On separable-variable functions

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Abstract

In this note we generalize Scott's result and give a way how to factorize separable-variable functions.

1 Introduction

Scott [2] in 1985 gave a necessary and sufficient condition for a two-variable function to be of separable variables. His result, of course, is very important; especially in the area of differential equations. In this work we generalize Scott's result to multi-variable functions ($n \geq 2$) and give a way how to factorize such separable variable functions. To make this note self contained we state the theorem by Scott. For the proof we refer the reader to [1] and [2].

Proposition

1) Suppose that f is a separable variable function on a domain D of $\mathbb{R}^2$; that is $f(x, y) = \phi(x)\psi(y)$. If ϕ and ψ are differentiable then

$$f\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} \tag{1.1}$$

2) Suppose that f, $\partial f/\partial x$, $\partial f/\partial y$, and $\partial^2 f/\partial x \partial y$ exist and are continuous on a domain D. If $f \neq 0$ and (1.1) holds then f is a separable variable function.

2 Main Result.

In this section we state and prove our main result. We first begin with the following

Lemma

Suppose that f, $\partial f/\partial x_i$, and $\partial^2 f/\partial x_i \partial x_n$, $\forall i = 1, 2, ..., n$, exist and are continuous on a domain D of $\mathbb{R}^n$ $(n \geq 2)$. If f is never zero on D and

$$f\frac{\partial^2 f}{\partial x_i \partial x_n} = \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_n}, \quad i = 1, 2..., n-1 \tag{2.1}$$

then there exist functions ϕ and g such that

$$f(x_1, ..., x_n) = \phi(x_n) g(x_1, ..., x_{n-1}) \tag{2.2}$$

Proof. We prove this lemma by induction. For $n = 2$, the result is trivial by Scott's theorem.

Suppose that (2.2) holds for $n-1$. To establish it for n, we first note that by virtue of (2.1) we have

$$\frac{\partial}{\partial x_n}\left[\frac{\partial f/\partial x_{n-1}}{f}\right] = \frac{f\partial^2 f/\partial x_{n-1}\partial x_n - \partial f/\partial x_{n-1}\partial f/\partial x_n}{f^2} = 0.$$

So by integrating with respect to x_n, we get

$$\frac{\partial f/\partial x_{n-1}}{f} = q(x_{1,}..., x_{n-2}, x_{n-1}).$$

Again an integration with respect to x_{n-1}, leads to

$$\ln|f(x_1, ..., x_n)| = \int q(x_{1,}..., x_{n-2}, x_{n-1}) dx_{n-1} + r(x_1, ..., x_{n-2}, x_n);$$

hence we have

$$f(x_1, ..., x_n) = Q(x_{1,}..., x_{n-2}, x_{n-1}) R(x_1, ..., x_{n-2}, x_n).$$

By taking derivatives with respect to x_i, $i \neq n-1$, we obtain

$$\frac{\partial f}{\partial x_n} = Q\frac{\partial R}{\partial x_n}, \quad \frac{\partial f}{\partial x_i} = Q\frac{\partial R}{\partial x_i} + R\frac{\partial Q}{\partial x_i}, \quad \forall i = 1, ..., n-2.$$

$$\frac{\partial^2 f}{\partial x_i \partial x_n} = Q\frac{\partial^2 R}{\partial x_i \partial x_n} + \frac{\partial Q}{\partial x_i}\frac{\partial R}{\partial x_n}, \quad \forall i = 1, ..., n-2.$$

By sustituting in (2.1), we get

$$R\frac{\partial^2 R}{\partial x_i \partial x_n} = \frac{\partial R}{\partial x_i}\frac{\partial R}{\partial x_n}, \quad \forall i = 1, 2..., n-2. \tag{2.3}$$

Since R is a $(n-1)$-variable function, which never vanishes on D and satisfies (2.3) then there exist functions ϕ and S such that

$$R(x_1, ..., x_{n-2}, x_n) = \phi(x_n)S(x_1, ..., x_{n-2}).$$

Consequently

$$\begin{aligned} f(x_1, ..., x_n) &= Q(x_1,..., x_{n-2}, x_{n-1})\phi(x_n)S(x_1, ..., x_{n-2}) \\ &= \phi(x_n)g(x_1, ..., x_{n-1}). \end{aligned} \tag{2.4}$$

Theorem 1 .

