# 1 Sobolev Spaces

## 1.1 Motivation

Let us consider the problem

$$\left.\begin{array}{ll} -u''(x) + u(x) = f(x), & 0 < x < 1 \\ u(0) = u(1) = 0. & \end{array}\right\} \tag{1}$$

The objective is to find a solution $u \in C^2([0,1])$ for $f \in C([0,1])$. This problem is solvable by standard calculus methods.

Suppose that $\varphi \in C_0^1((0,1))$; that $\varphi$ is continuously differentiable and $\varphi(0) = \varphi(1) = 0$. So

$$\int_0^1 (-u'' + u)\varphi(x)dx = \int_0^1 f\varphi dx.$$

Using integration by parts, we obtain

$$\int_0^1 u'\varphi' + u\varphi = \int_0^1 f\varphi dx. \tag{2}$$

We notice here that (2) is valid if $u$ and $u' \in L^1((0,1))$ or "simply" $u \in C^1([0,1])$. In this case, we say that $u$ is a weak solution of (1). It satisfies (1) in the weak or in the variational sense (2).

(2) is the variational equation for (1).

**Theorem**: If $f \in C^0$, then any weak solution $u \in C^1$ is $C^2$.

**Proof**: It suffices to note that $u'' = u - f \in C$. So $u \in C^2$.

**Theorem**: If $u \in C^1((0,1))$ with $u(0) = u(1) = 0$ satisfying (2) for all $\varphi \in C_0^\infty((0,1))$. Then $u$ is a classical solution; that $u$ satisfies (1).

**Remark** In some instance, we can show that a weak solution is in fact a classical solution. This is called the regularity theory. It will be a part of our course.

## 1.2 Sobolev Space $W^{1,p}(I)$

**Definition**: Given an open interval $I$ and $1 \leq p \leq +\infty$ (bounded or not). We define the Sobolev space $W^{1,p}(I) = \{u \in L^p(I)$ such that there exists $g \in L^p(I)$, for which we have $\int_I u\varphi' = -\int_I g\varphi, \quad \forall\, \varphi \in C_0^1(I)\}$.

**Remark** : For $p = 2$, we denote $W^{1,2}(I) = H^1(I)$.

Remark : In such a case, we write $u' = g$ in the weak sense; that is

$$\int_I (u' - g)\varphi dx = 0, \quad \forall\, \varphi \in C_0^1(I).$$

**Example**: Let $I = (-1,1)$ and $u(x) = \frac{1}{2}(x + |x|)$. It is clear that $u$ is not differentiable at $x = 0$ (hence on $I$). Let $\varphi \in C_0^1(I)$, then

$$\begin{aligned} \int_{-1}^1 u\varphi' &= \int_{-1}^0 0.\varphi' dx + \int_0^1 x\varphi' dx = \int_0^1 x\varphi' dx \\ &= x\varphi|_0^1 - \int_0^1 \varphi(x)dx = -\int_0^1 \varphi(x)dx = -\int_{-1}^1 g\varphi(x)dx \end{aligned}$$

where

$$g(x) = \begin{cases} 0, & -1 \leq x < 0 \\ 1, & 0 \leq x \leq 1. \end{cases}$$

This is the Heavyside function denoted by $H$. So $u' = H$ in the weak sense.
Note that $u, H \in L^p(I), \quad \forall\, p \geq 1 \Rightarrow u \in W^{1,p}(I), \quad \forall\, 1 \leq p \leq \infty$.
Now let us consider $H(x)$ and test if it is a $W^{1,p}(I)$ function. For this we take $\varphi \in C_0^1(I)$ as a test function

$$\int_{-1}^{1} H\varphi' dx = \int_0^1 \varphi'(x) = \varphi(1) - \varphi(0) = -\varphi(0) = -\delta(\varphi).$$

In this case, we cannot find $g \in L^p(I)$ such that

$$\int_{-1}^{1} g\varphi dx = -\varphi(0), \quad \forall\, \varphi \in C_0^1(I).$$

**Remark**: In this latter case, we say that $H^{'} = \delta$ in the distributional sense. $\delta$ is a distribution. This also motivates us to define $W^{1,p}(I)$ by using distributions.
**Definition**: We say that $u \in W^{1,p}(I)$ if $u \in L^p(I)$ and its distribution derivative $u'$ coincides with an $L^p(I)$ function $g$.
**Theorem**: The Sobolev space $W^{1,p}(I)$ equipped with the norm

$$\|W\|_{1,p} = \|u\|_p + \|u'\|_p$$

is a Banach space.
**Proof**: Let $(u_n)$ be a Cauchy sequence in $W^{1,p}(I)$. So $(u_n)$ and $(u_n')$ are Cauchy in $L^p \Rightarrow$

$$u_n \to u \text{ and } u_n' \to g \text{ in } L^p(I).$$

Now

$$\int_I u_n\varphi' = -\int_I u_n'\varphi, \quad \forall\, n, \quad \varphi \in C_0^1(I).$$

As $n \to \infty$,

$$\int_I u\varphi' = -\int_I g\varphi, \quad \forall\, \varphi \in C_0^1(I)$$

$\Rightarrow u \in W^{1,p}(I)$ with $u' = g$ (Definition) and $\|u_n - u\|_{1,p} \to 0$ as $n \to \infty$.
**Theorem**: The space $W^{1,p}(I)$ is reflexive for $1 < p < \infty$ and separable for $1 \leq p < \infty$.
**Proof**: We define the operator

$$\begin{aligned} &T : W^{1,p}(I) \to L^p(I) \times L^p(I) \\ &u \to (u, u'). \end{aligned}$$

