}) \in \overbrace{\mathrm{epi}F}^{\circ}$ So

$$\alpha(\bar{a} - F(v)) \geq 0 \Rightarrow \alpha \geq 0$$

Assume that $\alpha = 0$. Then $\langle w, u^* \rangle + \beta = 0$ for all $\langle w, a \rangle \in H$.

■

9 Lecture 9

We have seen in the previous lecture the if $F : V \longrightarrow \bar{I\!R}$ is convex, finite ($u \in V, |F(u)| < \infty$) and continuous at u. Then $\partial F(v) \neq \phi$ for all $v \in \overbrace{\text{dom}}^{\circ}$. The following inequality is satisfied for each $u^* \in \partial F(u)$

$$\langle v - u, u^* \rangle + \alpha(a - F(u)) \geq 0, \qquad \forall \quad (v, a) \in \text{epi}F$$

So for $(v, F(v))$ we have

$$\langle v - u, u^* \rangle + \alpha(F(v) - F(u)) \geq 0$$
$$F(v) \geq \langle v - u, -\frac{1}{\alpha}u^* \rangle + F(u)$$

So $-\frac{1}{\alpha}u^* \in \partial F(u)$; which shows that $\partial F(u) \neq \phi$

Relation with Gâteaux derivative

$F : V \longrightarrow \bar{I\!R}, u \in V$. If there exists $u^* \in V^*$ such that

$$F'(u, v) = \lim_{\lambda \to 0+} \frac{F(u + \lambda v) - F(u)}{\lambda} = \langle v, u^* \rangle, \qquad \forall v \in V$$

Then u^* is called the Gâteaux derivative of F at u, denoted b $F'(u)$. $F'(u, v)$ is called the directional derivative of F at u in the direction of v. If F is convex, then the above limits always exists; since $\frac{F(u+\lambda v)-F(u)}{\lambda}$ is nondecreasing function of λ (check it).

PROPOSITION 23
Let $F : V \longrightarrow \bar{I\!R}, u \in V$. If $F'(u)$ exists, then $\partial F(u) = \{F'(u)\}$. Conversely, if F is continuous and finite at u and $\partial F(u)$ consists of only one subgradient, then $F'(u)$ exists and $\partial F(u) = \{F'(u)\}$.

Proof. $F'(u)$ exists; that is

$$\langle v, F'(u) \rangle = \lim_{\lambda \to 0+} \frac{F(u - \lambda v) - F(u)}{\lambda} \leq \frac{F(u - \lambda v) - F(u)}{\lambda}, \qquad \forall \quad \lambda \geq 0$$

Let $u + \lambda v = w$, then

$$\langle \frac{w - u}{\lambda}, F'(u) \rangle \leq \frac{F(w) - F(u)}{\lambda}$$
$$\langle w - u, F'(u) \rangle + F(u) \leq F(w)$$
$$\therefore F'(u) \in \partial F(u)$$

Now, suppose $u^* \in \partial F(u)$

$$\langle v - u, u^* \rangle + F(u) \leq F(v), \qquad v \in V$$

Let $\lambda > 0$, put $v = u + \lambda w$. So we have for all $w \in V$ (using the convexity of F)

$$\langle w, u^* \rangle + F(u) \leq \frac{F(u + \lambda w) - F(u)}{\lambda} \leq \frac{F(u + \lambda_0 w) - F(u)}{\lambda_0} \qquad \text{where } \lambda_0 > \lambda$$

This shows that $F'(u)$ exists. Taking the limit as $\lambda \to 0+$ we have

$$\langle w, u^* \rangle \leq \langle w, F'(u) \rangle \qquad \forall \quad w \in V$$

So $u^* = F'(u)$ (since $\langle -w, u^* \rangle \leq \langle -w, F'(u) \rangle \Rightarrow \langle w, u^* \rangle \geq \langle w, F'(u) \rangle$) ■

LEMMA 24
Let $F : A \subseteq V \longrightarrow I\!R$, where A is a convex set, F is Gâteaux differentiable on A. Then $A =$ internal A.

Proof. Let $u \in A$. Since $F'(u)$ exists, then

$$\langle v, F'(u)\rangle = \lim_{\lambda \to 0+} \frac{F(u+\lambda v) - F(u)}{\lambda}$$

Hence, for any $v \in V, u + \lambda v \in A$ for sufficiently small λ. So u is an internal to A. ■

PROPOSITION 25
Let $F : A \subseteq V \longrightarrow I\!R$, where A is a convex set, F is Gâteaux differentiable on A. Then the following statements are equivalent.

(i) F (strictly) convex on A.

(ii) $F(v)(>) \geq F(u) + \langle F'(u), v - u\rangle$.

Proof. $(i) \Rightarrow (ii)$ Suppose that F is strictly convex.

$$\langle w, F'(u)\rangle = \lim_{\lambda \to 0+} \frac{F(u+\lambda w) - F(u)}{\lambda} \leq \frac{F(u+\lambda w) - F(u)}{\lambda}, \qquad \forall \quad \lambda > 0$$

Let $u + \lambda w = v$, then

$$\langle \frac{v-u}{\lambda}, F'(u)\rangle \leq \frac{F(v) - F(u)}{\lambda}$$

So,

$$F(v) \geq \langle v - u, F'(u)\rangle + F(u)$$

Since v is an internal point of A (by previous lemma). Then for $v = \alpha v_1 + (1-\alpha)u, \quad \alpha \in (0,1)$ we have

$$\begin{array}{ccl} \alpha F(v_1) + (1-\alpha)F(u) & > & F(v) \geq \langle \alpha v_1 + (1-\alpha)u - u, F'(u)\rangle + F(u) \\ \alpha F(v_1) & > & \alpha\langle v_1 - u, F'(u)\rangle + \alpha F(u) \\ F(v_1) & > & \langle v_1 - u, F'(u)\rangle + F(u) \end{array}$$

This proves the first direction. ■

10 Lecture 10

Let $F : A \subseteq V \longrightarrow I\!R$, where A is convex. F' exists on A. F is convex iff

$$F(v) \geq F(u) + \langle F'(u), v - u\rangle, \qquad \forall \quad u, v \in A$$

proposition proof continued. Let $u, v \in A$

$$\begin{array}{llcl} (1) & F(v) & \geq & F[u + \lambda(v - u)] + (1 - \lambda)\langle F'[u + \lambda(v - u)], v - u\rangle \\ (2) & F(u) & \geq & F[u + \lambda(v - u)] + \lambda\langle F'[u + \lambda(v - u)], u - v\rangle \end{array}$$

Multiplying (1) by λ and (2) by $1 - \lambda$ and adding we get

$$F[(1 - \lambda)u + \lambda v] \leq (1 - \lambda)F(u) + \lambda F(v)$$

which completes the proof of the proposition. ■

PROPOSITION 26
Let $F : A \subseteq V \longrightarrow I\!R$, A is convex, F' exists on A. Then F is convex F' is monotone. That is

$$\langle F'(u) - F'(v), u - v\rangle \geq 0, \qquad \forall \quad u, v \in A$$

Subdifferential Calculus

Let $F : V \longrightarrow \bar{I\!R}$. Then

- $\partial(\lambda F)(u) = \lambda\partial F(u), \qquad \forall \quad \lambda > 0.$
- $\partial(F_1 + F_2)(u) \supseteq \partial F_1(u) + \partial F_2(u).$

Now choose $u^* \in \partial F_1(u), v^* \in \partial F_2(u)$

$$\begin{array}{lcll} F_1(v) & \geq & F_1(u) + \langle v - u, u^*\rangle, & \forall \quad v \in V \\ F_2(v) & \geq & F_2(u) + \langle v - u, v^*\rangle, & \forall \quad v \in V \\ \hline \text{Adding} & & & \\ (F_1 + F_2)(v) & \geq & (F_1 + F_2)(u) + \langle v - u, u^* + v^*\rangle, & \forall \quad v \in V \end{array}$$

PROPOSITION 27
Let $F_1, F_2 \in \Gamma(V), \bar{u} \in \text{dom}F_1 \cap \text{dom}F_2$, F_1 is continuous at $\bar{u}$, then

$$\partial(F_1 + F_2)(u) = \partial F_1(u) + \partial F_2(u)$$

Proof. Let $u^* \in \partial(F_1 + F_2)(u)$. Then

$$-\langle v - u, u^* + v^*\rangle - F_1(u) + F_1(v) \geq F_2(u) - F_2(v)$$

Let $G(v) = -\langle v - u, u^* + v^*\rangle - F_1(u) + F_1(v)$ and define

$$\begin{array}{lcl} C_1 & = & \{(v, a) \in V \times I\!R : G(v) \leq a\} = \text{epi}G \\ C_1 & = & \{(v, a) \in V \times I\!R : F_2(u) - F_2(v) \geq a\} \end{array}$$

$\overset{\circ}{C_1} \neq \phi, \overset{\circ}{C_1} \cap C_2 = \phi$ (If not, let $(v, a) \in \overset{\circ}{C_1} \cap C_2$. Then $G(v) < a$ and $F_2(u) - F_2(v) \geq a$). Therefore there exist $v^* \in V^*, \alpha, \beta \in I\!R$ such that

$$\begin{array}{lcll} \langle v, v^*\rangle + \alpha a + \beta & \geq & 0, & \forall \quad (v, a) \in C_1 \\ \langle v, v^*\rangle + \alpha a + \beta & \leq & 0, & \forall \quad (v, a) \in C_2 \end{array}$$

Since $(u, 0) \in C_1 \cap C_2$. Then $\langle u, v^*\rangle + \beta = 0 \Rightarrow \beta = -\langle u, v^*\rangle$ and

$$\begin{array}{lcll} \langle v - u, v^*\rangle + \alpha a & \geq & 0, & \forall \quad (v, a) \in C_1 \\ \langle v, v^*\rangle + \alpha a & \leq & 0, & \forall \quad (v, a) \in C_2 \end{array}$$

We can show that $\alpha > 0$. Now for $(v, G(v)) \in C_1$, we have

$$\langle v-u, v^* \rangle + \alpha G(v) \geq 0 \Rightarrow G(v) \geq \langle v-u, -\frac{1}{\alpha} v^* \rangle$$

That is

$$-\langle v-u, u^*+v^* \rangle - F_1(u) + F_1(v) \geq \langle v-u, -\frac{1}{\alpha} v^* \rangle \Rightarrow F_1(v) \geq F_1(u) + \langle v-u, u^* - \frac{1}{\alpha} v^* \rangle$$

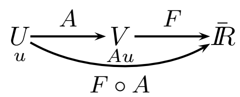
Thus $u^* - \frac{1}{\alpha} v^* \in \partial F_1(u)$. On the other hand, for $(v, F_2(u) - F_2(v)) \in C_2$

$$\langle v-u, v^* \rangle + \alpha (F_2(u) - F_2(v)) \leq 0 \Rightarrow F_2(v) \geq F_2(u) + \langle v-u, \frac{1}{\alpha} v^* \rangle$$

Therefore $\frac{1}{\alpha} v^* \in \partial F_2(u)$ and so $u^* \in \partial F_1(u) + \partial F_2(u)$. ∎

PROPOSITION 28
$A : U \longrightarrow V$ is a continuous linear operator, $F \in \Gamma(V)$. If F is continuous and finite at Au, then

$$\partial F \circ A = A^* \partial F(Au)$$



Proof. Suppose $u^* \in A^* \partial F(Au)$ and let $u^* = A^* v^*$ where $v^* \in \partial F(Au)$. Then

$$F(v) \geq F(Au) + \langle v - Au, v^* \rangle, \qquad \forall \quad v \in V$$

In particular for $v = Aw, w \in U$

$$F(Aw) \geq F(Au) + \langle Aw - Au, v^* \rangle, \qquad \forall \quad w \in U$$

So,

$$(F \circ A)(w) \geq (F \circ A) + \langle w - u, A^* v^* \rangle$$

Therefore $A^* v^* = u^* \in \partial\partial (F \circ A)(u)$.
Conversely, let $u^* \in \partial (F \circ A)(u)$. Then

$$(F \circ A)(v) \geq (F \circ A)(u) + \langle v - u, u^* \rangle, \qquad \forall \quad v \in U$$

Let $C_1 = \{(Av, \langle v-u, u^* \rangle + F(Au)) : v \in U\}$. Clearly C_1 is convex and $C_1 \cap \overset{\circ}{\overbrace{\text{epi}F}} = \phi$. Hence there exist $v^* \in V^*, \alpha, \beta \in I\!R$ such that

$$\begin{array}{rcll} \langle v, v^* \rangle + \alpha a + \beta & \geq & 0 & \forall \quad (v,a) \in \text{epi}F \\ \langle v, v^* \rangle + \alpha a + \beta & \leq & 0 & \forall \quad (v,a) \in C_1 \end{array}$$

Now for $(Au, F(Au)))$ we get

$$\begin{array}{rcl} \langle Au, v^* \rangle + \alpha F(Au) + \beta & = & 0 \\ \beta & = & -\langle Au, v^* \rangle - \alpha F(Au) \end{array}$$

So

$$\begin{array}{rcll} \langle v - Au, v^* \rangle + \alpha (a - F(Au))) & \geq & 0 & \forall \quad (v,a) \in \text{epi}F \\ \langle v - Au, v^* \rangle + \alpha (a - F(Au))) & \leq & 0 & \forall \quad (v,a) \in C_1 \end{array}$$

We can show in the same manner as before that $\alpha > 0$. Since $(Av, \langle v-u, u^* \rangle + F(Au)) \in C_1$ we have

$$\langle Av - Au, v^* \rangle + \alpha \langle v - u, u^* \rangle \leq 0 \Rightarrow \langle v - u, A^* v^* - \alpha u^* \rangle \leq 0 \quad \forall \quad \in V.$$

Therefore $A^* v^* + \alpha u^* = 0$ (since a linear functional that keeps the same sign for the whole space must be zero). So

$$u^* = A^*(-\frac{1}{\alpha} v^*) \in A^* \partial F(Au)$$

Which completes the proof. ∎

11 Lecture 11

Minimization of Convex Functions and Variational Inequalities

Recall that :

1. a normed vector space X is called reflexive if $X = X^{**}$.
2. A Banach space is reflexive if its unit ball is compact in the weak topology.
3. Hilbert spaces and L^p spaces $(1 < p < \infty)$ are reflexive.

Let V be a reflexive Banach space (with norm$\|\|$) and $\phi \neq C$ is closed convex subset of V. The function $F : C \to \mathbf{R}$, is convex and $l.s.c$ and proper. $\hat{F} : V \to \bar{\mathbf{R}}$ is the convex extension of F to all V.

$$\hat{F}(u) = \begin{cases} F(u) & \text{if } u \in C \\ +\infty & \text{if } u \notin C \end{cases}$$

$\hat{F}$ is convex and $l.s.c.$
Consider the minimization problem:

$$\alpha = \inf_{v \in C} F(v) = \inf_{v \in V} \hat{F}(v) \qquad ((*))$$

DEFINITION 29
an element $u \in C$, s.t. $F(u) = \alpha$ is called a solution of the problem (*).

PROPOSITION 30 (1)
The set of solution of (*) is closed and convex set (possibly empty).

Proof. Proof. Consider the set

$$\left\{ u \in V : \hat{F}(u) \leq \alpha \right\}$$

since $\hat{F}$ is convex and $l.s.c$ the set is convex and closed. ■ ■

PROPOSITION 31 (2)
If C is bounded or F is coercive , then (*) has at least one solution. It has a unique solution if F is strictly convex.

Proof. Let $\{u_n\}$ be a sequence in C s.t.

$$F(u_n) \to \alpha = \inf_{v \in C} F(v)$$

- If C is bounded then $\{u_n\}$ is bounded.
- If F is coercive then $F(u_n) \to \alpha \neq \infty$, then$F(u_n)$ is bounded above, the subsequence $\{u_{n_k}\} \overset{weakly}{\to} u$.
- C is closed $\Rightarrow C$ is weakly closed $\Rightarrow u \in C$.
- F is convex and $l.s.c \Rightarrow F$ weakly $l.s.c.$
- $F(u_n) \leq \underline{\lim} F(u_n) = \lim F(u_n) = \alpha >$
- Then $F(u) = \alpha$. u is a solution.

■

PROPOSITION 32 (3)
Consider $F : C \to \mathbf{R}$,$F' exists, u \in C$, The following are Equivalent:

(i)
$$u \text{ minimizes } F \ \ on \ \ C.$$
(ii)
$$< F'(u), v-u > \geq 0 \quad \forall v \in C.$$
(iii)
$$< F'(v), v-u > \geq 0 \quad \forall v \in C.$$

Proof. (i)$\Rightarrow$ (ii)
$< F'(u), v-u >= \lim_{\gimel \to 0} \frac{F(u+\gimel(v-u))-F(u)}{\gimel} \geq 0$
(ii)$\Rightarrow$ (iii)
$$F(u) \geq F(v) + < F'(v), u-v >$$
$$F(v) \geq F(u) + < F'(u), v-u >$$
Adding them
$$0 \geq < F'(v), u-v > + < F'(u), v-u >$$
$$< F'(v), v-u > \geq < F'(u), v-u > \geq 0$$
(iii)$\Rightarrow$ (ii)
$$F(v) \geq F(\gimel u + (1-\gimel)v) + < F'(\gimel u + (1-\gimel)v), \gimel(v-u) >$$
$$= F(\gimel u + (1-\gimel)v) + \frac{\gimel}{1-\gimel} < F'(\gimel u + (1-\gimel)v), (1-\gimel)(v-u) >$$
$$\geq F(\gimel u + (1-\gimel)v) = \phi(\gimel)$$
$$F(v) \geq \phi(1) = F(u)$$
$$\Rightarrow F(u) \text{ is a minimum.}$$

■

REMARK 33
$F(u) = a(u,u) - 2 < l, v >$

- $a(.,.)$ is continuous bilinear form ($|a(u,v)| \leq \|u\| \, \|v\|$) ,
- $a(u,u) \geq \gamma \|u\|^2, \gamma > 0.$
- $l \in V^*$ (continuous linear functional)
- F is strictly convex , Coercive, Then F has a unique minimum.
- if $u \neq v$
$$a(u,v) + a(v,u) < a(u,u) + a(v,v)$$
$$0 < a(u-v, u-v)$$
- F is strictly convex, $u \neq v$, $\lambda \in (0,1)$,
$$F(\gimel u + (1-\gimel)v) = a(\gimel u + (1-\gimel)v, \gimel u + (1-\gimel)v - 2 < l, \gimel u + (1-\gimel)v >$$
$$= \gimel^2 a(u,u) + \gimel(1-\gimel)(a(u,v) + a(v,u)) + (1-\gimel)^2 a(v,v)$$
$$-2\gimel < l, u > -2(1-\gimel) < l, v >$$
$$< \gimel^2 a(u,u) + \gimel(1-\gimel)(a(u,u) + a(v,u) + (1-\gimel)^2 a(v,v)$$
$$-2\gimel < l, u > -2(1-\gimel) < l, v >$$
$$= \gimel a(u,u) + \gimel(1-\gimel)a(v,v) - 2\gimel < l, u > -2(1-\gimel) < l, v >$$
$$\Rightarrow F \text{ is strictly convex}$$
- F is coercive,

$$F(u) = a(u,u) - 2 < l, u >$$
$$\geq \gamma \|u\|^2 - 2 < l, u >$$
$$\geq \gamma \|u\|^2 - 2\|l\|\|u\| \to \infty, \text{ as } \|u\| \to \infty$$

Then we have a unique minima.