Let f be such that f, $\partial f/\partial x_i$, and $\partial^2 f/\partial x_i \partial x_j$, $\forall i, j = 1, 2, ..., n$, $i \neq j$, exist and are continuous on a domain D of $\mathbb{R}^n$ $(n \geq 2)$. If f is never zero on D and

$$f\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}, \quad i, j = 1, 2..., n, \quad i \neq j \tag{2.5}$$

then there exist differentiable functions $\phi_1, ..., \phi_n$ such that

$$f(x_1, ..., x_n) = \phi_1(x_1)\phi_2(x_2)...\phi_n(x_n). \tag{2.6}$$

Proof. We prove this theorem by induction. For $n = 2$, the result is trivial by Scott's theorem.

Suppose that (2.6) holds for $n-1$. To prove it for n, we first note that

$$f(x_1, ..., x_n) = \phi_n(x_n)g(x_1, ..., x_{n-1}),$$

by virtue of the lemma. As a consequence we have

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 g}{\partial x_i \partial x_j}\phi_n(x_n), \quad i, j = 1, 2..., n-1, \quad i \neq j.$$

Therefore a substitution in (2.5) yields

$$g\frac{\partial^2 g}{\partial x_i \partial x_j} = \frac{\partial g}{\partial x_i}\frac{\partial g}{\partial x_j}, \quad i, j = 1, 2..., n-1, \quad i \neq j. \tag{2.7}$$

Since g is a $(n-1)$-variable function, which never vanishes on D and sataifies (2.7) then there exist functions $\phi_1, ..., \phi_{n-1}$ such that $g(x_1, ..., x_{n-1}) = \phi_1(x_1)...\phi_{n-1}(x_{n-1})$, which implies (2.6).

Remark 1. It is easy to see that any separable-variable function satisfies (2.5) provided that the relevant partial derivatives exist and are continuous.

Theorem 2.

Let f be a separable variable function satisfying $f(\xi) = 1$, for some point ξ in the domain D. Then $f(x) = \phi_1(x_1)\phi_2(x_2)...\phi_n(x_n)$, where

$$\phi_i(x_i) = f(\xi_1, ..., \xi_{i-1}, x_i, \xi_{i+1}, ..., \xi_n)$$

Proof.

Since $f(\xi) = 1$ we can take $\phi_i(\xi_i) = 1, \quad \forall i = 1, ..., n.$
Therefore

$$f(\xi_1, ..., \xi_{i-1}, x_i, \xi_{i+1}, ..., \xi_n) = \phi_1(\xi_1)...\phi_{i-1}(\xi_{i-1})\phi_i(x_i)\phi_{i+1}(\xi_{i+1})...\phi_n(\xi_n) = \phi_i(x_i).$$

Remark 2. It is clear that the condition $f(\xi) = 1$ can be replaced by $f(\xi) \neq 0$.
Example Let $f(x, y) = [cos(x-y) - cos(x+y)]/2$. It is easy to verify that f satisfies (2.5), for $n = 2$ and $f(\pi/2, \pi/2) = 1$. Therefore

$\phi(x) = f(x, \pi/2) = [cos(x - \pi/2) - cos(x + \pi/2)]/2 = [\sin(x) + \sin(x)]/2 = \sin(x)$

$\psi(y) = f(\pi/2, y) = [cos(\pi/2 - y) - cos(\pi/2 + y)]/2 = [\sin(y) + \sin(y)]/2 = \sin(y)$

By using the trigonometric identities, we easily see that ϕ and ψ are the right factors of f.

References.

1. Derrick W. R. and S. I Grossman, A first course in differential equations, $\mathbf{3}^d$ Edition, West Publishing Company, USA 1987.

2. Scott D., When is an ordinary differential equation separable, Ameri. Math. Monthly **92** (1985), 422 - 423.