

**Lemma. (Partition of Unity)**

Let  $S \subset \mathbb{R}^N$  be compact and  $O_1, O_2, \dots, O_k$  be open covering  $S$ ; that is  $S \subset \bigcup_{i=1}^k O_i$ .

Then there exist functions  $\psi_0, \psi_1, \dots, \psi_k \in C^\infty(\mathbb{R}^N)$  such that

(i)  $0 \leq \psi_i \leq 1, \quad \forall i = 0, 1, 2, \dots, k$  and  $\sum_{i=0}^k \psi_i = 1$ .

(ii)  $\text{supp } \psi_i \subset O_i, \quad \forall i = 1, 2, \dots, k$  and  $\text{supp } \psi_0 \subset \mathbb{R}^N \setminus S$

**Proof.** For each  $x \in S$ , there exists  $r_x$  such that the ball  $B(x, 2r_x) \subset O_i$ , for some  $i \in \{1, 2, \dots, k\}$ .

Since  $S$  is compact then there exist a finite number of balls  $(B(x_i, r_{x_i}))$  covering  $S$ ; that is  $S \subset \bigcup_{i=1}^m B(x_i, r_{x_i})$ .

Let  $\varepsilon = \min_{1 \leq i \leq m} r_{x_i}$  and set  $O_{i,\varepsilon} = \{x \in O_i / \text{distance}(x, \partial O_i) > \varepsilon\}, i = 1, 2, \dots, k$ . It is easy to check that  $\{O_{i,\varepsilon}\}_{i=1}^k$  is a covering of  $S$ . Let

$$O'_1 = O_{1,\varepsilon}, \quad O'_j = O_j \setminus \bigcup_{i=1}^{j-1} O_{i,\varepsilon}, \quad \forall j = 2, \dots, k \quad \text{and} \quad O_0 = \bigcup_{i=1}^k O_i = \bigcup_{i=1}^k O_{i,\varepsilon}$$

Let  $\chi_i = \chi_{O'_i}$  (the characteristic function), so  $\sum_{i=1}^k \chi_i = 1$  on  $O_0 \supset S$  since  $(O'_i)$  are pairwise disjoint. define

$$\psi_i = \rho_h * \chi_i, \quad h < \min \left\{ \frac{\varepsilon}{2}, \text{dist}(S, \partial O_0) \right\}$$

So

$$\psi_i \in C^\infty(\mathbb{R}^N) \quad \text{and} \quad \psi_i(x) = 0, \quad \forall x / \text{dist}(x, O'_i) > h$$

hence  $\psi_i \in C_0^\infty(O_i)$ . Also

$$\begin{aligned} \sum_{i=1}^k \psi_i &= \sum_{i=1}^k \int_{\mathbb{R}^N} \rho_h(x-y) \psi_i(y) dy \\ &= \int_{\mathbb{R}^N} \rho_h(x-y) \sum_{i=1}^k \psi_i(y) dy = \int_{\mathbb{R}^N} \rho_h = 1, \quad \forall x \in O_0 \supset S. \end{aligned}$$

**Proposition (Change of variables)**

Let  $H : \Omega' \rightarrow \Omega$  be a bijection with  $\Omega, \Omega'$  opens of  $\mathbb{R}^N$  and such that

$$H \in C^1(\Omega'), \quad H^{-1} \in C^1(\Omega), \quad \text{Jac } H \in L^\infty(\Omega'), \quad \text{Jac } H^{-1} \in L^\infty(\Omega)$$

If  $u \in W^{1,p}(\Omega)$  then  $u \circ H \in W^{1,p}(\Omega')$  with

$$\frac{\partial}{\partial y_i} (u \circ H)(y) = \sum_{i=1}^N \frac{\partial u}{\partial x_i} (H(y)) \frac{\partial H_i}{\partial y_j} (y), \quad 1 \leq i \leq N.$$

Here  $H(y) = x$ .

