1 Embedding

Sobolev Inequalities

In the one-dimensional case, we saw that $W^{1,p}(I)$ is embedded continuously in $L^\infty(I)$. However, in the higher dimensional case we saw examples, for which this type of embedding is no longer true. To address this issue we start with the situation where $\Omega = \mathbb{R}^N$.

Lemma. Suppose, for $N \geq 2$, that $f_1, f_2, \ldots, f_N \in L^{N-1}(\mathbb{R}^{N-1})$. Then

$$f(x) = f_1(x_2, \ldots x_N)\, f_2(x_1, x_3, \ldots, x_{_N}) \ldots f_N\left(x_1, x_2, \ldots, x_{_{N-1}}\right) \in L^1(\mathbb{R}^N)$$

and

$$||f||_{L^1(\mathbb{R}^N)} \leq \prod_{i=1}^{N} ||f_i||_{L^{N-1}(\mathbb{R}^{N-1})}$$

Proof. The case $N = 2$ is trivial. Consider the case $N = 3$

$$\begin{aligned} \int |f(x)| dx_3 &= |f_3(x_1, x_2)| \int |f_1(x_2, x_3)| \; |f_2(x_1, x_3)| \, dx_3 \\ &\leq |f_3(x_1, x_2)| \left(\int |f_1(x_2, x_3)|^2 \, dx_3\right)^{\frac{1}{2}} \left(\int |f_2(x_1, x_3)|^2 \, dx_3\right)^{\frac{1}{2}}. \end{aligned}$$

We then integrate with respect to x_1

$$\begin{aligned} \int \int |f(x)| dx_3 dx_1 &\leq \int |f_3(x_1, x_2)| \left(\int |f_1|^2 dx_3\right)^{\frac{1}{2}} \left(\int |f_2|^2 dx_3\right)^{\frac{1}{2}} dx_1 \\ &\leq \left(\int |f_1|^2 dx_3\right)^{\frac{1}{2}} \int |f_3| \left(\int |f_2|^2 dx_3\right)^{\frac{1}{2}} dx_1 \end{aligned}$$

By using Cauchy-Schwarz, we get

$$\int \int |f| dx_3 \, dx_1 \leq \left(\int |f_1|^2 dx_3\right)^{\frac{1}{2}} \left(\int |f_3|^2 dx_1\right)^{\frac{1}{2}} ||f_2||_{L^2(\mathbb{R}^2)}.$$

Now, we integrate with respect to x_2, so we obtain

$$\begin{aligned} \int \int \int |f| dx &\leq ||f_2||_{L^2(\mathbb{R}^2)} \int \left(\int |f_1|^2 dx_3\right)^{\frac{1}{2}} \left(\int |f_3|^2 dx_1\right)^{\frac{1}{2}} dx_2 \\ &\leq ||f_2||_{L^2(\mathbb{R}^2)} ||f_1||_{L^2(\mathbb{R}^2)} ||f_3||_{L^2(\mathbb{R}^2)}. \end{aligned}$$

For $N > 3$, we use induction. Assume that the assertion of the lemma is true for $N - 1$ and prove it for N.

$$\int_{\mathbb{R}^{N-1}} |f| dx_1 \ldots dx_{N-1} = \int_{\mathbb{R}^{N-1}} |f_1| \ldots |f_{N-1}| \; |f_N| dx_1, \ldots dx_{N-1}$$

$$\leq \left(\int_{\mathbb{R}^{N-1}} |f_N|^{N-1} dx_1 \ldots dx_{N-1}\right)^{\frac{1}{N-1}} \left(\int_{\mathbb{R}^{N-1}} |f_1|^{N^1} \ldots |f_{N-1}|^{N^1} dx_1 \ldots dx_{N-1}\right)^{\frac{1}{N1}}$$

by Holder's inequality, where

$$\frac{1}{N-1} + \frac{1}{N^1} = 1 \Rightarrow N^1 = \frac{N-1}{N-2}.$$

Since $f_i \in L^{N-1}(\mathbb{R}^{N-1}) \Rightarrow$

$$|f_i|^{\frac{N-1}{N-2}} \in L^{N-2}(\mathbb{R}^{N-2}), \text{ for each fixed } , x_N.$$

