Section 6.1 - Area under curves

Assume that $f(x)$ and $g(x)$ are continuous functions such that $0 \leq g(x) \leq f(x)$. Then the area A of the region R between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is given by

$$A = \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b [f(x) - g(x)]dx$$

Sometimes, it is easy to change the variable of integration. It depends largely on the geometry of the problem.

The general strategy is to consider whether which is easier to evaluate.

Example: Sketch the region enclosed by the curves $y = |x|$ and $y = x^2 - 2$. Find the area of this region.

Solution:
First, we need to determine the region itself (through calculation).

When $x < 0$, $y = -x$ and $y = x$ when $x > 0$.

First intersection point in $x < 0$: $-x = x^2 - 2 \to x = -2, 1$. So the relevant point is at $x = -2$

Second intersection point in $x > 0$: $x = x^2 - 2 \to x = 2, -1$. So the relevant point is at $x = 2$.

$$A = \int_{-2}^{0} [-x - (x^2 - 2)]dx + \int_{0}^{2} [x - (x^2 - 2)]dx$$

$$A = [-\frac{x^2}{2} - \frac{x^3}{3} + 2x]^0_{-2} + [\frac{x^2}{2} - \frac{x^3}{3} + 2x]^2_0$$

$A = -[-\frac{2^2}{2} - \frac{(-2)^3}{3} + 2(-2)] + [\frac{2^2}{2} - \frac{2^3}{3} + 2(2)] = 4.$

Example: Find the values of c such that the area of the region enclosed by the parabolas $y = x^2 - c^2$ and $y = c^2 - x^2$ is 576.

Solution:
Again, we need to determine the region itself.

Intersection points occur when $x^2 - c^2 = c^2 - x^2 \to 2x^2 = 2c^2 \to x = \pm c$

So, $A = \int_{-c}^{c} [(c^2 - x^2) - (x^2 - c^2)]dx = [2xc^2 - \frac{2x^3}{3}]_{-c}^{c}$

$A = 576 = 8c^3/3$

$c = 6.$