Section 6.2 - Volumes

The basic definition of a volume = cross-sectional area $\times$ height. In integral terms, this translates to $V = \int A(x)dx$ or $V = \int A(y)dy$ where A is the cross-sectional area function expressed in either x or y depending on orientation.

Example: Find the volume of a frustum of a pyramid with a square base of side b. square top of side a and height h.

Solution:
First, we need to determine the cross-sectional area function. Note that the approach will be similar to that of example 8 (textbook page 451).

The slope of the side is $\frac{a-b}{2h}$. So at a particular value of y, the length of the square will be $2 \times [\frac{a-b}{2h}y + b/2] = \frac{y(a-b)}{h} + b$. Check: at $y = 0$, the length of square is b and at $y = h$, the length of the square is a. CORRECT!!!

Thus, the cross-sectional area function is $A = (\frac{y(a-b)}{h} + b)^2$.

So the volume of the frustum $V = \int_0^h A dy = \dfrac{h^3(a-b)^2}{3h^2} + hb^2 + \dfrac{h^2(a-b)b}{h}$

$V = \frac{h(a-b)^2}{3} + hb^2 + \frac{h(a-b)b}{1} = \frac{h(a^2+ab+b^2)}{3}$

A solid of revolution is generated by revolving a plane area about a line, called the axis of rotation in the plane. This volume of a solid of revolution may be found with one of three procedures. In this section, we will go through the disc method and the washer method. Next week, we will go through the shell method.

Disc method:

$V = \int_a^b \pi(f(x))^2 dx$ or $V = \int_c^d \pi(g(y))^2 dy$ depending on whether the axis of rotation is the y-axis for the latter and the x-axis for the former. Note that the functions $f(x)$ and $g(y)$ are rotated about the x-axis and y-axis respectively.

Example: Find the volume of a cap of a sphere with radius r and height h.

Solution:
Let us consider at the cross-sectional area. The sphere is essentially a rotation of the semi-circle $x^2 + y^2 = r^2$, about the y-axis.

Since the cap is oriented vertically, we express this as a function of y and hence rotate the curve about the y-axis. So, $g(y) = \sqrt{r^2 - y^2}$ and the ranges of integration are from $y = r - h$ to $y = r$. Note that r is fixed since we are dealin with a sphere after all.

$$V = V = \int_c^d \pi(r^2 - y^2)dy = \pi(rh^2 - \frac{h^3}{3}).$$

Washer method:

$V = \int_a^b \pi[(f(x))^2 - (h(x))^2]dx$ or $V = \int_c^d \pi[(p(y))^2 - (q(y))^2]dy$ depending on whether the axis of rotation is the y-axis for the latter and the x-axis for the former.

Example: Find the volume generated by revolving the area cut off from the parabola $y = 4x - x^2$ by the x-axis about the line $y = 6$.

Solution:
Sketch the line and the the parabola. Identify the region first. It is the region between the parabola and the x-axis. What is making it complicated however is the fact that the axis of rotation is not the x-axis but $y = 6$. The parabola cuts the x-axis when $y = 4x - x^2 = 0$. Hence, $x = 0,\ 4$

So the volume of interest is generated by taking the cylinder (x-axis rotated about $y = 6$) and subtracting the volume generated by the region between parabola and $y = 6$. So, the first volume $V_1 = \int_0^4 \pi(6)^2 dx$ and the second volume is $V_2 = \int_0^4 \pi[6 - (4x - x^2)]^2 dx$.

So, the volume of interest, $V = V_1 - V_2$. You may continue the computation. You should get $\frac{1408\pi}{15}$ cubic units.