**Proof:** For  $1 \leq p < \infty$ , we choose a sequence  $(u_n) \in C_0^\infty(\mathbb{R}^N)$  such that  $u_n \rightarrow u$  in  $L^p(\Omega)$  and  $\nabla u_n \rightarrow \nabla u$  in  $[L^p(\omega)]^N$ ,  $\forall \omega \subset\subset \Omega$ . So

$$u_n \circ H \rightarrow u \circ H \text{ in } L^p(\Omega')$$

and

$$\frac{\partial u_n}{\partial x_i} \circ H \rightarrow \frac{\partial u}{\partial x_i} \circ H \text{ in } L^p(\omega'), \quad \forall \omega' \subset\subset \Omega'$$

By taking  $\phi \in C_0^1(\Omega')$ , we easily see that

$$\int_{\Omega'} (u_n \circ H) \frac{\partial \phi}{\partial y_j} = - \int_{\Omega'} \sum_{i=1}^N \left( \frac{\partial u_n}{\partial x_i} \circ H \right) \frac{\partial H_i}{\partial y_j} \phi$$

By letting  $n$  go to  $\infty$ , we arrive to the desired result. For  $p = +\infty$ , we proceed like the previous theorems.

**Theorem.** (Extension Theorem)

Suppose that  $\Omega$  is of class  $C^1$  with  $\partial\Omega$  bounded (or  $\Omega = \mathbb{R}_+^N$ ). Then there exists an extension operator

$$P : W^{1,p}(\Omega) \rightarrow W^{1,p}(\mathbb{R}^N)$$

linear and such that  $\forall u \in W^{1,p}(\Omega)$

(i)  $Pu|_\Omega = u$

(ii)  $\|Pu\|_{L^p(\mathbb{R}^N)} \leq C\|u\|_{L^p(\Omega)}$

(iii)  $\|Pu\|_{W^{1,p}(\mathbb{R}^N)} \leq C\|u\|_{W^{1,p}(\Omega)}$ .

$C$  is a constant depending on  $p$  and  $\Omega$ .

**Proof.** Since  $\partial\Omega$  is compact and of class  $C^1$ , then there exists  $k$  opens  $(O_i)_{i=1}^k$  such that  $\partial\Omega \subset \bigcup_{i=1}^k O_i$  and bijections  $H_i : Q \rightarrow O_i$  such that

$$H_i \in C^1(\bar{Q}), \quad H_i^{-1} \in C^1(\bar{O}_i), \quad H_i(Q_+) = O_i \cap \Omega$$

and

$$H_i(Q_0) = O_i \cap \partial\Omega$$

Consider the functions  $\theta_0, \theta_1, \dots, \theta_k$  seen in the partition of unity lemma. We then set

$$u = u \sum_{i=0}^k \theta_i = \sum_{i=0}^k u \theta_i = \sum_{i=0}^k u_i, \quad u_i = u \theta_i$$

**Extension of  $u_0$ :** Let

$$\tilde{u}_0(x) = \begin{cases} u_0(x) & , \quad x \in \Omega \\ 0 & , \quad x \in \mathbb{R}^N \setminus \Omega \end{cases}$$

and note that

$$\theta_0 \in C^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \quad 0 \leq \theta_0 \leq 1, \quad \nabla \theta_0 \in [L^\infty(\mathbb{R}^N)]^N$$

since

$$\nabla \theta_0 = - \sum_{i=1}^k \nabla \theta_i, \quad \text{supp } \theta_i \subset O_i, \quad \forall i = 1, 2, \dots, k,$$

and  $\text{supp } \theta_i$  is compact. Therefore

$$\tilde{u}_0 \in W^{1,p}(\mathbb{R}^N), \quad \frac{\partial}{\partial x_i} \tilde{u}_0 = \theta_0 \frac{\partial \tilde{u}}{\partial x_i} + \tilde{u} \frac{\partial \theta_0}{\partial x_i},$$

hence

$$\|u_0\|_{W^{1,p}(\mathbb{R}^N)} \leq C \|u\|_{W^{1,p}(\Omega)},$$

where  $C$  is constant depending on the  $L^\infty$  norm of  $\theta_0$  and  $\nabla \theta_0$ .