- If F is considered on a bounded set C, then we only required $a(u,u) > 0$.

12 Lecture 12

Assumptions : V is a reflexive Banach space, $\varnothing \neq C \subseteq V$ is closed and convex, $F : C \longrightarrow \mathbb{R}$ convex and lower simicontinuous.
Result: under the above assumptions if C is bounded or F is coercive, $a(u,u)$ is a bilinear continuous form satisfiying
$a(u,u) \geq \gamma \|u\|^2, \gamma > 0, l \in V^*$, then $F(u) = a(u,u) - 2\langle l, u\rangle$ has a unique minimazer.
Proposition1: If $F : \varnothing \neq C \longrightarrow \mathbb{R}$ covex, F' exists on C, $u \in C$. TFAE
(i) u minimize F on C
(ii) $\langle F'(u), v-u\rangle \geq 0$ for all $u \in C$
(iii) $\langle F'(v), v-u\rangle \geq 0$ for all $v \in C$

Finding the derivitive of $F(u) = a(u,u) - 2\langle l, u\rangle$,indeed;

$$\begin{aligned}
\lim_{\lambda \longrightarrow 0^+} \frac{F(u+\lambda v) - F(u)}{\lambda} &= \lim_{\lambda \longrightarrow 0^+} \frac{a(u+\lambda v, u+\lambda v) - 2\langle l, u+\lambda v\rangle - a(u,u) + 2\langle l, u\rangle}{\lambda} \\
&= \lim_{\lambda \longrightarrow 0^+} \frac{a(u,u) + \lambda a(u,v) + \lambda a(v,u) + \lambda^2 a(v,v) - 2\langle l,u\rangle - 2\lambda\langle l, v\rangle - a(u,u) + 2\langle l,u\rangle}{\lambda} \\
&= \lim_{\lambda \longrightarrow 0^+} \frac{\lambda a(u,v) + \lambda a(v,u) + \lambda^2 a(v,v) - 2\lambda\langle l, v\rangle}{\lambda}
\end{aligned}$$

$\Rightarrow \langle F'(u), v\rangle = a(v,u) + a(u,v) - 2\langle l, v\rangle$
Remark: if $a(u,u)$ is symmetric, then $\langle F'(u), v\rangle = 2a(u,v) - 2\langle l, v\rangle$
Characterization of the minimizer
$u \in C$ minimizes F iff
(i) $a(u, v-u) - \langle l, v-u\rangle \geq 0$
(ii)$a(v, v-u) - \langle l, v-u\rangle \geq 0$

proposition 2: let $F_1, F_2 : C \longrightarrow \mathbb{R}$ be convex functions,C convex, F_1' exists, $u \in C$,TFAE
(i) u minimizes $F = F_1 + F_2$
(ii) $\langle F_1'(u), v-u\rangle + F_2(v) - F_2(u) \geq 0$ for all $v \in C$
(iii) $\langle F_1'(v), v-u\rangle + F_2(v) - F_2(u) \geq 0$ for all $v \in C$
proof
(i)$\Longrightarrow$(ii)
$0 \leq \frac{F_1((1-\lambda)u+\lambda v) - F_1(u)}{\lambda} + \frac{F_2((1-\lambda)u+\lambda v) - F_2(u)}{\lambda} \leq \frac{F_1(u+\lambda(v-u)) - F_1(u)}{\lambda} + \frac{(1-\lambda)F_2(u) + \lambda F_2(v) - F_2(u)}{\lambda} \leq \frac{F_1(u+\lambda(v-u)) - F_1(u)}{\lambda} + F_2(v) - F_2(u)$ by taking the limit as $\lambda \longrightarrow 0^+$ we get (ii)

(ii) $\Longrightarrow$ (iii) using the convexity of F_1
$F_1(v) \geq F_1(u) + \langle F_1'(u), v-u\rangle$
$F_1(u) \geq F_1(v) + \langle F_1'(v), u-v\rangle$ by adding these two inequlities we obtain
$0 \geq \langle F_1'(u), v-u\rangle + \langle F_1'(v), u-v\rangle \Longrightarrow \langle F_1'(v), v-u\rangle \geq \langle F_1'(u), v-u\rangle \Longrightarrow \langle F_1'(v), v-u\rangle + F_2(v) - F_2(u) \geq \langle F_1'(u), v-u\rangle + F_2(v) - F_2(u) \geq 0$

(iii)$\Longrightarrow$(i)
since C is convex $\Longrightarrow \lambda u + (1-\lambda)v \in C$, $\lambda \in (0,1)$ using (iii) we have
$\langle F_1'(\lambda u + (1-\lambda)v), (1-\lambda)(v-u)\rangle + F_2(\lambda u + (1-\lambda)v) - F_2(u) \geq 0 \Longrightarrow$(by using the convexity of F_2)
$(1-\lambda)\langle F_1'(\lambda u + (1-\lambda)v), (v-u)\rangle + (1-\lambda)(F_2(v) - F_2(u)) \geq 0$ (dividinig by $(1-\lambda)$) we have
$\langle F_1'(\lambda u + (1-\lambda)v), (v-u)\rangle + F_2(v) - F_2(u) \geq 0 \Longrightarrow \langle F_1'(\lambda u + (1-\lambda)v), (v-u)\rangle \geq F_2(u) - F_2(v)$ but
$F_1(v) \geq F_1(\lambda u + (1-\lambda)v) + \langle F_1'(\lambda u + (1-\lambda)v), \lambda(v-u)\rangle \geq F_1(\lambda u + (1-\lambda)v) + \lambda(F_2(u) - F_2(v)) \Longrightarrow$
$F_1(v) + \lambda F_2(v) \geq F_1(\lambda u + (1-\lambda)v) + \lambda F_2(u)$ (by letting $\lambda \longrightarrow 1^-$) we get $F_1(v) + F_2(v) \geq F_1(u) + F_2(u) \Longrightarrow F(v) \geq F(u)$
which completes the proof.
Example1: Proximity Mapping

Lecture 12

Transcribed by:
Badar Humaidi

Let V be a Hilbert space, $x \in V, \varphi \in \Gamma_0(V)$. Define $F(u) = \frac{1}{2}\|u-x\|^2 + \varphi(u)$. set $F_1(u) = \frac{1}{2}\|u-x\|^2$ and $F_2(u) = \varphi(u)$

i) F is strictly convex since F_1 is strictly convex.

ii) F is coercive, indeed; since $\varphi \in \Gamma_0(V)$, there exists a $l \in V^*$, $\alpha \in \mathbb{R}$ such that $\varphi(u) \geq \langle l, u\rangle + \alpha \Longrightarrow F(u) \geq \frac{1}{2}\|u-x\|^2 + \langle l, u\rangle + \alpha \Longrightarrow$
$F(u) \geq \frac{1}{2}(\|u\| - \|x\|)^2 - \|l\|\|u\| - |\alpha| \Longrightarrow F(u) \longrightarrow \infty$ as $u \longrightarrow \infty$. hence F is coercive. By proposition 1 F has a unique minimazer.

Evaluating the derevitive of $F_1(u)$.

$\acute{F_1}(u) = \lim\limits_{\lambda \longrightarrow 0^+} \frac{F_1(u+\lambda v) - F_1(u)}{\lambda} = \lim\limits_{\lambda \longrightarrow 0^+} \frac{\frac{1}{2}\|u+\lambda v - x\|^2 - \frac{1}{2}\|u-x\|^2}{\lambda} = \lim\limits_{\lambda \longrightarrow 0^+} \frac{\frac{1}{2}\|u-x\|^2 + \lambda\langle u-x, v\rangle + \frac{1}{2}\lambda^2\|v\| - \frac{1}{2}\|u-x\|^2}{\lambda} = \langle u-x, v\rangle$

by using proposition 2 : u is a minimizer if and only if

i) $\langle u-x, v-u\rangle + \varphi(v) - \varphi(u) \geq 0$ and ii) $\langle v-x, v-u\rangle + \varphi(v) - \varphi(u) \geq 0$.

Special case: if C is a non empty closed convex subset of V, $x \in v$

Define $F(u) = \frac{1}{2}\|u-x\|^2 \Longrightarrow \widetilde{F}(u) = \frac{1}{2}\|u-x\|^2 + \chi_c(u)$ by using the above arqument we have
$\langle u-x, v-u\rangle + \chi_c(v) - \chi_c(u) \geq 0$ and $\langle v-x, v-u\rangle + \chi_c(v) - \chi_c(u) \geq 0 \Longrightarrow$
$\langle u-x, v-u\rangle \geq 0$ for all $v \in C$ and $\langle v-x, v-u\rangle \geq 0$ for all $v \in C$. the mapping $x \longrightarrow u$ is called aproximity mapping and we write
$u-$prox x.

13 Lecture 13

The Direct Study of Certain Variational Inequalities

$\langle Au - f, v - u\rangle + \Phi(v) - \Phi(u) \geq 0.\ \forall\, v \in V$ where V is a reflexive Banach space, A: V→V*, where f∈V* is given and $\Phi : V \to \overline{\overline{R}}$.

i) Φ is proper, lsc and convex.

ii) A is weakly continous on finite dimensional subspaces of V.

iii) A is a monotone. i.e. $\langle Au - Av, u - v\rangle \geq 0.\ \forall\, u, v \in V.$

iv) A is coercieve: $\exists\, v_\circ \in V$ such that: $\dfrac{\langle Av, v - v_\circ\rangle + \Phi(v)}{\|v\|} \to \infty$ as $\|\, v\, \| \to \infty$.

Problem:

Find u∈V such that $\langle Au - f, v - u\rangle + \Phi(v) - \Phi(u) \geq 0.\ \forall\, v \in V$ (call this *).

Theorem:

Problem (*) has at least one solution.

Proof:

step (1):

Assume V is finite dimensional (FD) and (dom Φ) is bounded. Also we assume here that V has a Hilbert space structure).

(*) may be rewritten as follows:

$\langle u - (u - Au + f), v - u\rangle + \Phi(v) - \Phi(u) \geq 0.\ \forall\, v \in V$

where $u = Prox_\Phi(u - Au + f)$

Define T:V→dom $\Phi \subseteq$ cl(dom Φ) by

Tu: $Prox_\Phi(u - Au + f)$

The idea here is to show that T has a fixed point. If we can show that $Prox_\Phi : V \to$domΦ is continuous then T has a fixed point by Brouwer's fixed point theorem. For that let $f_1, f_2 \in V$, $u_1 = Prox_\Phi\, f_1$, $u_2 = Prox_\Phi\, f_2$ then:

$\langle u_1 - f_1, v - u\rangle + \Phi(v) - \Phi(u) \geq 0$

$\langle u_2 - f_2, v - u\rangle + \Phi(v) - \Phi(u) \geq 0$

$\langle u_1 - f_1, u_2 - u_1\rangle + \Phi(u_2) - \Phi(u_1) \geq 0$

$\langle u_2 - f_2, u_1 - u_2\rangle + \Phi(u_1) - \Phi(u_2) \geq 0$ by summing the last two inequalities we get:

$\langle (u_1 - f_1) - (u_2 - f_2), u_2 - u_1\rangle \geq 0$ or by rearranging:

$\langle (u_1 - u_2) - (f_1 - f_2), u_2 - u_1\rangle \geq 0 \implies$

$\|u_2 - u_1\|^2 \leq -\langle f_1 - f_2, u_2 - u_1\rangle \leq \|f_1 - f_2\|\|u_2 - u_1\| \implies \|u_2 - u_1\| \leq \|f_1 - f_2\|$

Therefore it is continous and so T has a fixed point u $\in$ cl(domΦ) and because u = Tu $\in$ domΦ since range T is in domΦ

$\therefore$ (*) has a solution.

Step (2):

Now assume only that V is FD.

For n = 1,2,3,....., define $\Phi_n(u) = \begin{cases} \Phi(u) & \text{if } \|\, u\, \| \leq n \\ \infty & \text{if } \|\, u\, \| \gneqq n \end{cases}$

Note that dom$\Phi_n \subseteq \overline{B(0.n)}$.

By step (1) the problem $\langle Au - f, v - u\rangle + \Phi_n(v) - \Phi_n(u) \geq 0$ has a solution $u_n \in$ dom$\Phi_n \subseteq \overline{B(0.n)}$

i.e. $\langle Au_n - f, v - u_n\rangle + \Phi_n(v) - \Phi_n(u_n) \geq 0.\ \forall\, v \in V$.

Now calaim that $\{u_n\}$ is bounded. If we assume not then we have:

$\langle Au_n - f, v_\circ - u_n\rangle + \Phi_n(v_\circ) - \Phi(u_n) \geq 0$ (note here that $\Phi_n(u_n) = \Phi(u_n)$ since $\|u_n\| \leq n$)

$\Rightarrow \langle Au_n, u_n - v_\circ\rangle + \Phi(u_n) \leq \Phi_n(v_\circ) - \langle f, v_\circ - u_n\rangle$

note here that for sufficiently large $n \geq \|v_\circ\|$, we have $\Phi_n(v_\circ) = \Phi(v_\circ)$ and so

$\langle Au_n, u_n - v_\circ\rangle + \Phi(u_n) \leq \Phi(v_\circ) - \langle f, v_\circ - u_n\rangle$ and by dividing every thing by $\|u_n\|$ we get:

$\dfrac{\langle Au_n, u_n - v_\circ\rangle + \Phi(u_n)}{\|u_n\,\|} \leq \dfrac{\Phi(v_\circ)}{\|u_n\,\|} + \|f\,\|\ (1 + \dfrac{\|v_\circ\,\|}{\|u_n\,\|})$ which $\to \|f\,\| \lneqq \infty$ as $\|u_n\,\| \to \infty$ and this of course conradicts the coercevity. Hence, $\{u_n\}$ is bounded.

Now, since $\{u_n\}$ is bounded in a FD space, there exists a subsequence $\{u_{n_j}\}$ and a $u \in V$ such that $u_{n_j} \to u$ and ($A_{u_j} \to A_u$ by continuity of A).
Letting $v \in V \Rightarrow \langle Au_{n_j} - f, v - u_{n_j} \rangle + \Phi_{n_j}(v) - \Phi(u_{n_j}) \geq 0$
Then for sufficiently large n_j with $\|v\| \leq n_j$ we have $\Phi_{n_j}(v) = \Phi(v)$
$\therefore$ taking the limit of both sides as $j \to \infty$ we get
$\langle Au - f, v - u \rangle + \Phi(v) - \Phi(u) \geq 0$ and this completes the proof.

<u>Remark</u> :

If $Au_n \to Au$ then $\langle Au_n, u \rangle \to \langle Au, u \rangle$ but it not always true that $\langle Au_n, u_n \rangle \to \langle Au, u \rangle$ whenever $u_n \to u$.Acutually
this can not happen unless we impose the conition of boundedness on either Au_n or u_n. Note on the following:
$\langle Au_n, u_n \rangle = \langle Au_n, u - u + u_n \rangle = \langle Au_n, u \rangle + \langle Au_n, u_n - u \rangle \to \langle Au, u \rangle + \langle Au_n, u_n - u \rangle$
But $| \langle Au_n, u_n - u \rangle | \leq \| Au_n \| \| u_n - u \|$ .....$(**)$
And since $\| u_n - u \| \to 0$ as $u_n \to u$ then the r.h.s of (**) will not vanished unless $\| Au_n \|$ is bounded.
Similar argument can be done on Au_n to have u_n being bouned.

14 Lecture 14

The Direct Study of Certain Variational Inequalities (continue)

$\langle Au - f, v - u\rangle + \Phi(v) - \Phi(u) \geq 0.\ \forall\, v \in V$ where V is a reflexive Banach space, $A:\ V \to V*$, where $f \in V*$ is given and $\Phi : V \to \overline{\overline{R}}$.
i) Φ is proper, lsc and convex.
ii) A is weakly continuous on finite dimensional subspaces of V.
iii) A is a monotone. i.e. $\langle Au - Av, u - v\rangle \geq 0.\ \forall\, u, v \in V$.
iv) A is coercive: $\exists\, v_\circ \in V$ such that: $\dfrac{\langle Av, v - v_\circ\rangle + \Phi(v)}{\|v\|} \to \infty$ as $\|v\| \to \infty$.

Problem:
Find $u \in V$ such that $\langle Au - f, v - u\rangle + \Phi(v) - \Phi(u) \geq 0.\ \forall\, v \in V$ (call this *).

THEOREM 34 Problem (*) has at least one solution.

Proof:
step (3):
Assume V is of infinite dimension i.e. $dimV = \infty$
Let $\{V_n\}_{n=1}^{\infty}$ be a sequence of FD subspaces of V containing $v_\circ$ that satisfies $\dfrac{\langle Av, v - v_\circ\rangle + \Phi(v)}{\|v\|} \to \infty$ as $\|v\| \to \infty$ where $V_n \subseteq V_{n+1}$ and $\overset{\infty}{\underset{n=1}{\cup}} V_n$. (Note here that having $\{V_n,\ n = 1, 2, 3, ...\}$ being just a family of
subspaces is not enough to have such $v_\circ$ in all of $V_i,\ i = 1, 2, 3, ...$)

Now, for each n $\exists$ a $u_n \in V_n$ s.t.
$\langle Au_n - f, v - u_n\rangle + \Phi(v) - \Phi(u_n) \geq 0.\ \forall\, v \in V_n$

and by the discussion made before about the coercivity of A, we have $\{u_n\}$ is bounded.
$\therefore u_n \rightharpoonup u_\circ$ for some $u_\circ \in V$.