We then apply the induction hypothesis to get

$$\int_{\mathbb{R}^{N-1}} |f_1|^{\frac{N-1}{N-2}} \dots |f_{N-1}|^{\frac{N-1}{N-2}} dx_1, \dots dx_{N-1} \leq \prod_{i=1}^{N-1} ||f_i||^{\frac{N-1}{N-2}}_{L^{N-1}(\mathbb{R}^{N-2})}$$

Hence

$$\int_{\mathbb{R}^{N-1}} |f| dx_1 \dots dx_N \leq ||f_N||_{L^{N-1}(\mathbb{R}^{N-1})} \prod_{i=1}^{N-1} ||f_i||_{L^{N-1}(\mathbb{R}^{N-2})}.$$

The function

$$F_i\,(x_N) = ||f_i||_{L^{N-1}(\mathbb{R}^{N-2})}, \quad 1 \leq i \leq N-1,$$

belongs to $L^{N-1}(\mathbb{R})$ since

$$\int_{\mathbb{R}} |F_i\,(x_N)\,|^{N-1} dx_N = \int_{\mathbb{R}} \int_{\mathbb{R}^{N-2}} |f_i|^{N-1} dx_1 \dots dx_N < \infty$$

by hypothesis. Therefore, as a consequence of Holder's inequality, we have

$$\prod_{i=1}^{N-1} F_i \in L^1(\mathbb{R})$$

which gives, by integration,

$$\int_{\mathbb{R}} \prod_{i=1}^{N-1} F_i \, dx_N \leq \prod_{i=1}^{N-1} ||f_i||_{L^{N-1}(\mathbb{R}^{N-1})}$$

hence

$$\int_{\mathbb{R}^N} |f| dx \leq \prod_{i=1}^{N} ||f_i||_{L^{N-1}(\mathbb{R}^{N-1})}.$$

Theorem (Sobolev, Gagliardo, Nirenberg)

Suppose that $1 \leq p < N$. Then

$$W^{1,p}(\mathbb{R}^N) \subset L^{p^*}(\mathbb{R}^N), \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}.$$

Moreover there exists a constant $C = C(N,p)$ such that

$$||u||_{L^{p*}} \leq C||\nabla u||_{L^p}, \quad \forall u \in W^{1,p}(\mathbb{R}^N)$$

Proof. Let $v \in C_0^1(\mathbb{R}^N)$, so we have

$$|v(x_1, x_2, \ldots, x_N)| = \left| \int_{-\infty}^{t} \frac{\partial u}{\partial x_1}(t, x_2, \ldots, x_N)\, dt \right| \le \int_{-\infty}^{\infty} \left| \frac{\partial v}{\partial x_1}(t, x_2, \ldots, x_N) \right| dt$$

Similarly, we have for $1 \le i \le N$

$$|v(x)| \le \int_{-\infty}^{\infty} \left| \frac{\partial v}{\partial x_i}(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_N) \right| dt = f_i(\tilde{x}_i),$$

where $\tilde{x}_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$. Thus

$$|v(x)|^N \le \prod_{i=1}^{N} f_i(\tilde{x}_i),$$

or

$$|v(x)|^{\frac{N}{N-1}} \le \prod_{i-1}^{N} f_i^{\frac{1}{N-1}}(\tilde{x}_i).$$

Since each $g_i = f_i^{\frac{1}{N-1}}(x_i) \in L^{N-1}\left(\mathbb{R}^{N-1}\right)$, then $|v(x)|^{\frac{N}{N-1}} \in L^1(\mathbb{R}^N)$ by the previous lemma and

$$\int_{\mathbb{R}^N} |v_{(x)}|^{\frac{N}{N-1}} \le \prod_{i=1}^{N} ||f_i||_{L^1(\mathbb{R}^{N-1})}^{\frac{1}{N-1}} = \prod_{i=1}^{N} \left|\left| \frac{\partial v}{\partial x_i} \right|\right|_{L^1(\mathbb{R}^N)}^{\frac{1}{N-1}}$$

hence

$$||v||_{L^{\frac{N}{N-1}}(\mathbb{R}^N)} \le \prod_{i=1}^{N} \left|\left| \frac{\partial v}{\partial x_i} \right|\right|_{L^1(\mathbb{R}^N)}^{\frac{1}{N}}$$

