Section 5.2 - The Definite Integral

The definite integral is defined to be:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

where $[a, b]$ is known as the domain of integration, x_i^* is a sample point chosen within a rectangle, $\Delta x = \frac{(b-a)}{n}$ and n is the number of rectangles.

Note that the sample point can be chosen to be the left-end, right-end points or a point between the two former points.

a, b are known as the lower and upper limits of integration. $f(x)$ is known as the integrand.

The Riemann sum is defined to be:

$$\sum_{i=1}^{n} f(x_i^*)\Delta x.$$

So we see that when we take the limit of a Riemann sum as $n \to \infty$, then we end up with a definite integral.

If the curve dips below the x-axis (see textbook page 382 Figure 4), then the Riemann sum is the sum of areas of rectangles above the x-axis (positive) and below the x-axis (negative).

Hence, the definite integral can be interpreted as the **net area**, i.e. $\int_a^b f(x)dx = A_1 - A_2$ where A_1 is the area of region above the x-axis and A_2 is the area of region below the x-axis.

Example: Evaluate the Riemann sum for $f(x) = 2-x^2$, $0 \leq x \leq 2$, with four sub-intervals, taking the sample points to be right-end points. Explain with the aid of a diagram what the Riemann sum represents.

Solution:
The right-end points are then $x = 0.25, 0.5, 1, 1.5, 2$. Follow textbook page 384 Figure 5 as to how to draw the rectangles. The Riemann sum represents the difference in areas of regions above and below x-axis.

Now we begin to evaluate integrals. First, we discuss the Midpoint rule.

The Midpoint rule is simply choosing the sample point to be in the middle of the rectangle. Let us go through an example.

Example: Use midpoint rule with $n = 4$ to approximate the integral $\int_2^{10} \sqrt{x^3 + 1}dx$

Solution:
The domain of integration is $[2, 10]$. As $n = 4$, there are 4 rectangles and they have a width of $\Delta x = \frac{(10-2)}{4} = 2$. Rectangles can

be found on $x \in [2,4]$, $x \in [4,6]$, $x \in [6,8]$ and $x \in [8,10]$. The integrand is $f(x) = \sqrt{x^3+1}$

The midpoints of interest are $x = 3,\ 5,\ 7,\ 9$. So evaluate the integrand at these midpoints in order to find the height of these rectangles and we get $f(3),\ f(5), f(7),\ f(9)$. You can do the computations yourself.

Now, evaluate $[f(3)+f(5)+f(7)+f(9)]\times\Delta x$ and we get the required approximation.

Please note that you're required to memorize the following summation rules.

- $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

- $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

- $\sum_{i=1}^{n} i^3 = (\frac{n(n+1)}{2})^2$

Example: Prove that $\int_a^b x dx = \frac{b^2 - a^2}{2}$

Solution:
Use the idea of taking the limit of a Riemann sum:

$$\int_a^b x dx = \lim_{n \to \infty} \frac{(b-a)}{n} \sum_{i=1}^{n} (a + i\frac{(b-a)}{n})$$

$$\int_a^b x dx = \lim_{n \to \infty} \frac{(b-a)}{n} [an + \frac{(b-a)}{n} \frac{n(n+1)}{2}]$$

$$\int_a^b x dx = \lim_{n \to \infty} [(b-a)a + \frac{(b-a)^2}{2} \frac{n(n+1)}{n^2}]$$

$$\int_a^b x dx = ab - a^2 + \frac{b^2}{2} - ab + \frac{a^2}{2} = \frac{b^2 - a^2}{2}$$

Let us now consider properties of integrals.

1. $\int_b^a f(x)dx = -\int_b^a f(x)dx$

2. $\int_a^a f(x)dx = 0$

3. $\int_a^b c dx = c(b-a)$

4. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

5. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

6. $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$

Example: Evalute the integral $\int_{-3}^{0}(1+\sqrt{9-x^2})dx$ by interpreting it in terms of areas.

Solution:
First, we split the integral into

$$\underbrace{\int_{-3}^{0} 1dx}_{A} + \underbrace{\int_{-3}^{0} \sqrt{9-x^2})dx}_{B}.$$

The first integral A is a rectangle of area $= 3$. The second integral B is a quarter-circle of area $= \frac{\pi \times 3^2}{4} = \frac{9\pi}{4}$.

Finally, we discuss comparison properties of the integral.

1. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_{a}^{b} f(x)dx \geq 0$.

2. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

3. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$.

Example: Use the properties of integrals to verify the inequality without evaluating the integrals.

$$\int_1^2 \sqrt{5-x}dx \geq \int_1^2 \sqrt{x+1}dx$$

Solution:
Sketching the functions would make it clear that $\sqrt{5-x} \geq \sqrt{x+1}$. Then using the second comparison rule, the inequality is verified.