Digression to investigate monotonicity:
$\langle Au - Av, u - v\rangle \geq 0$

$\Rightarrow \langle Au, u - v\rangle \geq \langle Av, u - v\rangle$
By putting $u = u_m,\ v = u_\circ,$ we get $\langle Au_m, u_m - u_\circ\rangle \geq \langle Au_\circ, u_m - u_\circ\rangle$, and by taking the $\underline{\lim}$ of both sides as $m \to \infty$,we have(insert a note): $\underline{\lim}\langle Au_m, u_m - u_\circ\rangle \geq 0 \Rightarrow$
$\overline{\lim}\langle Au_m, u_m - u_\circ\rangle \geq 0$ ....(***)

Also, we already have: $\langle Au_n - f, v - u_n\rangle + \Phi(v) - \Phi(u_n) \geq 0.$
so, by fixing n and letting $m \geq n$ we have:

$\langle Au_m - f, v - u_m\rangle + \Phi(v) - \Phi(u_m) \geq 0 \quad \Rightarrow$
$\Phi(v) - \Phi(u_m) \geq \langle f, v - u_m\rangle + \langle Au_m, u_m - v\rangle$........(****)

Note here that
i) since Φ lsc and convex then $\Phi(u_\circ) = \underset{n\to\infty}{\underline{\lim}}\ \Phi(u_n)$.
ii) $\overline{\lim}(-\Phi(u_n)) = -\underline{\lim}\Phi(u_n)$
iii) $\overline{\lim}(a - \Phi(u_n)) = \overline{\lim}(a + (-\Phi(u_n)) = a + \overline{\lim}(-\Phi(u_n)) = a - \underline{\lim}\Phi(u_n)$

Now taking $\overline{lim}$ of both sides of (****) we get:

$\Phi(v) - \Phi(u_\circ) \geq \langle f, v - u_\circ\rangle + \overline{\lim}\langle Au_m, u_m - v\rangle$.
Note here that $\overline{\lim}$ of LHS of (****) $= \overline{\lim}(\Phi(v) - \Phi(u_m)) = \Phi(v) - \underline{\lim}\Phi(u_m) = \Phi(v) - \Phi(u_\circ)$.
Also, since n is arbitrary, we have the above inequality is true for all n.

Let $v \in V$ and let $v_n \to v$ then
$\Phi(v_n) - \Phi(u_\circ) \geq \langle f, v_n - u_\circ\rangle + \overline{\lim}\langle Au_m, u_m - v_n\rangle$.
Take $\underline{\lim}$ for both side as $n \to \infty$ we get:

$$\begin{aligned}\Phi(v) - \Phi(u_\circ) &\geq \langle f, v - u_\circ\rangle + \underline{\lim}_n\, \overline{\lim}_m\langle Au_m, u_m - v_n\rangle \\ &\geq \langle f, v - u_\circ\rangle + \overline{\lim}_m\, \underline{\lim}_n\, \langle Au_m, u_m - v_n\rangle \\ &= \langle f, v - u_\circ\rangle + \overline{\lim}_m\, \langle Au_m, u_m - v\rangle \ \ \forall\ v \in V.\end{aligned}$$

Now, if we let $v = u_\circ$ in the above inequality (since it is true $\forall\ v \in V$) we have:

$$0 \geq \overline{\lim}_m\, \langle Au_m, u_m - u_\circ\rangle$$

$\Rightarrow \quad 0 \geq \overline{\lim}_m\, \langle Au_m, u_m - u_\circ\rangle \geq \underline{\lim}\, \langle Au_m, u_m - u_\circ\rangle \geq 0$ (by(***)above) $\geq \overline{\lim}\langle Au_m, u_m - u_\circ\rangle$.

$\therefore \quad \lim\langle Au_m, u_m - u_\circ\rangle = 0$...........(*****)

By going back to monotonicity of A i.e. $\langle Au, u - v\rangle \geq \langle Av, u - v\rangle$ and letting $u = u_m$, $v = (1-\alpha)u_\circ + \alpha w$ then

$$\begin{aligned}u_m - v &= u_m - (1-\alpha)u_\circ - \alpha w \\ &= u_m - u_\circ + \alpha(u_\circ - w) \\ &= (1-\alpha)(u_m - u_\circ) + \alpha(u_m - w)\end{aligned}$$

and so

$$\langle Au_m, (1-\alpha)(u_m - u_\circ) + \alpha(u_m - w)\rangle \geq \langle Av, u_m - u_\circ + \alpha(u_\circ - w)\rangle$$

$$\Rightarrow \quad (1-\alpha)\langle Au_m, (u_m - u_\circ)\rangle + \alpha(Au_m, u_m - w)\rangle \geq \langle Av, u_m - u_\circ\rangle + \alpha\langle Av, u_\circ - w\rangle$$

taking $\underline{\lim}$ for both sides gives:

$$\alpha\underline{\lim}\langle(Au_m, u_m - w)\rangle \geq \alpha\underline{\lim}\langle Av, u_\circ - w\rangle \ \text{ or:}$$

$\underline{\lim}\langle(Au_m, u_m - w)\rangle \geq \underline{\lim}\langle Av, u_\circ - w\rangle$ and by taking $\lim_{\alpha\to 0}$ $\Rightarrow$ (by using the continuity $v \to u_\circ$ as $\alpha \to 0$
we have $v \rightharpoonup u_\circ$ and so $Av \rightharpoonup Au_\circ$).

$$\overline{\lim}\langle Au_m, u_m - w\rangle \geq \underline{\lim}\langle Au_m, u_m - w\rangle \geq \langle Au_\circ, u_\circ - w\rangle \quad \forall\, w \in V$$

and by (*****) we have:

$$\Phi(w) - \Phi(u_\circ) \geq \langle f, w - u_\circ\rangle + \langle Au_\circ, u_\circ - w\rangle \quad \text{or}$$

$$\langle Au_\circ - f, w - u_\circ\rangle + \Phi(w) - \Phi(u_\circ) \geq 0$$

i.e. it has a solution

Special Cases:

case (1):

$A : C \subseteq V \to V^*$. C is closed and convex. A is a monotone, weakly continuous on a FD subset of C and coercive. Then, there exists a $u \in C$ such that $\langle Au - f, v - u\rangle \geq 0$

proof:

By extending A to the whole space as

$$\overline{A}u = \begin{cases} Au & \text{if} \quad u \in C \\ \infty & \text{if} \quad u \notin C\end{cases}$$

and by using Φ being the indicator function on C, we have the result directly by the previous theorem.

---

case (2):

$A : V \to V^*$ with same assumption as above i.e. monotone.,,etc. $\Rightarrow \exists\, u \in V$ s.t. $Au = f$

proof:

By putting $V = C$ in case (1) and letting $v = u + w$ and so $v - u = w$ we get:

$\langle Au - f, w\rangle \geq 0 \quad \forall \; w \in V^* \;\Rightarrow$

$\langle Au - f, -w\rangle \geq 0 \quad \Rightarrow$

$\langle Au - f, w\rangle = 0 \quad \forall \; w \in V^* \;\Rightarrow$

$Au = f$

---

case (3):

$A : V \to V^*$ where V is a Hilbert space with $V = V^*.A$ is linear and bounded with $\langle Au, u\rangle \geq \alpha \parallel u \parallel^2$. Then given $f \in V$, there exists a unique $u \in V$ s.t. $Au = f$.

proof:

Note here that a bounded operator is continuous iff it is weakly continuous.

Monotonicity of A:

$$\langle Au - Av, u - v\rangle = \langle A(u - v), u - v\rangle \geq \alpha \parallel u - v \parallel^2 \geq 0$$

Coercivity of A:

$$\frac{\langle Au, u\rangle}{\|u \|} \geq \frac{\alpha\|u \|^2}{\|u \|} = \alpha\|u \| \to \infty \quad \text{as} \quad \|u \| \to \infty$$

So, by case (2) the existence is obtained. The uniqueness of u is obtained easily $\langle Au, u\rangle \geq \alpha \parallel u \parallel^2$

Assume $\exists\, u_1, u_2$ such that $Au_1 = f = Au_2$. Then:

$$\langle Au_1 - Au_2, u_1 - u_2\rangle = \langle A(u_1 - u_2), u_1 - u_2\rangle \geq \alpha \parallel u_1 - u_2 \parallel^2 \Rightarrow$$

$$0 = \langle f - f, u_1 - u_2\rangle \geq \alpha \parallel u_1 - u_2 \parallel^2 \quad \forall\, \alpha \quad \Rightarrow$$

$$0 = u_1 - u_2 \quad \Rightarrow u_1 = u_2 \quad \text{i.e. } u \text{ is unique.}$$

---

15 Lecture 15

Duality in convex optimization

Setting: V, Y are topological vector spaces, V^*, Y^* are their dual, $F : V \longrightarrow \bar{I\!R}$ and

$$(P) \qquad \inf_{u\in V} F(u)$$

- The inf for problem (P) will be denoted by $\inf P$.
- A solution of (P) is any $u \in V$ such that $F(u) = \inf P$.
- Problem (P) is called nontrivial if $\exists u_0 \in V$ such that $F(u_0) < \infty$. If $F \in \Gamma_0(V)$, then (P) is nontrivial.

Suppose $\Phi : V \times Y \longrightarrow \bar{I\!R}$ such that $\Phi(u, 0) = F(u)$. The problem

$$(P_p) \qquad \inf_{u\in V} \Phi(u,p)$$

is called the perturbed prolem of (P) with respect to Φ ($P_0 = P$). The problem

$$(P^*) \qquad \sup_{p^*\in Y^*} \{-\Phi(0,p^*)\}$$

is called the dual of (P) with respect to Φ [2].

PROPOSITION 35

$$-\infty \le \sup P^* \le \inf P \le \infty$$

Proof. $\sup P^* = \sup_{p^*\in V^*}\{-\Phi^*(0,p^*)\}$

$$\begin{aligned}\Phi^*(0,p^*) &= \sup_{(u,p)\in V\times Y)} \{\langle p,p^*\rangle - \Phi(u,p)\}\\ &\ge \sup_{u\in V} -\Phi(u,0)\\ &= -\inf_{u\in V} F(u)\end{aligned}$$

So, $\sup P^* \le \inf P$. ■

PROPOSITION 36
If P is nontrivial then

$$-\infty \le \sup P^* \le \inf P < \infty$$

If P^* is nontrivial then

$$-\infty < \sup P^* \le \inf P \le \infty$$

If P and P^* are nontrivial then

$$-\infty < \sup P^* \le \inf P < \infty$$

[2] $\Phi : V \times Y \longrightarrow \bar{I\!R}$, $\langle (v^*,p^*),(v,p)\rangle = \langle v,v^*\rangle + \langle p,p^*\rangle$

$$\Phi^*(v^*,p^*) = \sup_{(v,p)\in V\times Y} \langle (v^*,p^*),(v,p)\rangle - \Phi(u,p) = \sup_{(v,p)\in V\times Y} \langle v,v^*\rangle + \langle p,p^*\rangle - \Phi(u,p)$$

Reiteration of duality

The problem

$$(P^*_{u^*}) \qquad \sup_{p^* \in Y^*} \{-\Phi(u^*, p^*)\}$$

is called the associated perturbed problem of P^*. The bidual problem

$$(P^{**}) \qquad \inf_{u \in V} \{\Phi^{**}(u, 0)\}$$

This process terminates. Indeed, $P^{***} = P^*$.

- If $P^{**} = P$ ($Q^{**} = Q$), then P, P^* are the dual of each other.
- If $\Phi \in \Gamma(V, Y)$ then $P^{**} = P$ and P is nontrivial.

Normal problems and stable problems

$\Phi \in \Gamma_0(V \times Y)$ define $h(p) = \inf P_p = \inf \Phi(u, p)$.

LEMMA 37
$h : Y \longrightarrow I\!R$ is convex.

Proof. Let $p, q \in Y$ and $\lambda \in [0, 1]$. Assume that $\lambda h(p) + (1-\lambda)h(q)$ is defined. If either $h(p)$ or $h(q)$ is infinite, nothing to prove. Assume $h(p)$ and $h(q)$ are finite. Let $\epsilon > 0$ be given, there exists a $u_1 \in V$ such that

$$\Phi(u, p) \leq h(p) + \epsilon$$

and there exists $u_2 \in V$ such that

$$\Phi(u, q) \leq h(q) + \epsilon$$

Now, we have

$$\begin{aligned} h\left[\lambda h(p) + (1-\lambda)h(q)\right] &\leq Q\left[\lambda(u_1, p) + (1-\lambda)(u_2, q)\right] \\ &\leq \lambda Q(u_1, p) + (1-\lambda)Q(u_2, q) \\ &\leq \lambda h(p) + (1-\lambda)h(q) + \epsilon \end{aligned}$$

Since ϵ is arbitrary h is convex. ∎

LEMMA 38
For all $p^* \in V^*$

$$h^*(p^*) = \Phi^*(0, p^*)$$

Proof.

$$\begin{aligned} h^*(p^*) &= \sup_{p \in Y} \langle p^*, p\rangle - h(p) \\ &= \sup_{p \in Y}\{\langle p^*, p\rangle - \inf_{u \in V} \Phi(u, p)\} \\ &= \sup_{(u,p) \in V \times Y} \{\langle p^*, p\rangle - \Phi(u, p)\} \\ &= \sup_{(u,p) \in V \times Y} \{\langle u, 0\rangle + \langle p^*, p\rangle - \Phi(u, p)\} = \Phi^*(0, p^*) \end{aligned}$$

∎

LEMMA 39
$\sup P^* = h^{**}(0)$.

Proof.

$$\begin{aligned}\sup P^* &= \sup_{p^*\in Y^*}\{-\Phi^*(0,p^*)\} \\ &= \sup_{p^*\in Y^*}\{-h^*(p^*)\} \\ &= \sup_{p^*\in Y^*}\{\langle 0,p^*\rangle - h^*(p^*)\} = \Phi(0,p^*)\end{aligned}$$

■

REMARK 40

$$\sup P^* \leq \inf P \Leftrightarrow h^{**}(0) \leq h(0)$$

DEFINITION 41
The problem (P) is called normal if $h(0) \in \mathbb{R}$ and h is lsc at 0.

PROPOSITION 42
Problem (P) is normal iff $\sup P^* = inf P \in \mathbb{R}$.

Proof. Assume that (P) is normal. Let $\overline{h}$ be the lsc regularization of h. Then

$$h^{**} \leq \overline{h} \leq h \tag{3}$$

$\overline{h}(0) = h(0)$, $\overline{h}$ is convex, lsc and finite at 0. So

$$\overline{h} \not\equiv -\infty \Rightarrow \overline{h} \in \Gamma_0(Y) \Rightarrow \overline{h}^{**} = \overline{h}$$

From 3

$$h^* \leq \overline{h}^* \leq h^{***} = h^*$$

but $h^* = \overline{h}^*$. So $h^{**} = \overline{h}^{**} = \overline{h}$ and $h^{**}(0) = \overline{h}(0) = h(0)$. That is

$$\sup P^* = inf P$$

.
Now assume $\sup P^* = inf P \in \mathbb{R}$. Then $h^{**}(0) = h(0)$. Let $\overline{h}$ be the lsc regularization of h

$$h^{**} \leq \overline{h} \leq h$$

So h is lsc at 0, i.e.

$$h(0) = \overline{h}(0) = \liminf_{p\longrightarrow 0} h(p)$$

■

LEMMA 43
P^* is normal iff $\inf P = \sup P^*$

Proof. By proposition (42) P^* is normal iff $\inf P^{**} = \sup P^*$ i.e. $\inf P = \sup P^*$ ■

16 Lecture 16

(Stable Problems)

DEFINITION 44
Problem P is called stable if $h(0) \in \mathbf{R}$,$\partial h(0) \neq \phi$.

LEMMA 45
The set of solution of $\mathbf{P}^*$ coincides with $\partial h^{**}(0)$.

Proof. Suppose $p^*$is a solution of $\mathbf{P}^*$, then

$$-h^*(p^*) = -\Phi(0, p^*) = \sup_{q^* \in Y} -\Phi(0, q^*) = h^{**}(0).$$

Fix $p \in Y$, then,

$$\sup_{q^* \in Y} < p, q^* > -h^*(q^*) \geq h^*(p^*) + < p^*, p >$$

i.e.:

$$h^{**}(p) \geq -h^*(p^*) + < p^*, p >= h^{**}(0) + < p^*, p >$$

Then, $p^* \in \partial h^{**}(0)$.
On the other hand, let $p^* \in \partial h^{**}(0)$,then

$$h^{**}(p) \geq h^{**}(0) + < p^*, p > \quad \forall v \in V$$

$$-h^{**}(0) \geq < p^*, p > -h^{**}(p)$$
$$-h^{**}(0) \geq h^{***}(p^*) = h^*(p^*)$$
$$h^{**}(0) \leq -h^*(p^*)$$
$$\sup -h^*(q^*) \leq -h^*(p^*)$$

Therefore,

$$-h^*(p^*) = \sup -h^*(q^*) \qquad q^* \in Y$$

Then, $p^*$is a solution of $\mathbf{P}^*$. ■

PROPOSITION 46
P is stable iff **P** is normal and $\mathbf{P}^*$ has a solution.

Proof. Suppose **P** is stable, then **P** is normal (since $\partial h(0) \neq 0 \implies h$ is $l.s.c$ at 0). Furthermore, $p^* \in \partial h(0) = \partial h^{**}(0)$, therefore, $p^*$is a solution of $\mathbf{P}^*$ by previous lemma. Conversely if **P** is normal and $\mathbf{P}^*$ has a solution p^*, then

$$p^* \in \partial h^{**}(0) = \partial h(0),$$

since h is $l.s.c$ at 0.Then **P** is stable. ■

PROPOSITION 47
The Following Conditions are equivalent:
(I) **P** and $\mathbf{P}^*$are normal and have some solutions,
(II) **P** and $\mathbf{P}^*$are stable,
(III) **P** is stable and has some solutions.

Proof. (I) ⇒(II),
Assume (I) , $\mathbf{P}^*$ is normal and **P** has a solution ⇒ $\mathbf{P}^*$ is normal and $\mathbf{P}^{**}$ has a solution,⇒ $\mathbf{P}^*$is stable. Similarly, **P** is normal and $\mathbf{P}^*$ has a solution ⇒ **P** is stable. (II) ⇒(I) direct. (III) ⇒(I) follows directly from previous proposition. ■

Lecture 16

PROPOSITION 48
A stability criterion.
Assume Φ is convex, that $\inf \mathbf{P} \in \mathbf{R}$. $\Phi(u_0, .)$ is bounded above at 0 for some $u_0 \in \mathbf{V}$. Then $\mathbf{P}$ is stable.

Proof.

$$h(p) = \inf_{u \in \mathbf{V}} \Phi(u, p) \leq \Phi(u_0, p),$$

and $h(0) \in \mathbf{R} \Rightarrow h$ is bounded above at $0, \Rightarrow h$ is continuous at $0, \Rightarrow \partial h(0) \neq \phi$. Then $\mathbf{P}$ is stable. ■

17 Lecture 17

Summery
P: $\underset{u\in v}{inf} F(u)$
$\Phi : V \times Y \longrightarrow \overline{R}$ such that $\Phi(u, 0) = F(u)$
P: $\underset{u\in v}{inf} \Phi(u, 0)$
The dual problem

$\overset{*}{P}$: $\underset{\overset{*}{p}\in\overset{*}{y}}{sup} - \overset{*}{\Phi}(0, \overset{*}{p})$

$Sup\ \overset{*}{P} \leq \inf P$

$-\overset{*}{\Phi}(0, \overset{*}{p}) \leq \sup \overset{*}{P} \leq \inf P \leq \Phi(u, 0) \Rightarrow \Phi(u, 0) + \overset{*}{\Phi}(0, \overset{*}{p}) \geq 0$
$h(p) = \inf_{u\in v} \Phi(u, p)$
- If $h(0) \in \mathbb{R}$ and h is lwoer semicontinuous at $0 \Rightarrow P$ is normal
- P is normal $\Leftrightarrow \inf P = \sup \overset{*}{p} \Leftrightarrow \overset{*}{p}$ is normal
- $h(0) \in \mathbb{R}$, $\partial h(0) \neq \Phi \Longrightarrow$ P is stable
- P is stable iff $\overset{*}{P}$ is normal and has some solutions
- the set of solution of $\overset{*}{P}$ conicides with $\partial \overset{**}{h}(0)$
- $P, \overset{*}{P}$ are normal and have same solutions $\Leftrightarrow P$ and $\overset{*}{P}$ are stable $\Leftrightarrow$ P is stable and has solutions.

