Section 5.4 - Indefinite Integrals and Net Change Theorem

So far, we have considered definite integrals because the limits of integration were specified (regardless of whether they are functions or constants). An indefinite integral is defined by

$\int f(x)dx = F(x)$ where $F(x)$ is the anti-derivative of $f(x)$ and $F'(x) = f(x)$ as we have seen before.

Please refer to the Table of Indefinite Integrals in your textbook page 406. You are required to know them well and the only way to do this is to practise!! The questions in the textbook are quite straightforward but there are some nasty ones. We shall go through two of them here.

Example: Evaluate the integral $\int_0^{\frac{3\pi}{2}} |sin(x)|dx$

Solution:
It is a good idea to always sketch the integrand where possible. Note that the $||$ refers to absolute value, i.e. always made positive.

We can solve this question by splitting the integral as follows:

$$\int_0^{\frac{3\pi}{2}} |sin(x)|dx = \underbrace{\int_0^{\pi} sin(x)dx}_{A} + \underbrace{\int_{\pi}^{\frac{3\pi}{2}} (-sin(x))dx}_{B}$$

Integral B is similar to the $|sin(x)|$ curve over $[\pi, \frac{3\pi}{2}]$. So, we then have:

$$\int_0^{\frac{3\pi}{2}} |sin(x)|dx$$
$$= -cos(\pi) - (-cos(0)) + cos(\frac{3\pi}{2}) - cos(\pi)$$
$$= 1 + 1 + 0 - (-1) = 3.$$

Net Change Theorem:

The integral of a rate of change is the net change defined by

$$\int_a^b F'(x)dx = F(b) - F(a)$$

Example: If $w'(t)$ is the rate of growth of a child in pounds per year then what does $\int_5^{10} w'(t)dt$ represent?

Solution:
From the net change theorem, $\int_5^{10} w'(t)dt = w(10) - w(5)$. Since w is the weight of a child then the integral represents a change in the weight between 5 and 10 years.

Example: The velocity function $v(t) = t^2 - 2t - 8$ (in m/s) is given for a particle moving along a line. Find (a) the displacement and (b) the distance travelled by the particle during the given time interval: $1 \leq t \leq 6$.

Solution:

(a) Displacement here is not the total distance covered but the change in distance between 2 points. In Physics, we say that displacement is direction dependent. So we recall the net change theorem.

$$\text{Displacement} = \int_1^6 v(t)dt$$

$$= [\frac{t^3}{3} - t^2 - 8t]_1^6 = -\frac{10}{3}\text{m}.$$

Note that you are to specify the units in your solution. What does the negative sign mean? Discussion in class.

(b) Distance covered is not direction dependent, i.e. independent of direction. So,

$$\text{Distance} = \int_1^6 |v(t)|dt$$

The next step is to find out where does $v(t) < 0$. First, we have to solve $v(t) = 0$. After working it out, you should have $t = -2, 4$. Since $t = -2$ is not inside our interval, the only relevant value of t is $t = 4$. $v(t)$ is "bowl-shaped", so $v(t) < 0$ for $t \in [1, 4]$

As we did earlier, split the integral into two.

$$\text{Distance} = \int_1^4 -v(t)dt + \int_4^6 (v(t))dt$$

$$= -[\tfrac{t^3}{3} - t^2 - 8t]_1^4 + [\tfrac{t^3}{3} - t^2 - 8t]_4^6$$

$$= -[\tfrac{4^3}{3} - 4^2 - 32 - (\tfrac{1}{3} - 1 - 8)] + [\tfrac{6^3}{3} - 6^2 - 48 - (\tfrac{4^3}{3} - 4^2 - 32)]$$

$= \frac{-64+1+216-64}{3} + 16 + 32 + 1 + 8 - 36 - 48 + 16 + 32 = \frac{89}{3} + 21 = 50\frac{2}{3}\text{m}.$