Section 8.1 - Arc Length

The arc length formula can be defined as follows:

$L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2} dx$ if we are finding length of a **smooth** curve $y = f(x)$ between $x = a$ and $x = b$ or

$L = \int_c^d \sqrt{1 + (\frac{dx}{dy})^2} dy$ if we are finding length of a **smooth** curve $x = g(y)$ between $y = c$ and $y = d$

The formula is derived by summing up all the many "small" straight lines each of length $\Delta s = \sqrt{\Delta x^2 + \Delta y^2}$ over the required length of the curve. So, $L = \int \sqrt{dx^2 + dy^2}$ when we take infinitely many very very small lines $ds = \sqrt{dx^2 + dy^2}$.

We can extend the arc length formula above by using a dummy variable of integration to get:

$L(x) = \int_a^x (\sqrt{1 + (f'(t))^2} dt$ for a smooth curve $y = f(x)$ starting from $x = a$.

We can see that this can be easily extended for $x = g(y)$.

Example: Find the arc length function for the curve $y = 2x^{3/2}$ with starting point $P_o(1, 2)$

$f(x) = 2x^{3/2} \Rightarrow f'(x) = 3x^{1/2}$.

$$L(x) = \int_a^x \sqrt{1 + (f'(t))^2} dt = \int_1^x \sqrt{1 + 9t} dt$$

Now we can use the substitution rule. Set $u = 1 + 9t$.

Then, we will have $\displaystyle\int \sqrt{u} \frac{du}{9} = \frac{2u^{3/2}}{27} \to \frac{2(1 + 9t)^{3/2}}{27}$

$L(x) = [\frac{2(1+9t)^{3/2}}{27}]_1^x = \frac{2}{27}((1+9x)^{3/2} - 10^{3/2})$.