Criterion for stability
Φ is convex, $h(0) \in \mathbb{R}$, $\Phi(u, .)$ bounded above in a nbhd of 0 $\Longrightarrow P$ is stable
$h(p) \leq \Phi(u, p)$

Criterion for existence
V is a reflexive Banach space, $\Phi(., 0)$ is coercive $\Longrightarrow P$ has a solution

Extremality relation and Existence

Lemma1: $\overline{u} \in V$ is a solution of P and $\overset{*}{\overline{p}}$ is a solution of $\overset{*}{P}$ and $\inf P = \sup \overset{*}{P}$ iff $\Phi(\overline{u}, 0) + \overset{*}{\Phi}(0, \overset{*}{\overline{p}}) = 0$
Proof: if $\overline{u} \in V$ is a solution of P and $\overset{*}{\overline{p}}$ is a solution of $\overset{*}{P}$ and $\inf P = \sup \overset{*}{P}$, then $-\overset{*}{\Phi}(0, \overset{*}{\overline{p}}) = \sup \overset{*}{P}$ $= \inf P = \Phi(\overline{u}, 0) \Longrightarrow \Phi(\overline{u}, 0) + \overset{*}{\Phi}(0, \overset{*}{\overline{p}}) = 0$
conversly assume $\Phi(\overline{u}, 0) + \overset{*}{\Phi}(0, \overset{*}{\overline{p}}) = 0$ for some $\overline{u} \in V$ and some $\overset{*}{\overline{p}} \in \overset{*}{Y}$ then

$$-\overset{*}{\Phi}(0, \overset{*}{\overline{p}}) \leq \sup \overset{*}{P} \leq \inf P \leq \Phi(\overline{u}, 0) = -\overset{*}{\Phi}(0, \overset{*}{\overline{p}})$$

and hence, the result is obtained.

Lagrangians and Saddle points

Definition: $L : V \times \overset{*}{Y} \longrightarrow \overline{\mathbb{R}}$ defined by $-L(u, \overset{*}{P}) = \underset{p\in Y}{Sup} \left\langle p, \overset{*}{p} \right\rangle - \Phi(u, p)$ is called the Lagrangian .

Note: $-L(u, \overset{*}{P}) = \overset{*}{\Phi}_u(\overset{*}{p})$ where $\Phi_u(p) = \Phi(u, p)$
Lemma
1- for $u \in V$,$L(u, .)$ is concave and u.s.c.

2- if Φ is convex, then for any $\overset{*}{p} \in \overset{*}{Y}$, $L(., \overset{*}{P})$ is convex

Proof: (part 2)

$L(\lambda u+(1-\lambda)v,\overset{*}{p}) = \inf_{p\in Y} -\left\langle p,\overset{*}{p}\right\rangle+\Phi((\lambda u+(1-\lambda)v,p) \leq -\left\langle \lambda p+(1-\lambda)q,\overset{*}{p}\right\rangle+\Phi((\lambda u+(1-\lambda)v,\lambda p+(1-\lambda)q) \leq$ $\lambda(-\left\langle p,\overset{*}{p}\right\rangle+\Phi(u,p))+(1-\lambda)(-\left\langle q,\overset{*}{p}\right\rangle+\Phi(u,q))$

fix q and take the inf over $p \Longrightarrow$

$L(\lambda u+(1-\lambda)v,\overset{*}{p}) \leq \lambda L(u,\overset{*}{p})+(1-\lambda)(-\left\langle q,\overset{*}{p}\right\rangle+\Phi(u,q))$

now take inf over $q \Longrightarrow$

$L(\lambda u+(1-\lambda)v,\overset{*}{p}) \leq \lambda L(u,\overset{*}{p})+(1-\lambda)L(v,\overset{*}{p})$ and hence, $L(.,\overset{*}{P})$ is convex.

$\overset{*}{P}$ in terms of L

$\overset{*}{\Phi}(\overset{*}{u},\overset{*}{p}) = \sup_{u\in v,p\in Y} \left\langle u,\overset{*}{u}\right\rangle+\left\langle p,\overset{*}{p}\right\rangle-\Phi(u,p)$

$\overset{*}{\Phi}(0,\overset{*}{p}) = \sup_{u\in V}\sup_{p\in Y}\left\langle p,\overset{*}{p}\right\rangle-\Phi(u,p) = \sup_{u\in V} -L(u,\overset{*}{p}) = -\inf_{u\in V} L(u,\overset{*}{p}) \Longrightarrow -\overset{*}{\Phi}(0,\overset{*}{p}) = \inf_{u\in V} L(u,\overset{*}{p}) \Longrightarrow \overset{*}{P}:$ $\sup_{\overset{*}{p}\in\overset{*}{Y}}\inf_{u\in V} L(u,\overset{*}{p})$

P in terms of L

$\Phi(u,0) = \Phi_u(0) = \sup_{\overset{*}{p}\in\overset{*}{Y}}\left\langle 0,\overset{*}{p}\right\rangle-\overset{*}{\Phi}_u(\overset{*}{p}) = \sup_{\overset{*}{p}\in\overset{*}{Y}} -\overset{*}{\Phi}_u(\overset{*}{p}) = \sup_{\overset{*}{p}\in\overset{*}{Y}} L(u,\overset{*}{p}) \Longrightarrow P: \inf_{u\in V}\sup_{\overset{*}{p}\in\overset{*}{Y}} L(u,\overset{*}{p})$

Definition: (Saddle point)

$(\overline{u},\overline{\overset{*}{p}}) \in V\times\overset{*}{Y}$ is called a saddle point of L if $L(\overline{u},\overset{*}{p}) \leq L(\overline{u},\overline{\overset{*}{p}}) \leq L(u,\overline{\overset{*}{p}})$ for all $u\in V,\overset{*}{p}\in\overset{*}{Y}$.

Lemma2:$(\overline{u},\overline{\overset{*}{p}}) \in V\times\overset{*}{Y}$ is called a saddle point of L iff $\overline{u}$ is a solution of P and $\overline{\overset{*}{p}}$ is a solution of $\overset{*}{P}$ and $\inf P = \sup\overset{*}{P}$

Proof:

($\Longrightarrow$)assume $(\overline{u},\overline{\overset{*}{p}})$ is a saddle point $\Longrightarrow \Phi(\overline{u},0) = \sup_{\overset{*}{p}\in\overset{*}{Y}} L(\overline{u},\overset{*}{p}) \leq L(\overline{u},\overline{\overset{*}{p}}) \leq \inf_{u\in V} L(u,\overline{\overset{*}{p}}) = -\overset{*}{\Phi}(0,\overline{\overset{*}{p}}) \Longrightarrow \Phi(\overline{u},0)+$ $\overset{*}{\Phi}(0,\overline{\overset{*}{p}}) \leq 0$ but $\Phi(\overline{u},0)+\overset{*}{\Phi}(0,\overline{\overset{*}{p}}) \geqslant 0$

$\Longrightarrow \Phi(\overline{u},0)+\overset{*}{\Phi}(0,\overline{\overset{*}{p}}) = 0$ and we get the extremality condition, so $\inf P = \sup\overset{*}{P}$.

($\Longleftarrow$) assume $\overline{u}$ is a solution of P and $\overline{\overset{*}{p}}$ is a solution of $\overset{*}{P}$ and $\inf P = \sup\overset{*}{P}$

$\Phi(\overline{u},0) = \sup_{\overset{*}{p}\in\overset{*}{Y}} L(\overline{u},\overset{*}{p}) \geqslant L(\overline{u},\overline{\overset{*}{p}}) \geqslant \inf_{u\in V} L(u,\overline{\overset{*}{p}}) = -\overset{*}{\Phi}(0,\overline{\overset{*}{p}})$

$L(u,\overline{\overset{*}{p}}) \geqslant \inf_{u\in V} L(u,\overline{\overset{*}{p}}) = L(\overline{u},\overline{\overset{*}{p}}) = \sup_{\overset{*}{p}\in\overset{*}{Y}} L(\overline{u},\overset{*}{p}) \geqslant L(\overline{u},\overset{*}{p})$ and hence, $(\overline{u},\overline{\overset{*}{p}})$ is a saddle point.

18 Lecture 18

$J(u,p): V \times Y \to \overline{R}, \quad A \in \mathcal{L}(Y,Y)$

Define $F: V \to \overline{R}$ by $F(u) = J(u, Au)$
p $\inf_{u\in V} F(u)$
P $\inf_{u\in V} J(u, Au)$
Define $\Phi: V \times Y \to \overline{R}$ by $\Phi(u,p) = J(u, Au - p)$

Clearly if J is convex, then Φ is convex.
If $J \in \Gamma_0(V \times Y)$, then $\Phi \in \Gamma_0(V \times Y)$
To show that $\Phi(u,p) = J(u, Au-p)$ is l.s.c. we have
$\underline{\lim}_{(u,p)\to(u_0,p_0)} \Phi(u,p) = \underline{\lim}_{(u,p)\to(u_0,p_0)} J(u, Au-p)$ (note if we put $w = Au - p \Rightarrow w_\circ = Au_\circ - p_\circ$ and as $(u,p) \to (u_\circ, p_\circ)$ we have by continuity of A that $(u,w) - (u_\circ, w_\circ)$). So we get:
$\underline{\lim}_{(u,w)\to(u_0,w_0)} J(u,w) = J(u_0, w_0) = \Phi(u_0, w_0)$

The dual problem:
$\Phi(u^*,p) = \sup_{(u,p)} (\langle u,u^*\rangle + \langle p,p^*\rangle - J(u, Au-p))$ (set $q = Au - p$)
$= \sup_u \sup_q \langle u,u^*\rangle + \langle Au - q, p^*\rangle - J(u,q)$
$= \sup_u \sup_q \langle u, u^* + A^*p^*\rangle + \langle q, -p^*\rangle - J(u,q)$
$= J^*(u^* + A^*p^*, -p^*) \quad \Rightarrow \quad \Phi^*(0,p^*) = J^*(A^*p^*, -p^*)$
So the dual problem can be written as :
P* : $\sup_{P^*\in Y^*} - J^*(A^*p^*, -p^*)$

Stablitiy:

If inf P=h(p) is finit and $J(u_0,\cdot)$ is bounded above in a nbhd of 0, then P is stable, and $inf\ P = sup\ P^*$ and P^* has solutions.
Existence:
If V is a reflexive Banach space, $J(u, Au) \to \infty$ as $\|u\| \to \infty$. Then P has solutions

Extremality:

$\overline{u}$ is a solution of P and $\overline{p}^*$ is a solution of P* iff $J(\overline{u}, A\overline{u}) + J^*(A^*\overline{p}^*, -\overline{p}^*) = 0$ iff $(A^*\overline{p}^*, -\overline{p}^*) \in \partial J(\overline{u}, A\overline{u})$

Note: $F(u) + F^*(u^*) = \langle u, u^*\rangle$ iff $u^* \in \partial F(u)$
$\langle (\overline{u}, A\overline{u}), (A^*\overline{p}^*, -\overline{p}^*)\rangle = \langle \overline{u}, A^*\overline{p}^*\rangle + \langle A\overline{u}, -\overline{p}^*\rangle = 0$

Lagragian of P:
$-L(u,p^*) = \sup_{p\in Y}(\langle p,p^*\rangle - J(u, Au-p)) = \sup_{q\in Y}\langle Au - q, p^*\rangle - J(u,q) = \langle Au, p^*\rangle - \sup_{q\in Y}\langle q, -p^*\rangle - J_u(q) = \langle Au, p^*\rangle + J_u^*(-p^*)$

If $J(u,p) = F(u) + G(p)$
$J^*(u^*,p^*) = F^*(u^*) + G^*(p^*) = J^*(u^*,p^*) = \sup_{(u,p)}(\langle u,u^*\rangle + \langle p,p^*\rangle - J(u,p)) = \sup_u \sup_p(\langle u,u^*\rangle + \langle p,p^*\rangle - F(u) - G(p))$
$= F^*(u^*) + G^*(p^*)$

Lecture 18

Transcribed by:
Abdulkhaleg Al-Baiayt

P: $\inf_{u \in V} F(u) + G(Au)$

P*: $\sup_{p^* \in Y^*} -[F^*(A^*p^*) + G^*(-p^*)]$

Stability:
inf P is finite, $F(u_0) + G(\cdot)$ is bounded in a nbhd of Au_0

Exsistance:
V is reflexive Banach-space
$F(u) + G(Au) \to \infty$ as $\|u\| \to \infty$

Extremality:
$\overline{u}$ is a solution of P and $\overline{p}^*$ is a solution of P* iff

$J(\overline{u}, A\overline{u}) + J^*(A^*\overline{p}^*, -\overline{p}^*) = 0$
$F(\overline{u}) + G(A\overline{u}) + F^*(A^*\overline{p}^*) + G^*(-\overline{p}^*) = 0$
$[F(\overline{u}) + F^*(A^*p^*)] + [G(A\overline{u}) + G^*(-\overline{p}^*)] = 0$
$[F(\overline{u}) + F^*(A^*\overline{p}^*) - \langle \overline{u}, A^*\overline{p}^* \rangle] + [G(A\overline{u}) + G^*(-\overline{p}^*) - \langle A\overline{u}, -\overline{p}^* \rangle] = 0$
$\therefore F(\overline{u}) + F^*(A^*\overline{p}^*) = \langle \overline{u}, A^*\overline{p}^* \rangle$ and $G(A\overline{u}) + G^*(-\overline{p}^*) = \langle A\overline{u}, \overline{p}^* \rangle$ iff $A^*\overline{p}^* \in \partial F(\overline{u})$ and $-\overline{p}^* \in \partial G(A\overline{u})$

Now :
If $Y = \prod_1^m Y_i \qquad Y^* = \prod_1^m Y_i^*$
$p \in Y \quad \to \quad p = (p_1, p_2, ..., p_m)\ , \qquad p_i \in Y_i$
$G(p) = \sum_1^m G_i(p_i) \qquad G_i : Y_i \quad \to \overline{R}$
$A : V \to Y$
$Au = (A_1 u, A_2 u, ..., A_m u)$

The extremality condition takes the form:
$\overline{u}$ is a solution of P, $\overline{p}_i^*$ is a solution of P* iff

$$F(\overline{u}) + F^*(A^*\overline{p}^*) + \sum_1^m G_i(A_j\overline{u}) + \sum_1^m G_i^*(-\overline{p}_i^*) = 0$$

$F(\overline{u}) + F^*(A^*\overline{p}^*) = \langle \overline{u}, A^*\overline{p}^* \rangle,$
$G_i(A_i\overline{u}) + G_i^*(-\overline{p}_i^*) = \langle A_i\overline{u}, -\overline{p}_i^* \rangle, \qquad i = 1, 2, ..., m$

end of lec#18

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19 Lecture 19

Important Special Cases II

DEFINITION 49
Let C be a subset of a linear space Y, then C is

- a cone if $\lambda C \subset C$ for all $\lambda > 0$.
- a pointed cone if it is a cone containing zero.
- a salient cone if it is a pointed cone with $C \cap (-C) = \{0\}$.

DEFINITION 50
A cone C of a linear space Y induces a partial ordering defined by $p \geq 0$ iff $p \in C$.

This means if $p \leq q$, then $q - p \in C$. If C is salient, then $\leq$ is an ordering relation. If $\leq$ is an ordering relation on Y compatible with the linear structure of Y (That is: $\lambda p \leq \lambda q, \forall \lambda > 0$ and $p + v \leq q + v, \forall v \in Y$ if $p \leq q$). Then $\{p \in Y : p \geq 0\}$ is a salient pointed cone.

DEFINITION 51
The polar cone of a cone C is the set

$$C^* = \{p^* \in Y^* : \langle p^*, p\rangle \geq 0 \forall p \in C\}$$

LEMMA 52
If C is a convex pointed cone, then

(i) C^* is closed (in $\sigma(Y^*, Y)$).

(ii) $C^{**} = C$.

(iii) $p \in C$ iff $p \in C$ iff $\langle p^*, p\rangle \geq 0$ for all $p^* \in C^*$.

Proof.

1. To show that C^* is closed, we write

$$\begin{aligned} C^* &= \{p^* \in Y^* : \langle p^*, p\rangle \geq 0 \forall p \in C\} \\ &= \bigcap_{p \in C} \{p^* \in Y^* : \langle p^*, p\rangle \geq 0\} \\ &= \bigcap_{p \in C} \{p^{-1}[0, \infty)\} \end{aligned}$$

   Since p is continuous in the topology $\sigma(Y^*, Y)$; C is closed.

2. $C \subset C^{**}$ is clear. To show that $C^{**} \subset C$; let $q \in C^{**}$, then $\langle q, p^*\rangle \geq 0$ for all $p^* \in C^*$. Assume that $q \notin C$, so there exists $x \neq 0 \in Y^*$ such that $\langle x, p\rangle \geq \alpha$ for all $p \in C$ and $\alpha \in I\!R$ and $\langle x, q\rangle < \alpha$. Since $0 \in C$, then $\alpha \leq 0$. Hence $\langle x, q\rangle < 0$, but this can not happen; since $x \in C^*$. To show that, assume otherwise then there exists $p' \in C$ such that $\langle x, p'\rangle < 0 \Rightarrow \lambda\langle x, p'\rangle = \langle x, \lambda p'\rangle < 0$. But for sufficiently small λ, we have $\langle x, \lambda p'\rangle < \alpha$ which is a contradiction. So $x \in C^*$, but again this is a contradiction. Thus $q \in C$.

3. $p \in C \Rightarrow p \geq 0 \Rightarrow \langle p^*, p\rangle \geq 0 \forall p^* \in C^* \Rightarrow p \in C^{**} = C$.

■

The problem considered

Let $\phi \neq A \subset V$ be closed and convex, $J : V \longrightarrow I\!R$ convex and lsc, C closed convex cone in Y, $\leq$ the partial ordering induced by C. $B : A \longrightarrow Y$ satisfy the following:

(B1) B is convex with respect to $\leq$.

(B2) For each $p^* \in C^*$, $\langle p^*, B(\cdot)\rangle : A \longrightarrow I\!R$ is lsc.

(B3) The set $\{u \in A : B(u) \leq 0\} \neq \phi$.

