Section 6.2 - Volumes

The basic definition of a volume = cross-sectional area × height. In integral terms, this translates to \( V = \int A(x) \, dx \) or \( V = \int A(y) \, dy \), where \( A \) is the cross-sectional area function expressed in either \( x \) or \( y \) depending on orientation.

Example: Find the volume of a frustum of a pyramid with a square base of side \( b \), square top of side \( a \) and height \( h \).

Solution:
First, we need to determine the cross-sectional area function. Note that the approach will be similar to that of example 8 (textbook page 451).

The slope of the side is \( \frac{a-b}{2h} \). So at a particular value of \( y \), the length of the square will
be $2 \times \left[ \frac{a-b}{2h} y + \frac{b}{2} \right] = \frac{y(a-b)}{h} + b$. Check: at $y = 0$, the length of square is $b$ and at $y = h$, the length of the square is $a$. CORRECT!!!
go through the disc method and the washer method. Next week, we will go through the shell method.

Disc method:

\[ V = \int_a^b \pi (f(x))^2 \, dx \text{ or } V = \int_c^d \pi (g(y))^2 \, dy \]
depending on whether the axis of rotation is the y-axis for the latter and the x-axis for the former. Note that the functions \( f(x) \) and \( g(y) \) are rotated about the x-axis and y-axis respectively.

Example: Find the volume of a cap of a sphere with radius \( r \) and height \( h \).

Solution:
Let us consider at the cross-sectional area. The sphere is essentially a rotation of the semi-circle \( x^2 + y^2 = r^2 \), about the y-axis.
Since the cap is oriented vertically, we express this as a function of \( y \) and hence rotate the curve about the \( y \)-axis. So, \( g(y) = \sqrt{r^2 - y^2} \) and the ranges of integration are from \( y = r - h \) to \( y = r \). Note that \( r \) is fixed since we are dealin with a sphere after all.

\[
V = \pi \int_{r-h}^{r} (r^2 - y^2) \, dy = \pi (rh^2 - \frac{h^3}{3}).
\]

Washer method:

\[
V = \int_{a}^{b} \pi [(f(x))^2 - (h(x))^2] \, dx \quad \text{or} \quad V = \int_{c}^{d} \pi [(p(y))^2 - (q(y))^2] \, dy
\]

depending on whether the axis of rotation is the \( y \)-axis for the latter and the \( x \)-axis for the former.

Example: Find the volume generated by revolving the area cut off from the parabola \( y = 4x - x^2 \) by the \( x \)-axis about the line \( y = 6 \).

Solution:
Sketch the line and the parabola. Identify the region first. It is the region between the parabola and the x-axis. What is making it complicated however is the fact that the axis of rotation is not the x-axis but $y = 6$. The parabola cuts the x-axis when $y = 4x - x^2 = 0$. Hence, $x = 0, 4$

So the volume of interest is generated by taking the cylinder (x-axis rotated about $y = 6$) and subtracting the volume generated by the region between parabola and $y = 6$. So, the first volume $V_1 = \int_0^4 \pi (6)^2 dx$ and the second volume is $V_2 = \int_0^4 \pi [6 - (4x - x^2)]^2 dx$.

So, the volume of interest, $V = V_1 - V_2$. You may continue the computation. You should get $\frac{1408\pi}{15}$ cubic units.