We then take $v = u^{r-1}u$, for $r \ge 1$, we have

$$\begin{aligned} ||u||^r_{L^{r\frac{N}{N-1}}(\mathbb{R}^N)} &\le r \prod_{i=1}^{N} \left|\left| u^{r-1} \frac{\partial u}{\partial x_i} \right|\right|_{L^1(\mathbb{R}^N)}^{\frac{1}{N}} \\ &\le r ||u||^{r-1}_{L^{p'(r-1)}} \prod_{i=1}^{N} \left|\left| \frac{\partial u}{\partial x_i} \right|\right|_{L^p}^{\frac{1}{N}} \end{aligned}$$

We choose then r in such a way that

$$\frac{rN}{N-1} = p'(r-1);$$

which gives

$$r = \frac{N-1}{N} p^*, \qquad p^* = \frac{Np}{N-p}.$$

Consequently we have

$$||u||_{L^{P^*}} \le r \prod_{i=1}^{N} \left|\left| \frac{\partial u}{\partial x_i} \right|\right|_{L^p}^{\frac{1}{N}} = \frac{p(N-1)}{N-p} ||\nabla u||_{L_p} \quad \forall\, u \in C_0^1(\mathbb{R}^N).$$

Now for $u \in W^{1,p}(\mathbb{R}^N)$, there exists a sequence $(u_n) \subset C_0^1(\mathbb{R}^N)$ such that $u_n \longrightarrow u$ in $W^{1,p}(\mathbb{R}^N)$ and $u_n \longrightarrow u$ a.e. (taking a subsequence if needed). So

$$||u_n||_{L^{p*}} \leq \frac{p(N-1)}{N-p}||\nabla u_n||_{L^p}.$$

By using Fatou's lemma and taking n to ∞, we get

$$||u||_{L^{p*}} \leq \frac{p(N-1)}{N-p}||\nabla u||_{L^p}.$$

Corollary: For $1 \leq p < N$, then

$$W^{1,p}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N) \quad \forall q \in [p, \quad p^*].$$

Proof. $q = \alpha p + (1-\alpha)p^*, \quad 0 \leq \alpha \leq 1,$

$$|u|^q = |u|^{\alpha_p} \cdot |u|^{(1-\alpha)p*} \Rightarrow \int |u|^q = \int |u|^{\alpha p} \cdot |u|^{(1-\alpha)p^*}$$

We use Holder's inequality to get

$$\begin{aligned} \int |u|^q &\leq \left(\int u^p\right)^\alpha \left(\int |u|^{p^*}\right)^{1-\alpha} \\ &\leq C||u||_{L^p}^{\alpha p}||\nabla u||_{L^p}^{(1-\alpha)p^*} \leq C||u||_{W^{1,p}}^q \end{aligned}$$

Hence

$$||u||_{L^q} \leq C||u||_{W^{1,p}}.$$

Corollary: (Case $p = N$)

$$W^{1,N}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N), \quad \forall\, q \in [N, \quad +\infty)$$

with continuous embedding.
Proof. Suppose that $u \in C_0^1(\mathbb{R}^N)$, we then use the inequality

$$||u||^r_{L^{r\frac{N}{(N-1)}}} \leq r||u||^{r-1}_{L^{(r-1)N'}} \prod_{i=1}^N \left|\left|\frac{\partial u}{\partial x_i}\right|\right|^{\frac{1}{N}}_{L^N}$$

hence we have

$$||u||^r_{L^{r\frac{N}{(N-1)}}} \leq r||u||^{r-1}_{L^{(r-1)\frac{N}{(N-1)}}}||\nabla u||_{L^N}, \quad \forall_{r\geq 1}$$

Young's inequality then gives

$$\begin{aligned} ||u||^r_{L^{\frac{rN}{(N-1)}}} &\leq C_1||u||^r_{L^{(r-1)\frac{N}{(N-1)}}} + C_2||\nabla u||^r_{L^N} \\ &\leq C\left(||u||_{L^{(r-1)\frac{N}{(N-1)}}} + ||\nabla u||_{L^N}\right)^r \end{aligned}$$