**Extension of  $u_i$ ,  $1 \leq i \leq k$  :**

We consider the restriction of  $u$  to  $O_i \cap \Omega$  and we "transport" it over  $Q_+$  by using  $H_i$ . For this, we define

$$v_i(y) = u(H_i(y)), \quad \forall y \in Q_+.$$

It is easy to verify that  $v_i \in W^{1,p}(Q_+)$ . We then extend  $v_i$  to  $Q$  by reflection and denote this extension by  $v_i^*$ , which belongs to  $W^{1,p}(Q)$ . We then "retransport back"  $v_i^*$  over  $O_i$  by using  $H_i^{-1}$ . Let

$$w_i(x) = v_i^* [H_i^{-1}(x)], \quad \forall x \in O_i;$$

So

$$w_i \in W^{1,p}(O_i) \text{ and } w_i = u \text{ over } O_i \cap \Omega$$

with

$$\|w_i\|_{W^{1,p}(O_i)} \leq C \|u\|_{W^{1,p}(O_i \cap \Omega)}$$

Finally let

$$\hat{u}_i(x) = \begin{cases} \theta_i(x) w_i(x) & , \quad \forall x \in O_i \\ 0 & , \quad \forall x \in \mathbb{R}^N \setminus O_i \end{cases}$$

By the above lemma, we have

$$\hat{u}_i \in W^{1,p}(\mathbb{R}^N), \quad \hat{u}_i = u_i \text{ over } \Omega$$

and

$$\|\hat{u}_i\|_{W^{1,p}(\mathbb{R}^N)} \leq C \|u\|_{W^{1,p}(O_i \cap \Omega)}$$

**Conclusion.**  $Pu = \tilde{u}_0 + \sum_{i=1}^k \hat{u}_i$  has all desired properties.

**Corollary (Density):**

Suppose that  $\Omega$  is of class  $C^1$  and let  $u \in W^{1,p}(\Omega)$  be given with  $1 \leq p < +\infty$ . Then there exists a sequence  $(u_n) \subset C_0^\infty(\mathbb{R}^N)$  such that

$$u_{n|\Omega} \longrightarrow u \text{ in } W^{1,p}(\Omega).$$

**Proof.**

- (1) If  $\partial\Omega$  is bounded we then extend  $u$  to  $\mathbb{R}^N$  and then we take  $u_n = \xi_n(\rho_n * Pu)$ , which converges to  $Pu$  in  $W^{1,p}(\mathbb{R}^N)$ . In particular  $u_{n|\Omega} \longrightarrow u$  in  $W^{1,p}(\Omega)$ .
- (2) If  $\partial\Omega$  is unbounded, then we consider the sequence  $\xi_n u$ , where  $\xi_n = \xi\left(\frac{x}{n}\right)$  and  $\xi$  is the truncation function. We know that  $\xi_n u \longrightarrow u$  in  $W^{1,p}(\Omega)$ , so for  $\varepsilon > 0$ ,  $\exists n_0 \in \mathbb{N}$  such that  $\|\xi_{n_0} u - u\|_{W^{1,p}(\Omega)} < \frac{\varepsilon}{2}$ .

Since  $\text{supp } \xi_{n_0} u$  is included in a large ball we then extend  $\xi_{n_0} u$  to  $\mathbb{R}^N$  and apply the above to get a function  $v_\varepsilon$  in  $C_0^\infty(\mathbb{R}^N)$  such that  $\|v_\varepsilon - \xi_{n_0} u\|_{W^{1,p}} < \frac{\varepsilon}{2}$ . Consequently

$$\|u_0 - v_\varepsilon\|_{W^{1,p}(\Omega)} < \varepsilon$$

We then construct the sequence  $(v_\varepsilon) \subset C_0^\infty(\mathbb{R}^N)$  which converges to  $u$  in  $W^{1,p}(\Omega)$ .