1 Dual of $W_0^{1,p}(\Omega)$

<u>Notation:</u> We denote by $W^{-1,P'}(\Omega)$, the dual space of $W^{1,P}(\Omega)$, $1 \leq p < +\infty$; where $\frac{1}{p} + \frac{1}{p'} = 1$.
We denote by $H^{-1}(\Omega)$ the dual of $H_0^1(\Omega)$.
By identifying $L^2(\Omega)$ to its dual, we obtain

$$H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega).$$

where the embedding is continuous and dense.
Proposition Let $F \epsilon W^{-1,p'}(\Omega)$. Then there exist $f_0,\ f_1, \dots ,\ f_N$ Such that

$$\langle F, \phi \rangle = \int_\Omega f_0 \phi + \sum_{i=1}^{N} \int_\Omega f_i \frac{\partial \phi}{\partial x_i}, \quad \forall \phi \ \in \ H_0^1(\Omega)$$

with

$$||F|| = \max_{0 \leq i \leq N} ||f_i||_{L^]}.$$

Moreover, if Ω is bounded then $f_0 = 0$.

2 Boundary-value problems

Let Ω be a bounded open set of $\mathbb{R}^N$ and let $\Gamma = \partial\Omega$.
We are looking a function $u : \overline{\Omega} \longrightarrow \mathbb{R}$ satisfying

$$(P_1) \begin{cases} -\Delta u + u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{cases}$$

where f is a given function defined in Ω.
$u = 0$ on Γ is called a "homogeneous" Dirichlet condition.
Definition (Weak solution): By a weak solution of (P), we mean a function $u \ \in \ H_0^1(\Omega)$ satisfying

$$\int_\Omega (\nabla u \ .\nabla v + uv) dx = \int_\Omega f v dx, \quad \forall v \ \in \ H_0^1(\Omega).$$

<u>**Existence and Uniqueness**</u>

To prove the existence of a unique (weak) solution. It suffices to note that

$$(u, v) \longrightarrow \int_\Omega (\nabla u \ .\nabla v + uv) dx$$

is a scalar product on $H_0^1(\Omega)$ and

$$v \longrightarrow \int_\Omega fv \text{ is a continuous bilinear form}$$

is a bounded linear form. So, by the Riesz representation theorem $\exists ! u \,\epsilon\, H_0^1(\Omega)$ such that

$$\int_\Omega (\nabla u \,.\nabla v + uv)dx = \int_\Omega fvdx, \quad \forall\ v \in\ H_0'(\Omega).$$

Proposition (classical solution)

Any weak solution, $u \,\epsilon\, C^2(\overline{\Omega})$, is a classical solution of (P).

Proof.

If $u \,\epsilon\, H_0^1(\Omega)\bigcap C(\overline{\Omega})$ then $u = 0$ on Γ (previous lemma). Hence the boundary condition is satisfied.

Let $\phi \,\epsilon\, C_0^\infty(\Omega)$ then we have

$$\int_\Omega (-\Delta u + u)\phi = \int f\phi, \quad \forall\ \phi \in\ C_0^\infty(\Omega).$$

Thus,

$$-\Delta u + u = f \qquad a.e.\ in\ \Omega$$

since C_0^∞ is dense in $L^2(\Omega)$. Thus, we have $-\Delta u + u = f$ in Ω since $u \,\epsilon\, C^2(\overline{\Omega})$.

Second-order elliptic equation:

Let $\Omega \subset \mathbb{R}^N$ be bounded and open. Given $a_{ij}(x) \in C^1(\overline{\Omega})$, $i \leq i, j \leq N$, satisfying the elliptic condition

$$\sum_{e1_j=1}^{N} ai_j(x)\xi_i\xi_j \geq \alpha|\xi|^2, \quad \forall x \in \Omega, \quad \forall \xi \in \mathbb{R}^N, \quad \alpha > 0.$$

Let $a_0(x) \in C(\overline{\Omega})$ also be given, with $a_0(x) \geq 0, \quad \forall x \in \Omega$. We would like to find $u : \overline{\Omega} \longrightarrow \mathbb{R}$ satisfying

$$(P_2) \begin{cases} -\sum_{i,j=1}^N \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + a_0 u = f \text{ in } \Omega \\ \qquad\qquad\qquad\qquad u = 0 \text{ on } \Gamma \end{cases}$$