Therefore

$$||u||_{L^{r\frac{N}{(N-1)}}} \leq C\left[||u||_{L^{(r-1)\frac{N}{(N-1)}}} + ||\nabla u||_{L^N}\right]$$

By choosing $r = N$, we obtain

$$||u||_{L^{\frac{N}{2}(N-1)}} \leq C||u||_{W^{1,N}}$$

By using the interpolation result, then it comes that

$$||u||_{L^q} \leq C||u||_{W^{1,N}}, \quad \forall q \in \left[N, \quad \frac{N^2}{N-1}\right]$$

we then take $r = N+1, \quad N+2, \ldots,$ etc. to obtain

$$||u||_{L^q} \leq C||u||_{W^{1,N}}, \quad \forall q \in [N, \quad +\infty).$$

Theorem.(Morrey)

Let $p > N$, then $W^{1,p}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$ with continuous embedding. Moreover, we have

$$|u(x) - u(y)| \leq C|x-y|^\alpha ||\nabla u||_{L^p}, \ \ a.e. \ \ x, y \in \mathbb{R}^N$$

where

$$\alpha = 1 - \frac{N}{p} \ \text{ and } \ C = C(N, p).$$

Remark. The above inequality implies the existence of a function $\tilde{u} \in C^{0,\alpha}(\mathbb{R}^N)$ such that $u = \tilde{u}$ for almost every $x, y \in \mathbb{R}^N$. We then say that $W^{1,p}$, for $p > N$, functions are Holder continuous.

Proof. Let $u \in C_0^1(\mathbb{R}^N)$ and Q_0 be a cube containing the origin with sides parallel to the axes, with length $= r$. So for $x \in Q$, we have

$$\begin{aligned} |u(x) - u(0)| &= \left| \int_0^1 \frac{d}{dt}(u(tx))\,dt = \int_0^1 \sum_{i=1}^N x_i \frac{\partial u}{\partial x_i}(tx)\,dt \right| \\ &\leq \int_0^1 \sum_{i=1}^N |x_i| \left| \frac{\partial u}{\partial x_i}(tx) \right| dt \leq r \sum_{i=1}^N \int_0^1 \left| \frac{\partial u}{\partial x_i}(tx) \right| dt. \end{aligned}$$

If

$$\bar{u} = \frac{1}{|Q_0|} \int_Q u(x)dx, \qquad |Q_0| = \int_{Q_0} dx = r^N$$

then

$$\begin{aligned} |\bar{u} - u(0)| &= \frac{1}{|Q_0|} \left| \int_{Q_0} (u(x) - u(0))\,dx \right| \leq \frac{1}{|Q_0|} \int_{Q_0} |u(x) - u(0)|\,dx \\ &\leq \frac{r}{|Q_0|} \int_0^1 \sum_{i=1}^N \int_{Q_0} \left| \frac{\partial u}{\partial x_i}(tx) \right| dx\,dt \\ &\leq \frac{1}{r^{N-1}} \int_0^1 \sum_{i=1}^N \int_{tQ_0} \left| \frac{\partial u}{\partial x_i}(y) \right| \frac{1}{t^N}\,dydt, \quad (tx = y) \end{aligned}$$

We then use Holder's inequality to estimate

$$\begin{aligned} \int_{tQ_0} \left| \frac{\partial u}{\partial x_i}(y) \right| dy &\leq \left(\int_{tQ_0} \left| \frac{\partial u}{\partial x_i} \right|^p \right)^{\frac{1}{p}} \left(\int_{tQ_0} 1 \right)^{\frac{1}{p'}} \\ &\leq \left\| \frac{\partial u}{\partial x_i} \right\|_p (tr)^{\frac{N}{p'}} = \left\| \frac{\partial u}{\partial x_i} \right\|_p (tr)^{\frac{N(p-1)}{p}} \end{aligned}$$

Therefore we get

$$\begin{aligned} |\bar{u} - u(0)| &\leq \frac{1}{r^{N-1}} r^{N(p-1)/p} ||\nabla u||_{L^p(Q_0)} \int_0^1 \frac{t^{N(p-1)/p}}{t^N}\, dt \\ &\leq \frac{r^{\frac{N}{p}}}{1-\frac{N}{p}} ||\nabla u||_{L^p(Q_0)} \end{aligned}$$