This is an isometry. So $T(W^{1,p}(I))$ is a closed subspace of $L^p(I) \times L^p(I)$; hence $T(W^{1,p}(I))$ is reflexive for $1 < p < \infty$ since $L^p \times L^p$ is reflexive for $1 < p < \infty$.
The same thing holds for separability.
**Theorem**: Given $u \in W^{1,p}(I)$. There exists $\tilde{u} \in C(\overline{I})$ such that $\tilde{u} = u$ a.e. in $I$ and

$$\tilde{u}(y) - \tilde{u}(x) = \int_x^y u'(t)dt, \quad \forall\, x, y \in I.$$

**Proof**: It is clear that

$$\tilde{u}(x) = u(x_0) + \int_{x_0}^{x} u'(t)dt, \quad x_0 \in I,$$

is absolutely continuous $\Rightarrow \tilde{u} \in C(\overline{I})$. Moreover, $\tilde{u}' = u'$ a.e. in $I \Rightarrow \tilde{u} = u + c$ a.e. in $I$. But $c = 0$ since $\tilde{u}(x_0) = u(x_0)$

**Remark**: This theorem shows that $W^{1,p}(I)$ functions can be represented by absolutely continuous functions defined on $\overline{I}$. This is why we usually say that if $u \in W^{1,p}(I)$ then $u \in C(\overline{I})$.

**Remark**: If $u$ is absolutely continuous with $u' \in L^p(I)$, this does not mean that $u \in W^{1,p}(I)$ unless $I$ is bounded.

Let

$$g(t) = \frac{1}{1+t^2} \in L^1($$

but

$$u(x) = \int_0^x g(t)dt = \tan^{-1} x \notin L^1($$

since

$$\int_0^R \tan^{-1} x dx = R\left[\tan^{-1} R - \frac{\log(1+R^2)}{2R}\right].$$

which gives

$$\int_0^\infty \tan^{-1} x dx = \infty.$$

**Definition**: Let $u$ be defined in $I$; for $x$ and $h$ such that $x + h \in I$, we denote by

$$\tau_h f(x) = f(x+h)$$

.

**Theorem**: Let $u \in L^p(I), \quad 1 < p \leq \infty$. The following properties are equivalent:

(i) $u \in W^{1,p}(I)$.

(ii) There exists a constant $C > 0$ such that

$$\left|\int_I u\varphi'\right| \leq C\|\varphi\|_{p'}, \quad \forall\, \varphi \in C^{\infty_0}(I), \quad \frac{1}{p'} + \frac{1}{p} = 1.$$

(iii) There exists a constant $C' > 0$ such that for each open $\omega \subset\subset I$ ($\overline{\omega} \subset I$) and for each $h$ with $|h| <$ distance $(\omega, \partial I)$, we have

$$||\tau_h u - u\|_p \leq C'|h|.$$

**Proof**: (i) $\Rightarrow$ (ii) by Holder's inequality with $c = \|u\|_p$.

(ii) $\Rightarrow$ (i) why?

We define $F : C(I) \to R$ by

$$F(\varphi) = \int_I u\varphi'$$

which is bounded and continuous. By Hahn Banach theorem this form is extended to $L^{p'}(I)$. So

$$|F(\varphi)| \leq C\|\varphi\|_{L^{p'}}, \quad \forall\, \varphi \in L^{p'}.$$

By Riesz representation theorem, there exists $g \in L^p(I)$ such that

$$\int_I u\varphi' = \int_I g\varphi, \quad \forall\ \varphi \in L^{p'}(I);$$

in particular, we have

$$\int_I u\varphi' = \int_I g\varphi, \quad \forall\ \varphi \in C_0^\infty(I).$$

So, $u \in W^{1,p}(I)$ with $u' = -g$

(i) $\Rightarrow$ (iii) why ?

$$u(x+h) - u(x) = \int_x^{x+h} u'(t)dt.$$

Now for $p = +\infty$ we have

$$|u(x+h) - u(x)|_\infty \leq \left| \int_x^{x+h} |u'|_\infty dt \right| \ \|\tau_n u\|_\infty \leq \|u\|_\infty |h|.$$

Next $1 < p < \infty$. We write

$$u(x+h) - u(x) = \int_0^1 hu'(x+sh)ds,$$

which implies

$$|u(x+h) - u(x)|^p \leq |h|^p \int_0^1 |u'(x+sh)|^p ds$$

Integrationg over $\omega$ :

$$\begin{aligned} \int_\omega |u(x+h) - u(x)|^p dx &\leq |h|^p \int_\omega \int_0^1 |u'(x+sh)|^p ds dx \\ &\leq |h|^p \int_0^1 \|u'\|_p^p ds = |h|^p \ \|u'\|_p^p \end{aligned}$$

Therefore

$$\|\tau_h u - u\|_p \leq |h| \ \|u\|_{1,p}.$$

(iii) $\Rightarrow$ (ii) why?

Let $\varphi \in C_0^\infty(I) \Rightarrow$ supp $\varphi \subset \omega \subset\subset I$. Take $h$ such that $|h| <$ distance $(\omega, \partial I)$, so

$$\int_I [u(x+h) - u(x)]\varphi(x)dx = \int_I [\varphi(x) - \varphi(x-h)]u(x)dx$$

This leads to

$$\left| \int_I [\varphi(x) - \varphi(x-h)]u(x)dx \right| \leq ||\tau_h u||_p ||\varphi||_{p'} \leq C|h|||\varphi||_{p'}$$

Divide by $h$ and let $h \to 0$ to obtain

$$\left\| \int_I u(x)\varphi'(x)dx \right\| \leq C\|\varphi\|_{p'}.$$

This completes the proof.

**Remark**: For $p = 1$, we only have (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii).

**Remark**: (ii) $\Rightarrow$ (i) is in general false because of the nonseparability of $L^\infty(I)$.