Primal problem

$$\inf_{\substack{u \in A \\ Bu \leq 0}} J(u)$$

Perturbation problem

$$\Phi(u,p) = \begin{cases} J(u) & \text{if } u \in A, Bu \leq p, \\ +\infty & \text{otherwise .} \end{cases}$$

LEMMA 53
The set $\mathcal{E} = \{(u,p) \in V \times Y : u \in A, Bu \leq p\}$ is closed and convex.

Proof.

$$\begin{aligned} \mathcal{E} &= \{(u,p) \in V \times Y : u \in A, \langle p^*, Bu - p\rangle \leq 0 \forall p^* \in C\} \\ &= \bigcap_{p^* \in C^*} \{(u,p) \in V \times Y : u \in A, \langle p^*, Bu - p\rangle \leq 0\} \cap (A \times Y) \end{aligned}$$

which is closed; since $u \longmapsto \langle p^*, Bu - p\rangle$ is lsc by (B2). To show the convexity of C, let $(u,p), (v,q) \in \mathcal{E}$ where $u, v \in A$ and $p, q \in Y$ and $\lambda \in [0,1]$. Then

$$\lambda(u,p) + (1-\lambda)(v,q) = (\lambda u + (1-\lambda)v, \lambda p + (1-\lambda)q)$$

Since A is convex $\lambda u + (1-\lambda)v \in A$. Now B is convex

$$B[\lambda u + (1-\lambda)v] \leq \lambda Bu + (1-\lambda)Bv \leq \lambda p + (1-\lambda)q.$$

Hence $\lambda u + (1-\lambda)v \in \mathcal{E}$ which proves that $\mathcal{E}$ is convex. ■
We can rewrite ϕ as

$$\phi(u,p) = \hat{J}(u) = \chi_{\mathcal{E}} \quad \text{where } \hat{J}(u) = \begin{cases} J(u), & u \in A \\ +\infty, u \notin A \end{cases}$$

PROPOSITION 54
$\phi \in \Gamma_0(V \times Y)$

1. ϕ does not take the value $-\infty$.
2. $\phi \not\equiv +\infty (\phi(u,0) < +\infty)$.
3. ϕ is convex.
4. ϕ is lsc.

20 Lecture 20

Important Special Case (II)

The dual problem

For $p^* \in Y$,

$$\Phi^*(0,p^*) = \sup_{u\in V} \sup_{p\in Y} < p,p^* > -\Phi(u,p)$$
$$= \sup_{u\in V} \sup_{p\in Y} < p,p^* > -\hat{J}(u) - \chi_\epsilon(u,p)$$
$$= \sup_{u\in A} \sup_{Bu\leq p} < p,p^* > -J(u);$$

Let $q = p - Bu$,we get

$$\Phi^*(0,p^*) = \sup \ \sup < q + Bu, p^* > -J(u)$$
$$= \sup_{u\in A} \sup_{q\geq 0} < q,p^* > + < Bu,p^* > -J(u)$$
$$= \sup_{u\in A} \ < Bu,p^* > -J(u) + \sup_{q\geq 0} < q,p^* >$$
$$= \sup_{u\in A} \ < Bu,p^* > -J(u) + \chi_{C^*}(-p),$$

then,

$$-\Phi^*(0,p^*) = \inf_{u\in A} \ - < Bu,p^* > +J(u) - \chi_{C^*}(-p),$$

Thus the dual problem is

$$P^* \qquad \sup_{p^*\in Y^*} \inf_{u\in A} \ - < Bu,p^* > +J(u) - \chi_{C^*}(-p)$$
$$\sup_{p^*\leq 0} \inf_{u\in A} \ - < Bu,p^* > +J(u).$$

Stability

$\inf P \in \mathbf{R}$,for some $u_0 \in A$, $Bu \in -C^\circ$(the interior of C).Then P is stable.

Existence

Assume V is a reflexive Banach space, $J(u) \to \infty$ as $\|u\| \to \infty, u \in A$, Then P has a solution.

Extremality

$$\inf P = \sup P^*,$$

the extremality relation

$$< B\bar{u}, \bar{p}^* >= 0.$$

because:

$$\inf P = J(\bar{u}), \qquad \bar{u} \in A,\ B\bar{u} \leq 0,$$

(*) $$\sup P^* = \inf_{u\in A} - < Bu, \bar{p}^* > J(u)\ , \qquad \bar{p}^* < 0$$

and

(**) $$J(\bar{u}) = \inf_{u\in A} - < Bu, \bar{p}^* > +J(u) \leq - < B\bar{u}, \bar{p}^* > +J(\bar{u})$$

then we have

$$< B\bar{u}, \bar{p}^* >\leq 0,$$

from (*) and (**) we have

$$< B\bar{u}, \bar{p}^* >\geq 0,$$

Then,we have the extremality relation

$$< B\bar{u}, \bar{p}^* >= 0.$$

The Lagrangian

$$-L(u,p^*) = \sup_{p\in Y} < p,p^* > -\Phi(u,p)$$

$$= \sup_{p\in Y} < p,p^* > -\hat{J}(u) - \chi_\epsilon(u,p)$$

$$= -\hat{J}(u) + \sup_{Bu\le p} < p,p^* >$$

$$= -\hat{J}(u) + \sup_{q\ge 0} < Bu,p^* > - < q,p^* >$$

$$= -\hat{J}(u) + < Bu,p^* > + \chi_{C^*}(-p^*).$$

Then,

$$L(u,p^*) = \hat{J}(u) - < Bu,p^* > - \chi_{C^*}(-p^*).$$

Proposition $(\bar{u},\bar{p}^*) \in V \times Y^*$ is a saddle point of L if and only if $\bar{u} \in A, \bar{p}^* < 0$,and

((1)) $$J(\bar{u}) - < B\bar{u},p^* > \le J(\bar{u}) - < B\bar{u},\bar{p}^* > \le J(u) - < Bu,\bar{p}^* >, \quad \forall u \in A, \forall p^* \le 0.$$

Proof: assume $(\bar{u},\bar{p}^*)$ is a saddle point of L, (let $u \in A$ and $p^* \le 0$)

$$- < B\bar{u},p^* > + \hat{J}(\bar{u}) - \chi_{C^*}(-p^*) \le - < B\bar{u},\bar{p}^* > + \hat{J}(\bar{u}) - \chi_{C^*}(-\bar{p}^*)$$

$$\le - < Bu,\bar{p}^* > + \hat{J}(u) - \chi_{C^*}(-\bar{p}^*),$$

then

$$-\infty < - < B\bar{u},p^* > + \hat{J}(\bar{u}) \le - < B\bar{u},\bar{p}^* > + \hat{J}(\bar{u}) - \chi_{C^*}(-\bar{p}^*)$$

$$\le - < Bu,\bar{p}^* > + \hat{J}(u) - \chi_{C^*}(-\bar{p}^*),$$

the left most and right most parts of the inequalities give $\bar{p}^* \le 0$,and the second and the third parts give $\bar{u} \in A$.

$$- < B\bar{u},p^* > + \hat{J}(\bar{u}) \le - < B\bar{u},\bar{p}^* > + \hat{J}(\bar{u}) \le - < Bu,\bar{p}^* > + \hat{J}(u).$$

Assume $\bar{u} \in A$ and $\bar{p}^* \le 0$ and (1) is satisfied,

$$L(\bar{u},\bar{p}^*) = - < B\bar{u},\bar{p}^* > + \hat{J}(\bar{u}),$$

$$L(u,\bar{p}^*) = - < Bu,\bar{p}^* > + \hat{J}(u),$$

$$L(\bar{u},p^*) = - < B\bar{u},p^* > + \hat{J}(\bar{u}) - \chi_{C^*}(-p^*),$$

then

$$L(\bar{u},p^*) \le L(\bar{u},\bar{p}^*) \le L(u,\bar{p}^*),$$

then $(\bar{u},\bar{p}^*)$ is a saddle point of L.

Kuhn-Tucker theorem $V = V^* = \mathbf{R}^n, Y = Y^* = \mathbf{R}^m, A \subseteq \mathbf{R}^n$ is closed convex set.

$$J : A \to \mathbf{R}, \quad \text{convex and } l.s.c.$$

the cone C,

$$C = \{p \in \mathbf{R}^m : p_i \ge 0, i = 1,2,...,m\}.$$

$C^* = C$,
the function $B : A \to \mathbf{R}^m$ is defined by $Bu = (B_1u, B_2u, ..., B_mu)$, and

$$B_i : A \to \mathbf{R} \quad \text{convex and } l.s.c.$$

$$B_iu_0 < 0, \quad i = 1,2, ...m \text{ for some } u_0 \in A.$$

the primal problem is

$$P \quad \inf_{u\in A, Bu\leq 0} J(u)$$

$\bar{u} \in A$ is a solution of P iff there exists $\bar{p} \in \mathbf{R}^m, \bar{p} \leq 0$ such that $(\bar{u}, \bar{p})$ is a saddle point of L, in this case

$$\sum_{i=1}^{m} p_i B_i \bar{u} = 0,$$

note that P is stable, if $\bar{u}$ is a solution of P therefor P^* has a solution $\bar{p} \leq 0$,and $(\bar{u}, \bar{p})$ is a saddle point of L.On the other hand if $\bar{p} \leq 0$ such that $(\bar{u}, \bar{p})$ is a saddle point of L, $\bar{u}$ is a solution of P. By the previous proposition, $\bar{u} \in A$.

$$\bar{p} \leq 0 \Rightarrow \bar{p}_i \leq 0 \quad \forall i$$

$$B\bar{u} \leq 0 \Rightarrow B_i\bar{u} \leq 0 \quad \forall i$$

$$\sum_{i=1}^{m} p_i B_i \bar{u} = 0 \Rightarrow p_i B_i \bar{u} = 0,$$

if $B_i\bar{u} < 0$ then $p_i = 0$ and if $p_i < 0$ then $B_i\bar{u} = 0$.

21 Lecture 21

Applications of Duality to the calculus of varitions

Preliminaries

Let $\Omega \subseteq \mathbb{R}^n$ be open, sometimes we require reqularity on Ω.
Regularity: Ω is said to be of class C^r if the boundary Γ is an $r-$times continuously differential maiifold of dimention $(n-1)$ and Ω lies locally in one side of Γ.
For $x \in \Gamma$, $\nu(x) = (\nu_1(x), \nu_2(x), ..., \nu_n(x))$ will denote the outward normal to Ω.

Differentiation, Multiindex Notation.
for $j = (j_1, j_2, ..., j_n) \in \mathbb{N}^n$,
$D^j u = D^j D^{j_2} ... D^{j_n} u = \frac{\partial^{|j|}}{\partial x_1^{j_1} \partial x_2^{j_2} ... \partial x_n^{j_n}}$ where $|j| = j_1 + j_2 + ... + j_n$.
Examle: let $j = (1, 2, 4, 0) \in \mathbb{N}^4$,
$D^j u = \frac{\partial^7 u}{\partial x_1 \partial x_2^2 \partial x_3^4}$

Remark: $D^{(0,0,...,0)} = I$
Space $L^\alpha(\Omega), 1 \leq \alpha < \infty$
$L^\alpha(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R} : \int\limits_\Omega |u(x)|^\alpha \, dx < \infty \right\}$is a Banach space under the norm $\|u\|_{L^\alpha(\Omega)} = \left(\int\limits_\Omega |u(x)|^\alpha \, dx \right)^{\frac{1}{\alpha}}$.
Space $L^\infty(\Omega)$
$L^\infty(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R} : Ess.\sup\limits_{x \in \Omega} |u(x)| < \infty \right\}$ is a Banach space under the norm $\|u\|_{L^\infty(\Omega)} = Ess.\sup\limits_{x \in \Omega} |u(x)|$

The Dual spaces of $L^\alpha(\Omega)$
$(L^\alpha(\Omega))^* = L^{\alpha'}(\Omega)$ where $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$
Special case: if $\alpha = 2 \Longrightarrow \alpha' = 2, L^2(\Omega)$ is a Hilbert space with inner product $\langle u, v \rangle = \int\limits_\Omega u(x) v(x) dx$
The Soblev Spaces $\overset{m,\alpha}{w}(\Omega), \overset{m,\alpha}{w_0}(\Omega)$ where $1 \leq \alpha < \infty$ and $m \geqslant 1$ is an integer.
$\overset{m,\alpha}{w}(\Omega) = \{ u \in L^\alpha(\Omega) : D^k u \in L^\alpha(\Omega), |k| \leq m \}$ is a Banach Space under the norm $\|u\|_{\overset{m,\alpha}{w}(\Omega)} = \left(\sum\limits_{|j| \leq m} \int\limits_\Omega \left| D^j u(x) \right|^\alpha dx \right)^{\frac{1}{\alpha}}$
$\overset{m,\alpha}{w_0}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the norm of $\overset{m,\alpha}{w}(\Omega)$.
The Trace Operator : suppose $\Omega \in C^{m+2}$
The operator $\gamma : (\gamma_0, \gamma_1, ..., \gamma_{m-1}) : \overset{m,\alpha}{w}(\Omega) \longrightarrow L^\alpha(\Gamma)$ defined by
$\gamma_0 u = u|_\Gamma, \gamma_1 u = \frac{\partial u}{\partial \nu}|_\Gamma ....\gamma_{n-1} u = \frac{\partial^{m-1} u}{\partial \nu^{m-1}}|_\Gamma$where $\frac{\partial u}{\partial \nu} = \nabla u . \nu|_\Gamma$and $\frac{\partial^k u}{\partial \nu^k} = \frac{\partial}{\partial \nu} \frac{\partial^{k-1} u}{\partial \nu^{k-1}}|_\Gamma = \nabla \left(\frac{\partial^{k-1} u}{\partial \nu^{k-1}} \right) .\nu|_\Gamma$is called the Trace Operator.
γ is linear and continuous operator, also $Ker \ \gamma = \overset{m,\alpha}{w_0}(\Omega)$

Poincare' Inequality (assume Ω to be bounded)

For all $u \in \overset{1,\alpha}{w_0}(\Omega)$, $\|u\|_{L^\alpha(\Omega)} \leq c \left\| D^i u \right\|_{L^\alpha(\Omega)}$ where c is a constant depends on Ω and α. i.e $c(\Omega, \alpha)$.
Green's Formula (Integration by Parts)
let $u \in \overset{1,\alpha}{w}(\Omega)$ and $v \in \overset{1,\alpha'}{w}(\Omega)$, then $\int\limits_\Gamma u v \nu_i d\,\Gamma = \int\limits_\Omega (u D_i v + v D_i u) \, dx$ (1) where ν_i is the i_{th} component of ν.
if we replace v by $D_i v$ in (1)
$\int\limits_\Gamma u \, D_i v \, \nu_i d\Gamma = \int\limits_\Omega \left(u D_i^2 v + D_i v D_i u \right) dx$, sum for $i = 1, 2, ..., n$, we get $\int\limits_\Gamma u \, \frac{\partial v}{\partial \nu} d\,\Gamma = \int\limits_\Omega (u \bigtriangleup v + \nabla u . \nabla v) \, dx$
also if interchanged u and v we get $\int\limits_\Gamma v \, \frac{\partial u}{\partial \nu} d\,\Gamma = \int\limits_\Omega (v \bigtriangleup u + \nabla u . \nabla v) \, dx$ subtracting we get, $\int\limits_\Gamma (u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu}) d$ $\Gamma = \int\limits_\Omega (u \bigtriangleup v - v \bigtriangleup u) \, dx$
also, replace v by v_i in (1)v $\Longrightarrow \int\limits_\Gamma u \, v_i \, \nu_i d\,\Gamma = \int\limits_\Omega (u D_i v_i + v_i D_i u) \, dx$ or $\int\limits_\Gamma u \, v . \, \nu d\,\Gamma = \int\limits_\Omega (u \nabla . v + v . \nabla u) \, dx$.

22 Lecture 22

Carathéodory Mappings

DEFINITION 55 (CARATHÉODORY MAPPINGS)
Let $\Omega \in I\!R^m$ be an open Borel set [3], E and F Banach spaces, $g : \Omega \times E \longrightarrow F$. g is called a Carathéodory mapping if

1. $g(\cdot, \zeta)$ is measurable for each $\zeta \in E$.
2. $g(x, \cdot)$ is continuous for almost all $x \in \Omega$.

Let $\mathcal{M}(\Omega, E)$ be the set of measurable functions $u : \Omega \longrightarrow E$, $\mathfrak{m}(\Omega, F)$ the set of measurable functions $v : \Omega \longrightarrow F$. Define $K : (\Omega, F) \longrightarrow \mathfrak{m}(\Omega, F)$ by

$$(Ku)(x) = g(x, u(x))), \qquad x \in \Omega$$

PROPOSITION 56
If $K : L^p(\Omega, E) \longrightarrow L^r(\Omega, F)$. Then K is continuous [4]
with respect to the norms of $L^p(\Omega, E), L^r(\Omega, F)$.

For $E = I\!R^m, F = R, u = \Omega \longrightarrow I\!R^m[u(x) = (u_1(x), u_2(x), \cdots, u_n(x))]$, assume $u \in L^{\alpha_1} \times L^{\alpha_2} \times \cdots \times L^{\alpha_n} = V$. Also assume $Ku(x) = g(x, u(x))$ maps V into $L'(\Omega)$. We can then define $G : V \longrightarrow I\!R$ by

$$G(u) = \int_\Omega Ku(x)dx = \int_\Omega g(x, u(x))dx$$

The conjugate function $G^* : V^* \longrightarrow R$ where

$$V^* = L^{\alpha_1'} \times L^{\alpha_2'} \times \cdots \times L^{\alpha_n'}$$

where $\frac{1}{\alpha_i} + \frac{1}{\alpha_i'} = 1$ for all i is given through the following proposition.

PROPOSITION 57

$$G^*(u^*) = \int_\Omega g^*(x, u^*(x))dx$$

where

$$g^*(x, y) = \sup_{\eta \in I\!R^m} \eta \cdot y - g(x, u)$$

First Examples
$\Omega \subseteq I\!R$ open, given $f \in L^2(\Omega)$,

$$\begin{aligned} -\Delta u &= f \\ u &= 0 \quad \text{on } \Gamma \end{aligned}$$

Variational Form
$V = H_0^1(\Omega)$, let $v \in V$

$$\int_\Omega -\Delta uvdx = \int_\Omega fudx$$

[3] For any topological space X, the Borel sigma algebra of X is the σ–algebra $\mathcal{B}$ generated by the open sets of X. In other words, the Borel sigma algebra is equal to the intersection of all sigma algebras $\mathcal{A}$ of X having the property that every open set of X is an element of $\mathcal{A}$. An element of $\mathcal{B}$ is called a Borel subset of X, or a Borel set.

