

1 Sobolev Spaces in \mathbb{R}^N

Definition: Let Ω be an open domain of \mathbb{R}^N and $1 \leq p \leq +\infty$. We define the Sobolev space

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) \mid \exists g_1, g_2, \dots, g_N \in L^p(\Omega)\}$$

satisfying

$$\left. \int_{\Omega} u \frac{\partial \phi}{\partial x_i} = - \int_{\Omega} g_i \phi, \quad \forall \phi \in C_0^\infty(\Omega) \right\}$$

we denote by $H^1(\Omega) = W^{1,2}(\Omega)$. We have, in this case, $\nabla u = (g_1, \dots, g_N)$. We equip $W^{1,p}(\Omega)$ with the norm

$$\|u\|_{1,p} = \|u\|_p + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_p$$

or equivalently with

$$\|u\|_{1,p} = \left(\|u\|_p^p + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_p^p \right)^{\frac{1}{p}}, \quad 1 < p < \infty.$$

It is also clear that $H^1(\Omega)$ equipped with scalar product

$$\langle u, v \rangle = \langle u, v \rangle_{L^2} + \sum_{i=1}^N \left\langle \frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right\rangle_{L^2}$$

is a inner product space. The associated norm, in this case, is

$$\|u\|_{1,2} = \left(\|u\|_{L^2}^2 + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2}^2 \right)^{\frac{1}{2}}.$$

Theorem: $W^{1,p}(\Omega)$ equipped with the above norm is a Banach space.

Proof: Given a Cauchy sequence (u_n) in $W^{1,p}(\Omega) \Rightarrow (u_n)$ is Cauchy in $L^p(\Omega)$ and $\left(\frac{\partial u_n}{\partial x_i} \right)$ is Cauchy in $L^p(\Omega)$ for $1 \leq i \leq N$. So $u_n \rightarrow u$ in $L^p(\Omega)$ and $\frac{\partial u_n}{\partial x_i} \rightarrow g_i$ in $L^p(\Omega)$.

We have

$$\int_{\Omega} u_n \frac{\partial \phi}{\partial x_i} = - \int_{\Omega} \frac{\partial u_n}{\partial x_i} \phi, \quad \forall \phi \in C_0^\infty(\Omega)$$

By letting n go to ∞ , we get

$$\int_{\Omega} u \frac{\partial \phi}{\partial x_i} = - \int_{\Omega} g_i \phi, \quad \forall 1 \leq i \leq N$$

Therefore $u \in W^{1,p}(\Omega)$ with

$$\frac{\partial u}{\partial x_i} = g_i, \quad \forall 1 \leq i \leq N.$$

This shows that $W^{1,p}(\Omega)$ is a Banach space.

Corollary: $H^1(\Omega)$, equipped with the above scalar product, is a Hilbert space.

Theorem: $W^{1,p}(\Omega)$, $1 < p < \infty$ is reflexive and $W^{1,p}(\Omega)$, $1 \leq p < \infty$ is separable.

For the proof, we only repeat the argument of the one-dimensional case, taking the operator

$$T : W^{1,p}(\Omega) \rightarrow [L^p(\Omega)]^{N+1}$$

$$u \rightarrow T(u) = (u, \nabla u).$$

Exercise: Suppose $u_n \rightarrow u$ in $L^p(\Omega)$ and (∇u_n) remains bounded in $[L^p(\Omega)]^N$. Then $u \in W^{1,p}(\Omega)$.

Example: Let $\Omega = (-1, 1) \times (-1, 1)$ be an open of \mathbb{R}^2 and $\omega = (0, 1) \times (0, 1) \subset \Omega$. We set $u = xy \chi_\omega$, where χ_ω is the characteristic of ω . So for $\phi \in C_0^1(\Omega)$, we have

$$\begin{aligned} \int \int_{\Omega} u \phi_x dx dy &= \int_0^1 \int_0^1 u \phi_x dx dy \\ &= \int_0^1 y \left(\int_0^1 x \phi_x dx \right) dy = \int_0^1 y \left[x \phi \Big|_{x=0}^{x=1} - \int_0^1 \phi dx \right] dy \\ &= - \int_0^1 \int_0^1 y \phi(x, y) dx dy = - \int \int_{\Omega} y \chi_\omega \phi dx dy \end{aligned}$$

Similarly we have

$$\int \int_{\Omega} u \phi_y dx dy = - \int \int_{\Omega} x \chi_\omega \phi dx dy$$

Thus

$$u \in W^{1,p}(\Omega), \quad \forall 1 \leq p \leq +\infty,$$

with $\nabla u = (y\psi_\omega, x\psi_\omega)$.