Definition: A weak solution of (P_2) is a function $u \in H_0^1(\Omega)$ which satisfies

$$\sum_{i,j=1}^{N} \int_\Omega \left(a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\right) + \int_\Omega (a_0 uv)dx = \int_\Omega fv, \quad \forall v \in H_0^1(\Omega).$$

Existence: We define the bilinear form

$$B : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$$

by

$$B(u,v) = \sum_{i,j=1}^{N} \int_\Omega a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \int_\Omega a_0 uv$$

Since $a_0, a_{ij} \in C(\overline{\Omega})$ then

$$\begin{aligned} |B(u,v)| &\leq C \left(\int |\nabla u|^2 \right)^{\frac{1}{2}} \left(\int |\nabla v|^2 \right)^{\frac{1}{2}} + c_0 \|u\|_{L^2} \|v\|_{L^2} \\ &\leq \tilde{C} \left[\|u\|_{H_0^1} \|v\|_{H_0^1} \right] \end{aligned}$$

i.e. B is continuous. Also

$$B(u,u) \geq \alpha \|\nabla u\|_{L^2}^2 = \alpha \|\nabla u\|_{H_0^1}$$

So, B is coercive. Lax Milgram Lemma then guarantees the existence of a unique weak solution for (P_2).

Neumann conditions:

Let $\Omega \subset \mathbb{R}^N$ be open and bounded of class C^1. We look for a function, $u : \overline{\Omega} \to \mathbb{R}$, which satisfies

$$(P_3) \begin{cases} -\Delta u + u = f \text{ in } \Omega \\ \frac{\partial u}{\partial \eta} = 0 \text{ on } \Gamma \end{cases}$$

$\frac{\partial u}{\partial \eta} = \nabla u \cdot \eta$ is the normal derivative. $\vec{\eta}$ is the unit outer normal to Γ.

Definition: $\frac{\partial u}{\partial \eta} = 0$ is called "homogeneous" Neumann condition

Definition: A weak solution of (P_3) is a function $u \in H^1(\Omega)$ which satisfies

$$\int_\Omega (\nabla u \cdot \nabla v + uv)dx = \int_\Omega fv, \quad \forall v \in H^1(\Omega).$$

Definition: A classical solution is a function $u \in C^2(\Omega) \bigcap C^1(\bar{\Omega})$ satisfying (P_3).

Proposition. Every classical solution is a weak solution.

Proof: Since $u \in C^1(\bar{\Omega})$ then $u \in H^1(\Omega)$ [Ω is bounded]

$$\int_\Omega (-\Delta u + u)v = \int fv$$

By using Green's identity and the boundary condition, we arrive at

$$\int_{\Omega} (\nabla u \cdot \nabla v + uv) = \int_{\Omega} fv.$$

Since Ω is of class C^1 and $C^1(\bar{\Omega})$ is dense in $H^1(\Omega)$ then

$$\int_{\Omega} \nabla u \cdot \nabla v + uv = \int fv, \quad \forall v \in H^1(\Omega). \tag{2.1}$$

So, u is a weak solution.
Theorem: If $f \in L^2\Omega)$ then $\exists! u \in H^1(\Omega)$ such that (2.1) is satisfied. u is the unique weak solution of (P_3).
Proof: We use Lax-Milgram lemma.
Proposition if u is a weak solution of (P_3) such that $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$. Then u is a classical solution.
Proof: Since $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$. Then, we have

$$\int_{\Gamma} \frac{\partial u}{\partial \eta} v + \int_{\Omega} (-\Delta u + u) v = \int fv, \quad \forall v \in H^1(\Omega)$$

Since $C_0^1(\Omega) \subset H^1(\Omega)$ then

$$\int_{\Omega} (-\Delta u + u) v = \int_{\Omega} fv, \quad \forall v \in C_0^1(\Omega)$$

So, $\Delta u + u = f$ in $L^2(\Omega)$, hence a.e (then every where since $u \in C^2$). Consequently,

$$\int_{\Gamma} \frac{\partial u}{\partial \eta} v = 0, \quad \forall v \in C(\bar{\Omega})$$

$$\frac{\partial u}{\partial \eta} = 0 \text{ on } \Gamma.$$