But this last inequality remains valid, by translation, for any cube Q with sides of length r and parallel to the axes, hence we obtain for any x_0 in this cube

$$|\bar{u} - u(x_0)| \leq \frac{r^{1-\frac{N}{p}}}{1-\frac{N}{p}} ||\nabla u||_{L^p(Q)}$$

Thus, for $x_0, y_0 \in Q$, we get

$$|u(x_0) - u(y_0)| \leq 2\frac{r^{1-\frac{N}{p}}}{1-\frac{N}{p}} ||\nabla u||_{L_p(Q)}$$

Now for any $x, y \in \mathbb{R}^N$, we can find a cube Q with sides of length $r = 2|x-y|$ parallel to the axes and containing x, y; consequently

$$\begin{aligned} |u(x) - u(y)| &\leq 2\frac{2^{1-\frac{N}{p}}|x-y|^{1-\frac{N}{p}}}{1-\frac{N}{p}} ||\nabla u||_{L^p(Q)} \\ &\leq C|x-y|^{1-\frac{N}{P}} ||\nabla u||_{L^p(\mathbb{R})} \end{aligned}$$

for any $u \in C_0^1(\mathbb{R}^N)$.

For $u \in W^{1,p}(\mathbb{R}^N)$, we use a sequence $(u_n) \subset C_0^1(\mathbb{R}^N)$ such that $u_n \longrightarrow u$ in $W^{1,p}(\mathbb{R}^N)$ and $u_n \longrightarrow u$ a.e. x is $\mathbb{R}^N$. Hence the second assertion of the theorem is established.

To establish the L^∞ bound, we use

$$|u(x) - \bar{u}| \leq C||\nabla u||_{L^p(Q)},$$

which implies for a cube containing x and with $r = 1$,

$$|u(x)| \leq |\bar{u}| + \frac{1}{1-\frac{N}{p}} ||\nabla_u||_{L^p(Q)}$$

By using

$$|\bar{u}| \leq \frac{1}{|Q|} \int |u(y)|dy \leq ||u||_{L^p(Q)}|Q| = |u||_{L^p(Q)}$$

We arrive at

$$|u(x)| \leq \frac{1}{1-\frac{N}{p}} \left(||\nabla u||_{L^p(Q)} + ||u||_{L^p(Q)} \right) \leq \frac{1}{1-\frac{N}{p}} ||u||_{W^{1,p}(\mathbb{R}^N)}$$

for any $u \in C_0^\infty(\mathbb{R}^N)$.

If $u \in W^{1,p}(\mathbb{R}^N)$, we then approximate it by a sequence (u_n) in $C_0^\infty(\mathbb{R}^N)$ which converges to u in $W^{1,p}(\mathbb{R}^N)$ and almost everywhere. Thus we obtain the desired result.

Corollary. If $u \in W^{1,p}(\mathbb{R}^N)$, $N < p < \infty$. Then $\lim\limits_{|x|\to\infty} u(x) = 0$.

Proof. We approximate by a sequence $(u_n) \subset C_0^\infty(\mathbb{R}^N)$. So, $\lim\limits_{n\longrightarrow\infty} u_n = u$ in $W^{1,p}(\mathbb{R}^N)$ and $\lim\limits_{n\to\infty} ||u_n - u||_\infty = 0$. So
$\forall\, \varepsilon > 0$, $\exists\, n_0 \in \mathbb{R}$ sucthat $\forall\, n > n_0, \quad |u_n(x) - u(x)| < \varepsilon, \quad \forall\, x \in \mathbb{R}^N$, which implies that

$$|u(x)| < |u_n(x)| + \varepsilon, \quad \forall\, x \in \mathbb{R}^N, \quad \forall\, n \geq n_0.$$

If $|x| \longrightarrow \infty$ then $u_n(x) \longrightarrow 0$, therefore

$$\lim_{|x|\to\infty} |u(x)| < \varepsilon, \quad \forall \varepsilon > 0 \Rightarrow \lim_{|x|\to\infty} |u(x)| = 0.$$