[4] Given $\epsilon > 0, \exists \delta > 0$ such that for all $u, v \in L^p(\Omega, E)$ we have

$$||u - v||_{L^p(\Omega,E)} \leq \delta \Rightarrow ||Ku - Kv||_{L^r(\Omega,F)} \leq \epsilon$$

That is

$$\left(\int_\Omega ||u(x) - v(x)||_E^p dx\right)^p \leq \delta \Rightarrow \left(\int_\Omega ||Ku(x) - Kv(x)||_F^r dx\right)^r \leq \epsilon$$

Lecture 22

$\langle \nabla u, \nabla v \rangle = \langle f, u \rangle$ for all $v \in V$. This is equivalent to

$$\min \frac{1}{2}||\nabla u||^2 - \langle f, u \rangle$$

Side Notes:

- Green's Form
$$\int_\Omega \nabla u \nabla v dx = \int_\Omega f u dx$$
- $\langle u, v \rangle + \sum \langle p_i u, p_i v \rangle = \langle u, v \rangle + \langle \nabla u, \nabla v \rangle$.
- To find the Gâteaux derivative of $F(u)$, we evaluate
$$\frac{d}{dt} F(u + tv)\Big|_{t=0}$$

So

$$\frac{1}{2}||\nabla(u, tv)||^2 - \langle f, u + tv \rangle = \frac{1}{2}||\nabla u||^2 + t\langle u, \nabla v \rangle + \frac{1}{2}t^2||\nabla v||^2 - \langle f, u \rangle - \langle f, tv \rangle$$

Differentiating

$$\langle \nabla u, \nabla v \rangle + t||\nabla u||^2 - \langle f, v \rangle\Big|_{t=0} = \langle \nabla u, \nabla v \rangle - \langle f, v \rangle = \min J(u)$$

where

$$J(u) = -\langle f, u \rangle + \frac{1}{2}||\nabla u||^2 = F(u) + G(Au)$$

That is

$$F(u) = -\langle f, u \rangle, \qquad Au = \nabla u, \qquad G(p) = \frac{1}{2}||p||^2$$

Now, we have $V = H_0^1(\Omega), Y = [L^2(\Omega)]^n = Y^*, A : V \longrightarrow Y$ and $V^* = H^{-1}(\Omega)$ (just the dual space of V). Also

$$\phi(u, p) = F(u) + G(Au - p)$$

which belongs to $\Gamma_0(V \times Y)$; since F is convex and G is convex and continuous. We now find the dual problem; so we need to find first F^*.

$$F^*(u^*) = \sup_{u \in V} \langle u, u^* \rangle + \langle f, u \rangle = \sup_{u \in V} \langle u, u^* + f \rangle = \begin{cases} 0, & \text{if } u + f = 0 \\ +\infty & \text{otherwise} \end{cases}.$$

Then G^*. Since $G(p) = \frac{1}{2}\int_\Omega ||p(x)||^2 dx$, we have

$$G^*(p^*) = \int_\Omega \left(\frac{1}{2}|p(x)|^2\right)^* dx$$

To find $\left(\frac{1}{2}|p(x)|^2\right)^*$ let us define $g : \Omega \times I\!R^n \longrightarrow I\!R$ by

$$g(x, y) = \frac{1}{2}||y||^2$$

Then

$$g^*(x, y) = \sup_{\eta \in I\!R^n} \eta y - \frac{1}{2}|y|^2$$

To find the supremum, we shall find the derivative, then equate with zero. Let $\tilde{F}(\eta) = \eta y - \frac{1}{2}|\eta|^2$, then

$$\tilde{F}(\eta + t\zeta) = (\eta + t\zeta) \cdot y - \frac{1}{2}|\eta + t\zeta|^2 = \eta y + t\zeta y - \frac{1}{2}(|\eta|^2 + 2t\eta\zeta + t^2|\zeta|^2)$$

Therefore,

$$\begin{aligned} \frac{d}{dt}\tilde{F}(\eta + t\zeta)\Big|_{t=0} &= 0 \\ \eta y - \eta\zeta - t|\zeta|^2\Big|_{t=0} &= \zeta y - \zeta\eta = \zeta(y-\eta) = 0, \quad \forall\, \zeta \in I\!R^n \end{aligned}$$

So for $\eta = y$ we get

$$g^*(x,y) = |y|^2 - \frac{1}{2}|y|^2 = \frac{1}{2}|y|^2$$

$$\therefore G^*(p^*) = \int_\Omega \frac{1}{2}|p^*(x)|^2 dx = \frac{1}{2}||p^*(x)||^2$$

Let us find $A^* : Y^* \longrightarrow V^*$

$$\langle Au, p\rangle = \langle \nabla u, p\rangle = \int_\Omega \nabla u \cdot p dx \overset{\text{Green's}}{=} -\int_\Omega u \nabla p\, dx = \langle u, A^* p\rangle$$

So,

$$A^* p = -\nabla \cdot p$$

Summary:

$F(u)$	$=$	$-\langle f, u\rangle$
$F^*(u^*)$	$=$	$\begin{cases} 0 & u^* = -f \\ +\infty & \text{otherwise} \end{cases}$
$G(p)$	$=$	$\frac{1}{2}\vert\vert p\vert\vert^2$
$G^*(p^*)$	$=$	$\frac{1}{2}\vert\vert p^*\vert\vert^2$
$A(u)$	$=$	∇u
$A^*(p)$	$=$	$\nabla \cdot p$

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Dirichlet Problem:

$-\triangle u = f \quad$ on Ω
$u = 0 \quad$ on Γ
$\inf\ (\frac{1}{2}\| \nabla u \|^2 - \langle f, u\rangle)$
$V = H_0^1(\Omega), \quad Y = L^2(\Omega)^n, \quad V^* = H_0^{-1}(\Omega), \quad Y^* = Y$
$F : V \to R$ is defined by $F(n) = -\langle f, u\rangle$

$$F^*(u^*) = \begin{cases} 0 & \text{if } u^* = -f \\ \infty & \text{other wise} \end{cases}$$

$G(p) = \frac{1}{2}\|p\|^2$

$G^*(p^*) = \frac{1}{2}\|p^*\|^2$

Now P is given as:
$\inf\ \frac{1}{2}\| \nabla u \|^2 - \langle f, u\rangle$
where $J(u,p) = \frac{1}{2}\| \nabla u \|^2 - \langle f, u\rangle$ is continouos, coercive (by Poin Care' inequality), and strictly convex which implies that P has a unique solution and P is stable $\Rightarrow$ P^* has a solution and $inf\ P = sup\ P^*$.

Also,
$\Phi(0, p^*) = J^*(A^*p^*, -p^*) = F^*(A^*p^*) + G^*(-p^*)$
$\Rightarrow P^*$ is given by: $\sup\limits_{p^* \in Y^*} -[F^*(A^*p^*) + G^*(-p^*)] = \sup\limits_{A^*p^*=-f} -G^*(-p^*) = \sup\limits_{A^*p^*=-f} -\frac{1}{2}\|p^*\|^2$
and since $p^* \to \|p^*\|^2$ is continouos, coercive, strictly convex, P^* has a unique solution.

Note here that we can find the clear relation between P and P^* For the extramility condition as follows:

$F(\overline{u}) + F^*(A^*\overline{p}^*) = \langle \overline{u}, A^*\overline{p}^*\rangle \Rightarrow -\langle f, \overline{u}\rangle = -\langle f, \overline{u}\rangle$ (trivial equation)
and $G(A\overline{u}) + G^*(-\overline{p}^*) = \langle A\overline{u}, \overline{p}^*\rangle \quad \Rightarrow \frac{1}{2}\|\overline{u}^*\|^2 + \frac{1}{2}\|\overline{p}^*\|^2 + \langle \overline{p}^*, \nabla\overline{u}\rangle = 0$
$\Rightarrow \|\nabla\overline{u} + \overline{p}^*\|^2 = 0 \quad \Rightarrow \quad \nabla\overline{u} = -\overline{p}^*$
$\inf P = \sup P^* = -G^*(-\overline{p}^*) = -\frac{1}{2}\|\overline{p}^*\|^2 = -\frac{1}{2}\|\nabla\overline{u}\|^2$

The nonlinear Dirichlet Problem:
$\inf\ (\frac{1}{\alpha}\| \nabla u \|^\alpha - \langle f, u\rangle)$
with $u \in W_0^{1,\alpha}(\Omega), \quad f \in W_0^{-1,\alpha'}(\Omega), \quad \frac{1}{\alpha} + \frac{1}{\alpha'} = 1 \quad and \quad 1 < \alpha < \infty$

Lemma:
let $f : R \to R$ be defined by $f(x) = \frac{1}{\alpha}\ |x|^\alpha$ then
$f^*(y) = \sup\limits_{x\in R} xy - \frac{1}{\alpha}\ |x|^\alpha = \frac{1}{\alpha'}\ |y|^{\alpha'}$ and the sup occurs at $\overline{x}$
where $\overline{x}\ |\overline{x}|^{\alpha-2} = y$

Proof: (EFS)

$V = W^{1,\alpha}(\Omega), \quad Y = L^\alpha(\Omega)^n, \quad Y^* = L^{\alpha'}(\Omega)^n, \quad V^* = W^{-1,\alpha'}(\Omega)$

$F(n) = -\langle f, u\rangle$

$$F^*(u^*) = \begin{cases} 0 & \text{if } u^* = -f \\ \infty & \text{other wise} \end{cases}$$

$G(p) = \frac{1}{\alpha}\|p\|^\alpha_{L^{\alpha'}(\Omega)^n}$

$G^*(p^*) = \frac{1}{\alpha'}\|p^*\|^{\alpha'}_{L^{\alpha'}(\Omega)^n}$ (to show)

Define: $g(\eta) = \frac{1}{\alpha} \mid \eta \mid^{\alpha} \Rightarrow g^*(\eta) = \sup_{\eta \in Y} \eta \cdot y - g(\eta)$

$$= \sup_{\eta \in Y} \eta \cdot y - \frac{1}{\alpha} \mid \eta \mid^{\alpha}$$
$$= \sup_{\eta \in Y} \eta \cdot y - \frac{1}{\alpha} \sum \mid \eta_i \mid^{\alpha}$$
$$= \sup_{\eta \in Y} \sum \eta_i y_i - \frac{1}{\alpha} \mid \eta_i \mid^{\alpha}$$

and by equating all partial derivative to zero we get:

$$y_i = \mid \eta_i \mid^{\alpha-1} \frac{\eta_i}{\mid \eta_i \mid} = \mid \eta_i \mid^{\alpha-2} \eta_i \quad \Rightarrow$$
$$g^*(y) = \frac{1}{\alpha'} \mid y \mid^{\alpha'}_{\alpha'}$$

i.e. $G^*(p^*) = \frac{1}{\alpha'}\|p^*\|^{\alpha'}_{L^{\alpha'}(\Omega)^n}$ #

and so P^* becomes:

$\sup_{A^* p^* = -f} - \frac{1}{\alpha'}\|p^*\|^{\alpha'}_{L^{\alpha'}(\Omega)^n}$ note here as exactly as before (coercivity, strict convexisty...

we have P has unique solution, and P^* is so.
end of lec#23

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The non-linear Dirichlet problem

$$P^* \qquad \inf \tfrac{1}{\alpha} \|u\|^\alpha - < f, u >$$

$$u \in V = W_0^{1,\alpha}(\Omega), \quad f \in V^* = W^{-1,\alpha\prime}(\Omega), \quad Y = L^\alpha(\Omega)^n, \quad Y^* = L^{\alpha\prime}(\Omega)^n$$

$$F(u) = -< f, u >, \quad G(p) = \tfrac{1}{\alpha} \|p\|^\alpha \quad G^*(p^*) = \tfrac{1}{\alpha\prime} \|p^*\|^{\alpha'},$$

Extremality

$$G(A\bar{u}) + G^*(-\bar{p}^*) + < \bar{p}^*, A\bar{u}) >= 0$$

$$\frac{1}{\alpha} \|A\bar{u}\|^\alpha + \frac{1}{\alpha\prime} \|\bar{p}^*\|^{\alpha'} + < \bar{p}^*, A\bar{u}) >= 0$$

$$\frac{1}{\alpha} \int_\Omega \sum |D_i\bar{u}|^\alpha + \frac{1}{\alpha'} \int_\Omega \sum |\bar{p}_i^*|^{\alpha'} + \int_\Omega \sum \bar{p}_i^* D_i\bar{u} = 0$$

$$\sum \int_\Omega \frac{1}{\alpha} |D_i\bar{u}|^\alpha + \frac{1}{\alpha'} |\bar{p}_i^*|^{\alpha'} + \bar{p}_i^* D_i\bar{u} = 0,$$

$\Rightarrow$

$$\frac{1}{\alpha} |D_i\bar{u}|^\alpha + \frac{1}{\alpha'} |\bar{p}_i^*|^{\alpha'} + \bar{p}_i^* D_i\bar{u} = 0, \qquad i = 1, 2, ..., n$$

then the extremality relation

$$\bar{p}_i^* = -D_i\bar{u} \, |D_i\bar{u}|^{\alpha-2}.$$

now, $A = \nabla, A^* = -div$,

$$A^* p^* = -f$$

$$-\nabla \cdot \bar{p}^* = -f$$

$$\sum D_i \bar{p}_i^* = f$$

$$f = -\sum D_i(D_i\bar{u} \, |D_i\bar{u}|^{\alpha-2}), \qquad \gamma_0\bar{u} = 0$$

The Neumann Problem

$$V = H^1(\Omega), \quad V^* = (H^1(\Omega))^*, \quad Y = L^2(\Omega)^{n+1} = Y^*$$

$$P \qquad \inf_{u \in H^1(\Omega)} \tfrac{1}{2} \left(\|u\|^2 + \|\nabla u\|^2 \right) - < f, u >$$

$$F(u) = -< f, u >, \quad Au =< u, \nabla u > \quad , G(p) = \tfrac{1}{2} \|p\|^2,$$

$$F^*(u^*) = \begin{cases} 0 & \text{if } u^* = -f \\ \infty & \text{otherwise,} \end{cases}$$

as before we have,

$$G^*(p^*) = \frac{1}{2} \|p^*\|^2$$

$$P^* \qquad \sup_{A^*p^* = -f} - \tfrac{1}{2} \|p^*\|^2,$$

Extremality

$$G(A\bar{u}) + G^*(-\bar{p}^*) + < \bar{p}^*, A\bar{u}) >= 0$$

$$\frac{1}{2} \|A\bar{u}\|^2 + \frac{1}{2} \|\bar{p}^*\|^2 + < \bar{p}^*, A\bar{u}) >= 0,$$

or

$$\|A\bar{u} + \bar{p}^*\| = 0$$

$$\bar{p}^* = -A\bar{u} = - < \bar{u}, \nabla\bar{u} >$$
$$\bar{p}_1^* = -\bar{u}, \quad \underbrace{\bar{p}_2^* = -\nabla\bar{u}}_{n-\dim.}$$

Now, let $u \in H^1(\Omega), v \in Y$

$$< Au, v >=< (u, \nabla u), (v_1, v_2) >$$
$$=< u, v_1 > + < \nabla u, v_2 >$$
$$=< u, v_1 > + < u, -div\ v_2 > + < \gamma_0 u, \gamma_0 v_2.\upsilon >_\Gamma =< u, A^* v >$$

for $v = \bar{p}^*, A^*p^* = -f$

$$< u, A^*\bar{p}^* >= - < u, \bar{u} > + < u, \Delta\bar{u} > + < \gamma_0 u, \gamma_0 v_2.\upsilon >_\Gamma$$

$$< u, -f >= - < u, \bar{u} > + < u, \Delta\bar{u} > + < \gamma_0 u, \gamma_0 v_2.\upsilon >_\Gamma, \quad \forall u \in H^1(\Omega),$$

in particular, for $u \in H_0^1(\Omega)$

$$< u, -f >=< u, -\bar{u} + \Delta\bar{u} >$$
$$< u, -f + \bar{u} - \Delta\bar{u} >= 0, \ \ \forall u \in H_0^1(\Omega)$$

so we have

$$-\Delta\bar{u} + \bar{u} - \Delta\bar{u} = f, \quad \text{in } (H^1(\Omega))^*$$

and for $uu \in H^1(\Omega)$

$$< \gamma_0 u, \gamma_0(-\nabla\bar{u}.\upsilon >_\Gamma = 0,$$
$$< \gamma_0 u, -\gamma_0 \frac{\partial\bar{u}}{\partial\upsilon} >_\Gamma = 0, \frac{\partial\bar{u}}{\partial\upsilon} = 0 \quad \text{on } \Gamma.$$

The Stokes Problem

$$V = H_0^1(\Omega)^n, \quad V^* = H^{-1}(\Omega)^n, \quad Y = Y^* = L^2(\Omega),$$

Given $f \in V^*$,find $u \in V, p \in L^2(\Omega)$,such that

$$-\Delta u + \nabla p = f$$
$$\nabla.u = 0$$
$$u = 0 \quad \text{on } \Gamma.$$

Let

$$W = \{u \in H_0^1(\Omega)^n : \nabla.u = 0,$$

this is a Hilbert space.
The minimization problem

$$P \qquad \inf_{u\in W} \tfrac{1}{2} \|\nabla u\|^2 - < f, u >$$
$$= \inf_{u\in V} \tfrac{1}{2} \|\nabla u\|^2 - < f, u > + \chi_{\{0\}}(\nabla.u)$$

$$A = div, \quad F(u) = - < f, u > + \tfrac{1}{2} \|\nabla u\|^2$$

$$G(p) = \chi_{\{0\}}(p) = \begin{cases} 0 & \text{if } p = 0 \\ \infty & \text{otherwise} \end{cases}$$

$$G^*(p^*) = \sup_{p\in Y} < p, p^* > -G(p) = 0$$

$$F^*(u) = \sup_{u\in V} < u, u^* > + < f, u > -\frac{1}{2} \|\nabla u\|^2$$

$$= \sup_{u\in V} < u, u^* > + < f, u > -\frac{1}{2} \|u\|^2_{H_0^1(\Omega)^n}$$

$$= \sup_{u \in V} < u, u^* + f > -\frac{1}{2} \|u\|^2_{H_0^1(\Omega)^n} = \|u^* + f\|_{H^{-1}(\Omega)}$$

the problem

$$\sup_{u \in V} < u, v^* > -\frac{1}{2} \|u\|^2$$

$$= \sup_{\alpha} \sup_{\|u\|=\alpha} < u, v^* > -\frac{1}{2}\alpha^2$$

$$= \sup_{\alpha} \sup_{\|v\|=1} \alpha < v, v^* > -\frac{1}{2}\alpha^2$$

$$= \sup_{\alpha} \alpha \|v^*\| - \frac{1}{2}\alpha^2 = \frac{1}{2} \|v^*\|^2 .$$

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Theorem: $-\Delta : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ is an isometric isomorphism.
Proof:
we know that, for each $f \in H^{-1}(\Omega)$,

$$\begin{aligned} -\Delta u &= f \\ \gamma_0 u &= 0 \end{aligned}$$

has a unique solution $u \in H_0^1(\Omega)$.
This implies that $-\Delta$ is 1-to-1 and on-to. we need to show it is an isometry, indded;

$$\|-\Delta u\|_{H^{-1}(\Omega)} = \|f\|_{H^{-1}(\Omega)} = \sup_{\substack{v \in H_0^1(\Omega) \\ v \neq 0}} \frac{\langle f, v\rangle}{\|v\|_{H_0^1(\Omega)}} = \sup_{\substack{v \in H_0^1(\Omega) \\ v \neq 0}} \frac{\langle -\Delta u, v\rangle}{\|\nabla v\|} = \sup_{\substack{v \in H_0^1(\Omega) \\ v \neq 0}} \frac{\langle \nabla u, \nabla v\rangle}{\|\nabla v\|} = \|u\|_{H_0^1(\Omega)}$$

* Let $L_0^2(\Omega) = \left\{u \in L^2(\Omega) : \int u = 0\right\}$
note that $L_0^2(\Omega)$ is a Hilbert subspace of $L^2(\Omega)$,indeed;

$$\begin{aligned} \text{let } u_n &\in L_0^2(\Omega) \to u \\ u_n &\to u \Longrightarrow \langle u_n, v\rangle \to \langle u, v\rangle \quad \forall\ v \in L^2(\Omega) \\ &\Longrightarrow \langle u_n, 1\rangle \to \langle u, 1\rangle \Longrightarrow \int u = 0 \end{aligned}$$

Lemma: $\nabla. : H_0^1(\Omega)^n \to L_0^2(\Omega)$ is an isomorphism.