Example: Let

$$u(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & , (x, y) \neq (0, 0) \\ 0 & , (x, y) = (0, 0) \end{cases}$$

be defined on $\Omega = \{(x, y) | x^2 + y^2 < 1\}$.

It is easy to see that $u \in L^\infty(\Omega)$ so it is in $L^p(\Omega) \quad \forall p \geq 1$. Next, for $\phi \in C_0^1(\Omega)$ we have

$$\begin{aligned}
\int \int_{\Omega} u \phi_x dx dy &= \int_0^1 \int_0^{2\pi} u \left(\phi_r \cos \theta - \phi_\theta \frac{\sin \theta}{r} \right) r dr d\theta \\
&= \int_0^{2\pi} \cos \theta \int_0^1 u r \phi_r dr d\theta - \int_0^1 \int_0^{2\pi} (u \sin \theta) \phi_\theta d\theta dr \\
&= - \int_0^{2\pi} \cos \theta \int_0^1 [(r u_r \phi + u \phi) dr] d\theta \\
&\quad + \int_0^1 \int_0^{2\pi} \phi (u_\theta \sin \theta + u \cos \theta) d\theta dr \\
&= - \int_0^{2\pi} \int_0^1 r u_r \phi \cos \theta dr d\theta + \int_0^1 \int_0^{2\pi} \phi u_\theta \sin \theta d\theta dr \\
&= - \int_0^{2\pi} \int_0^1 \left(u_r \cos \theta - u_\theta \frac{\sin \theta}{r} \right) \phi r dr d\theta = \int \int_{\Omega} u_x \phi dx dy
\end{aligned}$$

We conclude that

$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) = \left(\frac{y(y^2 - x^2)}{(x^2 + y^2)^2}, \frac{x(x^2 - y^2)}{(x^2 + y^2)^2} \right)$$

Let's verify that $\nabla u \in [L^p(\Omega)]^2$:

$$\begin{aligned}
\int \int_{\Omega} \left| \frac{\partial u}{\partial x} \right|^p dx dy &= \int_0^{2\pi} \int_0^1 \frac{r^{3p} |\sin \theta (\sin^2 \theta - \cos^2 \theta)|^p}{r^{4p}} r dr d\theta \\
&\leq \int_0^{2\pi} \int_0^1 r^{1-p} dr d\theta < \infty
\end{aligned}$$

iff $p < 2$. Similarly we estimate $\int \int_{\Omega} \left| \frac{\partial u}{\partial y} \right|^p dx dy$. Therefore $u \in W^{1,p}(\Omega)$, $1 \leq p < 2$.

Remark: This example shows that functions in $W^{1,p}(\Omega)$ are not necessarily continuous if the space dimension $N \geq 2$.

Theorem: Let $u \in W^{1,p}(\Omega)$. Then

1)

$$\tilde{u} = \begin{cases} u & \text{on } \Omega \\ 0 & \text{on } \mathbb{R}^N \setminus \Omega \end{cases} \in L^p(\mathbb{R}^N)$$

2) If $\alpha \in C_0^1(\Omega)$ then $\tilde{\alpha u} \in W^{1,p}(\mathbb{R}^N)$ and

$$\frac{\partial}{\partial x_i}(\tilde{\alpha u}) = \frac{\partial \tilde{\alpha}}{\partial x_i} u + \alpha \frac{\partial u}{\partial x_i}.$$

Proof.

1) is trivial

2)

$$\int_{\mathbb{R}^N} \tilde{\alpha u} \phi_{x_i} = \int_{\Omega} \alpha u \phi_{x_i}, \quad \forall \phi \in C_0^1(\mathbb{R}^N)$$

$$\begin{aligned}
&= \int_{\Omega} u [(\alpha\phi)_{x_i} - \phi\alpha_{x_i}] \\
&= - \int_{\Omega} (u_{x_i}\alpha\phi + u\alpha_{x_i}\phi) = - \int_{\Omega} \phi (u_{x_i}\alpha + u\alpha_{x_i}) \\
&= - \int_{\mathbf{R}} \phi (\widetilde{\alpha u}_{x_i} + u\widetilde{\alpha}_{x_i})
\end{aligned}$$

Since $\alpha \frac{\partial u}{\partial x_i} + \frac{\partial \alpha}{\partial x_i} \in L^p(\mathbb{R}^N)$ then $\widetilde{\alpha u} \in W^{1,p}(\mathbb{R}^N)$ with

$$\frac{\partial}{\partial x_i}(\widetilde{\alpha u}) = \alpha \frac{\partial u}{\partial x_i} + u \frac{\partial \alpha}{\partial x_i}.$$

Lemma: Let $\rho \in L^1(\mathbb{R}^N)$ and $u \in W^{1,p}(\mathbb{R}^N)$. Then $\rho * u \in W^{1,p}(\mathbb{R}^N)$ and

$$\frac{\partial}{\partial x_i}(\rho * u) = \rho * \frac{\partial u}{\partial x_i}.$$

The proof of this lemma goes exactly like the one in the one-dimensional case.

