1 Classical L^p Spaces

1.1 Definitions and Properties

Definition: Let E be a measurable set of $\mathbb{R}^n$. We say that a function f is in $L^p(E)$, if f is measurable and $\int_E |f|^p < +\infty$.
Remark: The integral here is in the Lebesgue sense and $p \in (0, +\infty)$.
Proposition: $L^p(E)$ is a linear space.
Proof: f and g be in $L^p(E)$ and $\alpha \in \mathbb{R}$. It is clear that

$$\begin{aligned} \int_E |\alpha f|^p &= |\alpha|^p \int_E |f|^p < \infty \\ \int_E |f+g|^p &\leq 2^p \left(\int_E |f|^p + \int_E |g|^p \right) < \infty \end{aligned}$$

We equip $L^p(E)$ with the "natural" norm

$$\|f\|_p = \left(\int |f|^p \right)^{1/p}, \quad p \geq 1.$$

One can easily verify that this is, indeed, a norm.
Definition: We call $L^\infty(E)$ the set of all functions which are bounded on E, except maybe on a subset of measure zero.
Examples: 1) $f(x) = x$ is not in $L^\infty(\mathbb{R})$.
2)

$$g(x) = \begin{cases} x, & x \in Q \\ 1, & \text{otherwise} \end{cases}$$

is in $L^\infty(\mathbb{R})$. We have in this case $|g(x)| \leq 1$, a.e.
Definition: On $L^\infty(E)$, we define the norm by

$$\|f\|_\infty = ess \sup |f(t)| = \inf\{M : \text{meas}\,\{t \quad / \quad |f(t)| > M\} = 0\}.$$

In the previous example, $\|g\|_\infty = 1$.
Proposition: If $f = g$ a.e. on E, then

$$\|f\|_p = \|g\|_p, \quad \forall \;\; p \in [1, +\infty].$$

Exercise: If

$$f(x) = \begin{cases} x, & x \text{ irrational}, \\ 1, & x \text{ rational}, \end{cases} .$$

show that $\|f\|_\infty = \infty$.
Theorem: (Hölder's inequality) Let p and q be nonnegative extended real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(E)$ and $g \in L^q(E)$, then $fg \in L^1(E)$ and $\int_E |fg| \leq \|f\|_p \; \|g\|_q$.
Theorem: (Riesz-Fischer) The $L^p(E)$, $\quad 1 \leq p \leq \infty$, equipped with the "natural" norm is complete (Banach space).
Remark: $L^2(E)$ is a Hilbert space. The inner product is $\langle f, g \rangle = \int_E fg$.

1.2 Bounded Linear Functionals on L^p Spaces

Given a fixed g in $L^q(E)$; we define the functional $F : L^p(E) \to \mathbb{R}$ by $F(f) = \int_E fg$. This is well defined since $f \in L^p(E)$ and $L^q(E)$, for $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem: F is a linear functional such that $\|F\| = \|g\|_{L^q(E)}$.

Proof: let us remember that

$$\|F\| = \sup_{f \neq 0} \frac{|F(f)|}{\|f\|_p}, \qquad f \in L^p(E)..$$

So

$$|F(f)| = \left| \int fg \right| \leq \|f\|_p \cdot \|g\|_q \Rightarrow \|F\| \leq \|g\|_q \tag{1.1}$$

Next, we set for, $1 < p < \infty$

$$h = |g|^{q/p} \text{ sing } g = \begin{cases} |g|^{q/p}, & g(x) \geq 0 \\ -|g|^{q/p}, & g(x) < 0. \end{cases}$$

It is clear that $\int |h|^p = \int |g|^q \Rightarrow h \in L^p(E)$. So,

$$\begin{aligned} F(h) &= \int |g|^{q/p} g(\text{sing } g) = \int |g|^{q/p} \ |g| = \int |g|^q \\ &= \|g\|_q^q = \|g\|_q \cdot \|g\|_q^{q-1} = \|g\|_q \ \|h\|_p. \end{aligned}$$

Hence

$$\frac{|F(h)|}{\|h\|_p} = \|g\|_q \Rightarrow \|F\| \geq \|g\|_q. \tag{1.2}$$

For (1) and (2), we obtain $\|F\| = \|g\|_q$.

Lemma: Let g be measurable on E. Suppose there exists $M > 0$ such that

$$\left| \int fg \right| \leq M \|f\|_p, \quad \text{for all } \ f \text{ in } L^p(E).$$

Then

$$g \in L^q \text{ and } \|g\|_q \leq M; \qquad (1 \leq p \leq \infty).$$

Theorem: (Riesz Representation Theorem). Let F be a bounded linear functional on $L^p(E)$, $1 \leq p < \infty$. Then there exists g in $L^q(E), 1/q + 1/p = 1$, such that

$$F(f) = \int_E fg, \quad \forall \ f \in L^p(E).$$

Moreover, we have $\|F\| = \|g\|_q$.

Exercise: 1) Show that if E is a set of finite measure (bounded for example) and $f \in L^p(E)$, then $f \in L^r(E)$ for all $r \leq p$.

2) How about if E is of infinite measure (unbounded for example) ?