Proof:
1- $R(\nabla.) \subseteq L_0^2(\Omega)$, for $u \in H_0^1(\Omega)$ we need to show $\int \nabla.u = 0$

$$\int_\Omega \nabla.u = \langle \nabla.u, 1\rangle = \langle u, \nabla 1\rangle = 0$$

2- $\nabla.$ is bounded, indeed;

$$\|\nabla.u\|_{L^2}^2 = \left\|\sum \frac{\partial u_i}{\partial x_i}\right\|^2 = \int \left|\sum \frac{\partial u_i}{\partial x_i}\right|^2 \le n \sum \int \left|\frac{\partial u_i}{\partial x_i}\right|^2 \le n \sum \int |\nabla u_i|^2 = n\,\|u\|_{H_0^1(\Omega)^n}^2 \Longrightarrow \|\nabla.\| \le \sqrt{n}$$

3- $\nabla.$ is onto

$$(\nabla.)^* = -\nabla : L_0^2(\Omega) \to H^{-1}(\Omega)^n$$

we show that $-\nabla$ is 1-to-1

$$\begin{aligned} -\nabla u &= 0 \text{ in } H^{-1}(\Omega)^n \Longrightarrow u = c \text{ (constant)} \\ \int c &= 0 \Longrightarrow c \int 1 = 0 \Longrightarrow c = 0 \end{aligned}$$

4- $\nabla.$ is 1-to-1

$$\begin{aligned} \text{let } \nabla.u &= 0 \text{ for some } u \in H_0^1(\Omega)^n \Longrightarrow \langle \nabla.u, v\rangle = 0 \quad \forall\ v \in L_0^2(\Omega) \\ &\quad \text{since } \nabla. \text{ is onto} \\ &\Longrightarrow v = \nabla.w \text{ for some } w \in H_0^1(\Omega)^n \Longrightarrow \langle \nabla.u, \nabla.w\rangle = 0\ \forall\ w \in H_0^1(\Omega)^n \\ &\Longrightarrow \langle u, -\triangle w\rangle = 0\ \forall\ w \in H_0^1(\Omega)^n \Longrightarrow \langle u, f\rangle = 0\ \forall\ f \in H^{-1}(\Omega)^n \Longrightarrow u = 0 \end{aligned}$$

Stokes Problem

Let $V = H_0^1(\Omega)^n$, $V^* = H^{-1}(\Omega)^n, Y = L_0^2(\Omega) = Y^*$.
we need to find $u \in H_0^1(\Omega)^n, p \in L_0^2(\Omega)$ such that

$$\begin{cases} -\triangle u + \nabla p = f, \\ f \in H^{-1}(\Omega)^n \nabla.u = 0 \end{cases}$$

Let $W = \{u \in H_0^1(\Omega)^n : \nabla.u = 0\}$

$$P : \inf_{u \in H_0^1(\Omega)^n} \frac{1}{2} \|u\|^2_{H_0^1(\Omega)^n} - \langle f, u \rangle + \chi_{\{0\}}(\nabla.u)$$

$$F(u) = \frac{1}{2} \|u\|^2_{H_0^1(\Omega)^n} - \langle f, u \rangle$$

$$A : \nabla. : H_0^1(\Omega)^n \to L_0^2(\Omega)$$

$$A^* : -\nabla : L_0^2(\Omega) \to H^{-1}(\Omega)^n$$

$$G(p) = \chi_{\{0\}}(p) = \begin{cases} 0 & \text{if } \ p = 0 \\ \infty & \text{otherwise} \end{cases}$$

$$G^*(u^*) = 0$$

$$F^*(u^*) = \frac{1}{2} \|u^* + f\|^2_{H^{-1}(\Omega)^n}$$

$$P^* : \sup_{p^* \in Y^*} \frac{-1}{2} \|-\nabla p^* + f\|^2_{H^{-1}(\Omega)^n}$$

P has a unique solution and P^* has a unique solution .

Extremality Condition

$$F(\overline{u}) + F^*(A^* \overline{p}^*) = \langle A^* \overline{p}^*, \overline{u} \rangle$$

$$\frac{1}{2} \|\overline{u}\|^2_{H_0^1(\Omega)^n} - \langle f, \overline{u} \rangle + \frac{1}{2} \|-\nabla \overline{p}^* + f\|^2_{H^{-1}(\Omega)^n} = \langle -\nabla \overline{p}^*, \overline{u} \rangle$$

$$\frac{1}{2} \|\overline{u}\|^2_{H_0^1(\Omega)^n} + \frac{1}{2} \|-\nabla \overline{p}^* + f\|^2_{H^{-1}(\Omega)^n} = \langle -\nabla \overline{p}^* + f, \overline{u} \rangle$$

since $-\triangle$ is an isometry $\Longrightarrow$

$$\frac{1}{2} \|-\triangle \overline{u}\|^2_{H^{-1}(\Omega)^n} + \frac{1}{2} \|-\nabla \overline{p}^* + f\|^2_{H^{-1}(\Omega)^n} = \langle -\nabla \overline{p}^* + f, \overline{u} \rangle$$

$$\|-\triangle \overline{u} + \nabla \overline{p}^* - f\|^2_{H^{-1}(\Omega)^n} = 0$$

$$-\triangle \overline{u} + \nabla \overline{p}^* = f$$

The Direct Proof of The Existence of a Solution for P^*

Suppose P_m is a minimizing sequence

$$\|-\nabla p_m + f\|^2_{H^{-1}(\Omega)^n} \longrightarrow \alpha = \inf \|-\nabla p + f\|^2_{H^{-1}(\Omega)^n}$$

$$\Longrightarrow \|-\nabla p_m\|_{H^{-1}(\Omega)^n} \text{ is bounded} \Longrightarrow -\nabla p_m \rightharpoonup F \qquad \text{weak convergence}$$

$$\langle -\nabla p_m, v\rangle \rightharpoonup \langle F, v\rangle$$

$\Longrightarrow$

$$\{\langle -\nabla p_m, v\rangle\}$$

is bunded $\Longrightarrow$

$$\langle -\nabla p_m, v\rangle = \langle p_m, \nabla . v\rangle \quad \text{by Green's Formula}$$

$$\{\langle p_m, \nabla . v\rangle\} \quad \text{is bounded for each} \qquad v \in H_0^1(\Omega).$$

$$\Longrightarrow \{\langle p_m, w\rangle\} \quad \text{is bounded for each} \qquad w \in L_0^2(\Omega)$$

By the uniform boundedness principle, $\{p_m\}$ is uniformly bounded in $L_0^2(\Omega)$.so,

$$p_m \rightharpoonup p_0 \quad \text{(subsequnce)}$$

$$-\nabla p_m \rightharpoonup -\nabla p_0$$

claim:
$\|-\nabla p_0 + f\|^2_{H^{-1}(\Omega)^n} = \alpha$,indeed;

$$\|-\nabla p_0 + f\|^2_{H^{-1}(\Omega)^n} = \langle -\nabla p_0 + f, \; -\nabla p_0 + f\rangle = \varliminf_{m\to\infty} \langle -\nabla p_0 + f, \; -\nabla p_0 + f\rangle \leq \varliminf_{m\to\infty} \|-\nabla p_0 + f\|_{H^{-1}(\Omega)^n} \|-\nabla p_m + f\|_{H^{-}}$$

$\Longrightarrow$

$$\|-\nabla p_0 + f\|_{H^{-1}(\Omega)^n} \leq \sqrt{\alpha} \Longrightarrow \|-\nabla p_0 + f\|^2_{H^{-1}(\Omega)^n} \leq \alpha \Longrightarrow \|-\nabla p_0 + f\|^2_{H^{-1}(\Omega)^n} = \alpha$$

26 Lecture 26

Mosolev Problem

$$\begin{array}{llll} V = H_0^1(\Omega) & V^* = H^{-1}(\Omega) & Y = L^1(\Omega)^n & Y^* = L^\infty(\Omega)^n \\ A = \nabla & A^* = -\operatorname{div} & f \in V^* \text{ given} & \alpha, \beta > 0 \end{array}$$

Before we state the problem, we should verify that $A : V \longrightarrow Y$ is continuous. Indeed,

$$A : H_0^1(\Omega) \longrightarrow L^2(\Omega)^n$$

is so. When Ω is finite we have $L^2(\Omega) \subset L^1(\Omega)$ and from Hölder inequality we have

$$\begin{aligned} \int |f| &\leq \sqrt{\int |f|^2}\sqrt{\int 1} \\ \int |f| &\leq C\sqrt{\int |f|^2} \\ \|f\|_1 &\leq C\|f\|_2 \\ \therefore \|\nabla u\|_1 &\leq \|\nabla u\|_2 \\ &\leq k\|u\|_{H_0^1(\Omega)} \end{aligned}$$

So $A : V \longrightarrow Y$ is continuous. The primal problem is

$$\inf_{u \in V} \frac{\alpha}{2}\|u\|_V^2 + \beta\|\nabla u\|_Y - \langle f, u\rangle \left(= \inf_{u \in V}\left\{\int \frac{\alpha}{2}|\nabla u|^2 + \beta \int |\nabla u| - \int fu\right\}\right)$$

Now, let

$$\begin{aligned} F(u) &= \frac{\alpha}{2}\|u\|_V^2 - \langle f, u\rangle \\ F^*(u^*) &= \frac{1}{2\alpha}\|u^* + f\|_{V^*}^2 \\ G(p) &= \beta\|p\|_Y \end{aligned}$$

To find G^*, let $f(x) = \beta|x| \qquad (x \in I\!R^n)$. Then

$$f^*(y) = \sup_{x \in I\!R^n} x \cdot y - \beta|x|$$

Now let $h(x) = x \cdot y - \beta|x|$, then

$$\begin{array}{l} h'(x) = y - \beta\frac{x}{|x|}, \qquad |x| = \sqrt{x_1^2 + x_2^2 + \cdots x_n^2} \\ h'(x) = 0 \Rightarrow y = \beta\frac{x}{|x|} \Rightarrow |y| = \beta \\ x \cdot y - \beta|x| = x \cdot y - |y||x| \leq |y||x| - |y||x| = 0 \\ x = 0 \qquad \text{or} \qquad x = \gamma y \end{array}$$

So if $|y| = \beta, x = \gamma y$ then $f^*(y) = 0$. If $|y| \neq \beta$ we do not have critical points
<u>case 1:</u> $|y| < \beta$

$$xy - \beta|x| \leq |x||y| - \beta|x| = |x|(|y|\beta) < 0$$

in this case $f^*(y) = 0$ as well.
<u>case 2:</u> $|y| > \beta$
Take $x = \lambda y, \quad \lambda > 0$

$$xy - \beta|x| = \lambda|y|^2 - \beta\lambda|y| = \lambda|y|(|y| - \beta) > 0$$

so

$$f^*(y) = \begin{cases} 0, & |y| \leq \beta \\ \infty, & \text{otherwise} \end{cases}$$

and we have

$$G^*(p^*) = \begin{cases} 0, & |p^*(x)| \leq \beta \text{ a.e on } \Omega \\ \infty, & \text{otherwise} \end{cases} = \begin{cases} 0, & \|p^*(x)\|_\infty \leq \beta \\ \infty, & \text{otherwise} \end{cases}$$

The dual problem

$$\sup_{p^* \in Y^*} -F^*(A^*p^*) - G^*(p^*) = \sup_{\|p^*(x)\|_\infty \leq \beta} -\frac{1}{2\alpha}\| - \nabla \cdot p^* + f\|^2_{V^*}$$

This problem has solutions; because $\|-\nabla p^* + f\|^2_{V^*}$ is convex over a bounded closed convex set $\|p^*(x)\|_\infty \leq \beta$.
Extremality conditions

$$\begin{array}{lcllcl} F(\bar{u}) & + & F^*(A^*\bar{p^*}) & = & \langle A^*\bar{p^*} + f, \bar{u}\rangle \\ \frac{\alpha}{2}\|\bar{u}\|^2_V & + & \frac{1}{2\alpha}\|A^*\bar{p^*} + f\|^2_{V^*} & = & \langle A^*\bar{p^*} + f, \bar{u}\rangle \\ \| - \alpha\Delta\bar{u}\|^2_{V^*} & + & \| - \nabla \cdot \bar{p^*} + f\|^2_{V^*} & = & 2\langle -\nabla \cdot \bar{p^*} + f, -\alpha\Delta\bar{u}\rangle \\ -\alpha\Delta\bar{u} & + & \nabla \cdot \bar{p^*} & = & f \end{array}$$

27 Lecture 27

Mossolov's problem (another method of dualization).

The given Problem is
$\inf_{u\in H_0^1(\Omega)} \left(\frac{\alpha}{2}\|u\|^2_{H_0^1(\Omega)} + \beta\|\nabla u\|_{L^1(\Omega)^n} - \langle f,u\rangle\right)$ with the following assupmtions:

$V = H_0^1(\Omega), \quad Y = L^2(\Omega)^n, \quad V^* = H_0^{-1}(\Omega), \quad Y^* = Y,$
$A = \nabla, \quad A^* = -\div \quad$ where $\alpha, \beta \gneqq 0$

Note that in the previous lecture this Mossolov's problem was solved with a choice of F and G. Here in this lecture the choice of F and G is different. i.e (another method of dualization).
Now, let $F(u) = -\langle f, u\rangle$.
Then

$$F^*(u^*) = \begin{cases} 0 & \text{if } u^* = -f \\ \infty & \text{other wise} \end{cases}$$

This is done before. (see previous lectures)
$G(p) = \frac{\alpha}{2}\|p\|^2_{L^2(\Omega)^n} + \beta\|p\|_{L^1(\Omega)^n} = \int_\Omega (\frac{\alpha}{2} \mid p \mid^2 + \beta \mid p \mid)dx$

we want to find: $G^*(p^*)$. To do so we first start with the following lemma.

Lemma:

Let $g(x) = \frac{\alpha}{2} \mid x \mid^2 + \beta \mid x \mid$ where $g : R^n \to R$, then

$$g^*(y^*) = \frac{1}{2\alpha}(\mid y^* \mid - \beta)^2_+ \qquad \text{where } S_+ = \begin{cases} s & \text{if } s \geq 0 \\ 0 & \text{other wise} \end{cases}$$

and the sup is attained at $\bar{x} = \frac{y}{\alpha|y|}(\mid y \mid - \beta)_+$

Proof:
Let $f(x) = x \cdot y - \frac{\alpha}{2} \mid x \mid^2 - \beta \mid x \mid$ (note that $f : R^n \to R$ and $\acute{f}$ is the grad (∇))
then $\acute{f}(x) = y - \alpha x - \beta\frac{x}{|x|}$
By setting $\acute{f}(x) = 0$ we get: $y = \alpha x + \beta\frac{x}{|x|} = (\alpha + \frac{\beta}{|x|})x$ ......(1)
We want to solve for (x). To do so, multply (1) by x then by y (note: multiplying here means dot product).
Multiplying by x gives:

$$x \cdot y = \alpha \mid x \mid^2 + \beta \mid x \mid = (\alpha \mid x \mid + \beta) \mid x \mid$$

Multiplying by y gives:

$$\begin{aligned} \mid y \mid^2 &= (\alpha + \tfrac{\beta}{|x|})x \cdot y \\ &= (\alpha + \tfrac{\beta}{|x|})(\alpha \mid x \mid + \beta) \mid x \mid) \\ &= (\alpha \mid x \mid + \beta)^2 \\ \Rightarrow \quad \mid y \mid &= (\alpha \mid x \mid + \beta) \\ \Rightarrow \quad \mid x \mid &= \tfrac{1}{\alpha}(\mid y \mid - \beta) \end{aligned}$$

This requires that $\mid y \mid \geq \beta$. Otherewise there are no critical points.

Assume now that $\mid y \mid \geq \beta$.
From (1)

$$\begin{aligned} y &= (\alpha + \frac{\beta}{\frac{1}{\alpha}(|y|-\beta)})x \\ &= (\alpha + \frac{\alpha\beta}{(|y|-\beta)})x \\ &= \frac{\alpha|y|}{|y|-\beta}x \qquad \Rightarrow \; x = \frac{y}{\alpha|y|}(\mid y \mid - \beta) \quad \text{and} \end{aligned}$$

$$f_{\max} = (\alpha \mid x \mid + \beta) \mid x \mid - \frac{\alpha}{2} \mid x \mid^2 - \beta \mid x \mid$$

$$= \frac{\alpha}{2} \mid x \mid^2 = \frac{1}{2\alpha}(\mid y \mid - \beta)^2$$

Note here that for $\mid y \mid \lneqq \beta$, there is no critical values and $f_{\max} = 0$ since

$$\begin{aligned} x \cdot y - \frac{\alpha}{2} \mid x \mid^2 - \beta \mid x \mid &\leq \mid x \mid\mid y \mid - \frac{\alpha}{2} \mid x \mid^2 - \beta \mid x \mid \\ &\leq -\frac{\alpha}{2} \mid x \mid^2 \\ &\leq 0 \end{aligned}$$

Therefore:

$f_{\max} = \frac{1}{2\alpha}(\mid y \mid - \beta)_+^2$ and occurs when $\overline{x} = \frac{y}{\alpha|y|}(\mid y \mid - \beta)_+$

So,

$G^*(p^*) = \frac{1}{2\alpha}\int_\Omega (\mid p^* \mid - \beta)_+^2 dx = \frac{1}{2\alpha}(\|\mid p^* \mid - \beta)_+ \|^2_{L^2(\Omega)^n}$ and

the Dual Problem would be:

$$\text{P}^*: \sup_{p^* \in Y^*} - F(A^* p^*) - G^*(-p^*)$$

$$= \sup_{A^* p^* = -f} \frac{1}{2\alpha}(\|\mid p^* \mid - \beta)_+ \|^2_{L^2(\Omega)^n}$$

note that $A^* p^* = -f$ is closed and convex set

and also $\|\mid p^* \mid - \beta)_+ \|^2_{L^2(\Omega)^n}$ is continouos, coercive, strictly convex which all implies that P^* has a unique solution.