Theorem: (Friedrichs)

Suppose that $u \in W^{1,p}(\Omega)$, $1 \leq p < \infty$, then there exists a sequence $(u_n) \subset C_0^\infty(\mathbb{R}^N)$ such that

1)

$$u_n|_{\Omega} \longrightarrow u \quad \text{in } L^p(\Omega)$$

2)

$$\frac{\partial u_n}{\partial x_i}|_{\omega} \rightarrow \frac{\partial u}{\partial x_i}, 1 \leq i \leq N \quad \text{in } L^p(\omega), \quad \forall \omega \subset\subset \Omega.$$

Proof. Let

$$\tilde{u} = \begin{cases} u & \text{on } \Omega \\ 0 & \text{on } \mathbb{R}^N \setminus \Omega \end{cases}$$

and

$$\rho \in C_0^\infty(\mathbb{R}^N), \quad \rho \geq 0, \quad \text{supp } \rho \subset\subset B(0,1), \quad \int_{\mathbb{R}^N} \rho = 1.$$

We define a regularizing sequence $\rho_n(x) = n^N \rho(nx)$. It is clear that

$$\text{supp } \rho_n \subset B\left(0, \frac{1}{n}\right) \quad \text{and} \quad \int_{\mathbb{R}^N} \rho_n(x) = 1, \quad \forall n = 1, 2, 3, \dots$$

We then set $v_n = \rho_n * \tilde{u}$. So $v_n \in C^\infty(\mathbb{R}^N)$ and $v_n \longrightarrow \tilde{u}$ in $L^p(\mathbb{R}^N)$.

For $w \subset\subset \Omega$, let $\alpha \in C_0^1(\Omega)$ such that $0 \leq \alpha \leq 1$ and $\alpha \equiv 1$ on a neighborhood \cup , of w , contained in Ω .

Also for n so large, we have

$$\rho_n * \widetilde{\alpha u}|_{\omega} = \rho_n * \tilde{u}|_{\omega}, \quad (i)$$

since

$$\text{supp}(\rho_n * \widetilde{\alpha u} - \rho_n * \tilde{u}) = \text{supp}(\rho_n * (\widetilde{\alpha u} - \tilde{u}))$$

$$\subset \left(B\left(0, \frac{1}{n}\right) + \Omega \setminus \bigcup \right) \subset \omega^c,$$

for n large. From the above lemma, we have

$$\frac{\partial}{\partial x_i} (\rho_n * \widetilde{\alpha u}) = \rho_n * \left(\alpha \frac{\widetilde{\partial u}}{\partial x_i} + u \frac{\widetilde{\partial \alpha}}{\partial x_i} \right)$$

therefore

$$\frac{\partial}{\partial x_i} (\rho_n * \widetilde{\alpha u}) \longrightarrow \alpha \frac{\widetilde{\partial u}}{\partial x_i} + u \frac{\widetilde{\partial \alpha}}{\partial x_i} \text{ in } L^p(\mathbb{R}^N).$$

In particular

$$\frac{\partial}{\partial x_i} (\rho_n * \widetilde{\alpha u}) \longrightarrow \frac{\partial u}{\partial x_i} \text{ in } L^p(\omega).$$

Hence, by (i), we see that

$$\frac{\partial}{\partial x_i} (\rho_n * \widetilde{u}) \longrightarrow \frac{\partial u}{\partial x_i} \text{ in } L^p(\omega)$$

Conclusion $v_n \in C^\infty(\mathbb{R}^N)$ and

$$v_n|_\Omega \rightarrow u \text{ in } L^p(\Omega)$$

$$\frac{\partial}{\partial x_i} v_n \longrightarrow \frac{\partial u}{\partial x_i} \text{ in } L^p(\omega), \quad w \subset\subset \Omega, \quad 1 \leq i \leq N.$$

Finally we put $u_n = \xi_n v_n$, where ξ_n is a truncation sequence. We then easily verify that u_n has the properties required by the theorem.

Remark: In general, for $u \in W^{1,p}(\Omega)$ we cannot always find a sequence (u_n) in $C_0^\infty(\mathbb{R}^N)$ such that $u_n|_\Omega \rightarrow u$ in $W^{1,p}(\Omega)$.