***The clear relation between P and P^* can be found by using the extramility condition. The relation is given as: $\nabla \overline{u} = \frac{-\overline{p}^*}{\alpha|\overline{p}^*|}(\mid p^* \mid - \beta)_+$ and the justification is left as an exercise.**

end of lec#27

28 Lecture 28

Duality by the Minimax Theorem

Saddle points of a function: Properties

PROPOSITION 58
If $L : A \times B \to R$,

$$\inf_{u\in A}\sup_{p\in B} L(u,p) \geq \sup_{p\in B}\inf_{u\in A} L(u,p).$$

Proof.

$$L(v,p) \geq \inf_{u\in A} L(u,p) \quad \forall v \in A, \forall p \in B,$$

then

$$\sup_{p\in B} L(v,p) \geq \sup_{p\in B}\inf_{u\in A} L(u,p),$$

then we have

$$\inf_{u\in A}\sup_{p\in B} L(u,p) \geq \sup_{p\in B}\inf_{u\in A} L(u,p).$$

■

DEFINITION 59
a point $(\bar{u},\bar{p}) \in A \times B$ is called a saddle point of L on$A \times B$ if

$$L(\bar{u},p) \leq L(\bar{u},\bar{p}) \leq L(u,\bar{p}), \quad \forall u \in A, \forall p \in B.$$

PROPOSITION 60
if $\exists \alpha \in R$ s.t.

$$L(\bar{u},p) \leq \alpha \quad \forall p \in B,$$

and

$$L(u,\bar{p}) \geq \alpha \quad \forall u \in A,$$

then $(\bar{u},\bar{p})$ is a saddle point of L and

$$L(\bar{u},\bar{p}) = \alpha.$$

Proof.

$$L(u,\bar{p}) \geq \alpha \quad \forall u \in A$$

$\Longrightarrow$

$$L(\bar{u},\bar{p}) \geq \alpha$$

and

$$L(\bar{u},p) \leq \alpha \ \ \forall p \in B$$

then

$$L(\bar{u},\bar{p}) \leq \alpha$$

then we have

$$L(\bar{u},\bar{p}) = \alpha$$

■

PROPOSITION 61
1) if $(\bar{u},\bar{p}) \in A \times B$ is a saddle point of L,then

$$L(\bar{u},\bar{p}) = \max_{p\in B}\min_{u\in A} L(u,p) = \min_{u\in A}\max_{p\in B} L(u,p)$$

2)

$$\text{If } \max_{p\in B}\inf_{u\in A} L(u,p) = \min_{u\in A}\sup_{p\in B} L(u,p) = \alpha,$$

then L has a saddle point $(\bar{u},\bar{p}) \in A \times B$ and $L(\bar{u},\bar{p}) = \alpha$.

Proof. 1) suppose $(\bar{u}, \bar{p})$ is a saddle point of L,then

$$L(\bar{u}, \bar{p}) \leq L(u, \bar{p}) \implies$$

$$L(\bar{u}, \bar{p}) = \inf_{u \in A} L(u, \bar{p}) = \min_{u \in A} L(u, \bar{p}) \leq \sup_{p \in B} \min_{u \in A} L(u, p),$$

and

$$L(\bar{u}, \bar{p}) \geq L(\bar{u}, p) \implies$$

$$L(\bar{u}, \bar{p}) = \sup_{p \in B} L(\bar{u}, p) = \max_{p \in B} L(\bar{u}, p) \geq \inf_{u \in A} \max_{p \in B} L(u, p),$$

since, inf and sup are attained we have:

$$\begin{aligned} \inf_{u \in A} \max_{p \in B} L(u, p) &= \max_{p \in B} L(\bar{u}, p) = L(\bar{u}, \bar{p}) \\ &= \min_{u \in A} L(u, \bar{p}) = \sup_{p \in B} \min_{u \in A} L(u, p) \end{aligned}$$

$\implies$

$$L(\bar{u}, \bar{p}) = \max_{p \in B} \min_{u \in A} L(u, p) = \min_{u \in A} \max_{p \in B} L(u, p).$$

2) Assume

$$\max_{p \in B} \inf_{u \in A} L(u, p) = \min_{u \in A} \sup_{p \in B} L(u, p) = \alpha,$$

we have,

$$\alpha = \inf_{u \in A} L(u, \bar{p}) \leq L(u, \bar{p}), \qquad \forall u \in A$$

and

$$= \sup_{p \in B} L(\bar{u}, p) \geq L(\bar{u}, p), \qquad \forall p \in B,$$

then L has saddle point. ■

PROPOSITION 62
the set of saddle points of L on $A \times B$ is of the form $A_o \times B_o$.

Proof. We need to show that if (u_1, p_1) and $(u_2, p_2) \in A \times B$ are saddle points then (u_1, p_2) is saddle point. we know that

$$L(u_1, p_1) = L(u_2, p_2) = \alpha$$

now,

$$\begin{aligned} L(u_1, p_2) &\leq \alpha, \text{ and} \\ L(u_1, p_2) &\geq \alpha, \end{aligned}$$

then we have (u_1, p_2) is saddle point. ■

Assumptions on L :

Assume V, Z are reflexive Banach spaces,and

$A \subseteq V$ is closed, convex and non empty,

$B \subseteq Z$ is closed, convex and non empty,

the function L satisfies:

for each $u \in A, L(u, .)$ is concave, u.s.c. on B,

for each $p \in B, L(., p)$ is convex, l.s.c. on A.

PROPOSITION 63
Under the above assumptions, the set $A_o \times B_o$ is convex.if $L(u, .)$ is strictly concave, then B_o contains at most one element. if $L(., p)$ is strictly convex, then A_o contains at most one element.

Proof. Assume $A_o \times B_o \neq \Phi$,and let $(u_1, p_1), (u_2, p_2) \in A_o \times B_o, \lambda \in [0, 1]$.

$$\begin{aligned} & L(\lambda(u_1, p_1) + (1-\lambda)(u_2, p_2)) \\ = \;& L(\lambda u_1 + (1-\lambda)u_2, \lambda p_1 + (1-\lambda)p_2) \\ \leq \;& \lambda L(u_1, \lambda p_1 + (1-\lambda)p_2) + (1-\lambda)L(u_2, \lambda p_1 + (1-\lambda)p_2) \\ \leq \;& \lambda L(u_1, p_1) + (1-\lambda)L(u_2, p_2) = \alpha. \end{aligned}$$

If $L(u, .)$ is strictly concave, let $u \in A_o, p_1, p_2 \in B_o, \lambda \in (0, 1)$,we have

$$\begin{aligned} \alpha &= L(u, \lambda p_1 + (1-\lambda)p_2) > \lambda L(u, p_1) + (1-\lambda)L(u, p_2) \\ &= \lambda\alpha + (1-\lambda)\alpha = \alpha, \end{aligned}$$

which is impossible ($\alpha > \alpha$). Similarly If $L(., p)$ is strictly convex, let $u_1, u_2 \in A_o, p_1 \in B_o, \lambda \in (0, 1)$,we have

$$\alpha = L(u_1 + (1-\lambda)u_2, p_1) < \lambda L(u_1, p_1) + (1-\lambda)L(u_2, p_1) = \alpha.$$

■

Characterization of a saddle point (differentiable functions)

PROPOSITION 64
Assume $L = l + m$, where

$l(u, .)$ is concave , Gateaux-diff. w.r.t. p,
$l(., p)$ is convex , Gateaux-diff. w.r.t. u,
$m(u, .)$ is concave,
$m(., p)$ is convex,

then $(\bar{u}, \bar{p}) \in A \times B$ is a saddle point of L if and only if

$$\begin{aligned} \left\langle \frac{\partial l}{\partial u}(\bar{u}, \bar{p}), u - \bar{u} \right\rangle + m(u, \bar{p}) - m(\bar{u}, \bar{p}) &\geq 0, \ \forall u \in A, \\ \left\langle \frac{\partial l}{\partial p}(\bar{u}, \bar{p}), p - \bar{p} \right\rangle + m(\bar{u}, p) - m(\bar{u}, \bar{p}) &\leq 0, \ \forall p \in B, \end{aligned}$$

Proof. Assume $(\bar{u}, \bar{p})$ is a saddle point, $\lambda \in (0, 1]$

$$\begin{aligned} & \frac{1}{\lambda}\left[L(\bar{u} + \lambda(u - \bar{u}), \bar{p}) - L(\bar{u}, \bar{p})\right] \\ = \;& \frac{1}{\lambda}\left[l(\bar{u} + \lambda(u - \bar{u}), \bar{p}) - l(\bar{u}, \bar{p}) + m(\bar{u} + \lambda(u - \bar{u}), \bar{p}) - m(\bar{u}, \bar{p})\right] \geq 0, \end{aligned}$$

therefore

$$\begin{aligned} \frac{l(\bar{u} + \lambda(u - \bar{u}), \bar{p}) - l(\bar{u}, \bar{p})}{\lambda} + \frac{m(\bar{u} + \lambda(u - \bar{u}), \bar{p}) - m(\bar{u}, \bar{p})}{\lambda} &\geq 0 \\ \frac{l(\bar{u} + \lambda(u - \bar{u}), \bar{p}) - l(\bar{u}, \bar{p})}{\lambda} + \frac{\lambda m(u, \bar{p}) + (1-\lambda)m(\bar{u}, \bar{p}) - m(\bar{u}, \bar{p})}{\lambda} &\geq 0 \end{aligned}$$

cancelling and taking the limits as $\lambda \longrightarrow 0$, we get

$$\left\langle \frac{\partial l}{\partial u}(\bar{u}, \bar{p}), u - \bar{u} \right\rangle + m(u, \bar{p}) - m(\bar{u}, \bar{p}) \geq 0, \ \forall u \in A,$$

the proof of the second one is analogous.
on the other hand assume the inequalities hold, let $u \in A, \lambda \in (0, 1)$,

$$\begin{aligned} l(\bar{u} + \lambda(u - \bar{u}), \bar{p}) - l(\bar{u}, \bar{p}) &\leq \lambda l(u, \bar{p}) + (1-\lambda)l(\bar{u}, \bar{p}) - l(\bar{u}, \bar{p}) \\ &= \lambda\left[l(u, \bar{p}) - l(\bar{u}, \bar{p})\right]. \end{aligned}$$

now,

$$\begin{aligned} L(u,\bar{p}) - L(\bar{u},\bar{p}) &= l(u,\bar{p}) - l(\bar{u},\bar{p}) + m(u,\bar{p}) - m(\bar{u},\bar{p}) \\ &\geq \frac{l(\bar{u}+\lambda(u-\bar{u}),\bar{p}) - l(\bar{u},\bar{p})}{\lambda} + m(u,\bar{p}) - m(\bar{u},\bar{p}) \geq 0 \end{aligned}$$

then we have

$$L(u,\bar{p}) \geq L(\bar{u},\bar{p}),$$

in the same way, we could prove that

$$L(\bar{u},\bar{p}) \geq L(\bar{u},p),$$

so, $(\bar{u},\bar{p})$ is a saddle point. ■

COROLLARY 65
Assume $L(u,.)$ is concave, cateaux-differentiabe and $L(.,p)$ is convex, cateaux-differentiabe, then $(\bar{u},\bar{p})$ is a saddle point of L on $A \times B$ if and only if

$$\begin{aligned} \left\langle \frac{\partial L}{\partial u}(\bar{u},\bar{p}), u-\bar{u} \right\rangle &\geq 0,\ \forall u \in A, \\ \left\langle \frac{\partial L}{\partial p}(\bar{u},\bar{p}), p-\bar{p} \right\rangle &\leq 0,\ \forall p \in B. \end{aligned}$$

Proof. Let $m = 0$. ■

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Existence of Saddle points

Proposition 1
Assume V, Y are reflexive Banach spaces. $A \subset V$, $B \subset Y$ are convex, closed and nonempty. $L : A \times B \longrightarrow \mathbb{R}$.
(1) $L(u,.)$ is concave and upper semicontinuous for each $u \in A$
(2) $L(.,p)$ is convex and lower semicontinuous for each $p \in B$.
(3) If A and B are bounded, then L possesses at least one saddle point $(\overline{u},\overline{p}) \in A \times B$ such that

$$L(\overline{u},\overline{p}) = \underset{p\in B}{Max}\ \underset{u\in A}{Min}\ L(u,p) = \underset{u\in A}{Min}\ \underset{p\in B}{Max} L(u,p)$$

Proposition 2
If insted of (3) we have
(4)
  a) there exists a $p_0 \in B$ such that $L(u,p_0) \longrightarrow \infty$ as $\|u\| \longrightarrow \infty$, $u \in A$
  b) there exists a$u_0 \in A$ such that $L(u_0,p) \longrightarrow -\infty$ as $\|p\| \longrightarrow \infty$, $p \in B$,
then L possesses at least one saddle point $(\overline{u},\overline{p}) \in A \times B$ such that

$$L(\overline{u},\overline{p}) = \underset{u\in A}{Min}\ \underset{p\in B}{Sup}\ L(u,p) = \underset{p\in B}{Max}\ \inf_{u\in A} L(u,p)$$

Proposition 3
if instead of (3) we have A is either finite or 4(a) holds, then

$$\underset{u\in A}{Min}\ \underset{p\in B}{Sup}\ L(u,p) = \sup_{p\in B} \inf_{u\in A} L(u,p)$$

Proposition 4
if instead of (3) we have B is either finite or 4(b) holds, then

$$\inf_{u\in A} \underset{p\in B}{Sup}\ L(u,p) = \underset{p\in B}{Max}\ \inf_{u\in A} L(u,p)$$

Application to Duality

(P) $\underset{u\in V}{inf} F(u)$ or $\underset{u\in domF}{inf}\ F(u)$
we try to write

$$F(u) = \sup_{p\in B} L(u,p)$$

the Primal problem becomes

$$\inf_{u\in A} \underset{p\in B}{Sup}\ L(u,p)$$

How? we cosider two cases
Case (1) : $F(u) = F_0(u) + F_1(u)$ with $F_1(u)$ proper, lower semicontinuous and convex ($F_1 \in \Gamma_0(V)$)

$$\overset{**}{F_1}(u) = F_1(u) = \sup_{u^* \in \overset{*}{V}} \langle u, u^* \rangle - \overset{*}{F_1}(\overset{*}{u})$$

$$L(u,p) = \langle u,p \rangle - \overset{*}{F_1}(p) + F_0(u)$$

$$F(u) = \sup_{p\in \overset{*}{V}} \langle u,p \rangle - \overset{*}{F_1}(p) + F_0(u)$$

The primal problem becomes

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$$\inf_{u\in A}\ \sup_{p\in \overset{*}{V}}\left\{\langle u,p\rangle-\overset{*}{F}_1(p)+F_0(u)\right\}$$

Case (2) :$F(u)=F_0(u)+F_1(Su)$ with $S:V\longrightarrow Y$ ($\overset{*}{Y}=Z$) S can be nonlinear, $F_1\in\Gamma_0(V)$

$$F_1(Su)=\sup_{p\in Z}\langle Su,p\rangle-\overset{*}{F}_1(p)$$

$$\langle Su,p\rangle-\overset{*}{F}_1(p)+F_0(u)$$

The primal problem becomes

$$\inf_{u\in V}\sup_{p\in Z}L(u,p)$$

Example: The Mossolev Problem

$$\inf_{u\in H_0^1(\Omega)}\frac{\alpha}{2}\left\|\nabla u\right\|^2_{L^2(\Omega)^n}+\beta\left\|\nabla u\right\|_{L^1(\Omega)^n}-\langle f,u\rangle$$

$$F(u)=\int_\Omega\left(\frac{\alpha}{2}\left|\nabla u\right|^2+\beta\left|\nabla u\right|-fu\right)dx$$

$$S=-\nabla$$

$$F_1(p)=\int_\Omega\left(\frac{\alpha}{2}\left|p\right|^2+\beta\left|p\right|\right)dx$$

$$L(u,p)=\int_\Omega\left(-p.\nabla u-\frac{1}{2\alpha}\left(|p|-\beta\right)_+^2-fu\right)dx$$

the primal problem (P)

$$\inf_{u\in H_0^1(\Omega)}\ \sup_{p\in L^2(\Omega)^n}\int_\Omega\left(-p.\nabla u-\frac{1}{2\alpha}\left(|p|-\beta\right)_+^2-fu\right)dx$$

the dual problem $(\overset{*}{P})$

$$\sup_{p\in L^2(\Omega)^n}\ \inf_{u\in H_0^1(\Omega)}\int_\Omega\left(-p.\nabla u-\frac{1}{2\alpha}\left(|p|-\beta\right)_+^2-fu\right)dx$$

(P)

$$\inf_{u\in H_0^1(\Omega)}\int_\Omega\left(u\operatorname{div}p-fu\right)dx=\inf_{u\in H_0^1(\Omega)}\int_\Omega\left(\operatorname{div}p-f\right)u\,dx=\begin{cases}0 & \operatorname{div}p-f=0\\ -\infty & \text{other wise}\end{cases}$$

$(\overset{*}{P})$ is

$$\sup_{\substack{p\in L^2(\Omega)^n\\ \operatorname{div}p=f}}\ -\frac{1}{2\alpha}\int_\Omega\left(|p|-\beta\right)_+^2dx$$

Extremality

$$L(\overline{u},\overline{p})=\inf_{u\in V}\ \sup_{p\in Z}L(u,p)=\sup_{p\in Z}\inf_{u\in V}L(u,p)$$

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$$\int\limits_{\Omega} -\overline{p}\nabla\overline{u} - \frac{1}{2\alpha}\left(\left|\overline{p}\right| - \beta\right)_+^2 - f\ \overline{u}\ dx = \int\limits_{\Omega}\left(\frac{\alpha}{2}\left|\nabla\overline{u}\right|^2 + \beta\left|\nabla\overline{u}\right| - f\ \overline{u}\right) dx$$

$$\int\limits_{\Omega}\left(-\overline{p}\nabla\overline{u} - \frac{1}{2\alpha}\left(\left|\overline{p}\right| - \beta\right)_+^2 - \frac{\alpha}{2}\left|\nabla\overline{u}\right|^2 - \beta\left|\nabla\overline{u}\right|\right) dx = 0$$

$\Longrightarrow$

$$-\overline{p}\nabla\overline{u} - \frac{1}{2\alpha}\left(\left|\overline{p}\right| - \beta\right)_+^2 - \frac{\alpha}{2}\left|\nabla\overline{u}\right|^2 - \beta\left|\nabla\overline{u}\right| = 0$$

$$\nabla\overline{u} = \frac{-\overline{p}}{\alpha\left|\overline{p}\right|}\left(\left|\overline{p}\right| - \beta\right)_+$$

$$\operatorname{div}\overline{p} = f$$