Section 5.1 - Areas and Distances

Question: How to find an area under a curve?

Answer: Draw *n rectangles of equal width under the curve and sum the areas of these n rectangles.*

Three ways to draw these rectangles.

- *use left-end points.*
- *use right-end points.*
- *use sample points.*

The basic idea is that as we increase the number of rectangles, the computed value of the area will approach a certain value. The true area is determined by the formulae

- *left-end point:* $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x$

- *right-end point:* $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

- *sample point:* $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$
  where x_i^* *is chosen between left-end and right-end points.*

When the number of rectangles increases, the area of the individual rectangle gets smaller. This is because the area of the region remains **FIXED***.*

By this time, the left-end and right-end points become so close that you can hardly distinguish them. We shall explore this idea further in next week's lecture. For now, let us consider some examples.

Example: Find an expression for the area under the graph of $f(x) = \frac{ln(x)}{x}$ over $3 \leq x \leq 10$ as a limit. Do not evaluate the limit.

Solution:
As we're not told whether to use a left-end or right-end point, you are free to choose which one to follow. The main difference is the choice of x. If we choose the right-end point approach, then we see that $x \in [3 + \Delta x, 10]$. If we choose the left-end point approach, then $x \in [3, 10 - \Delta x]$.

Width of rectangle $= \frac{10-3}{n} = \frac{7}{n}$ where $n =$ number of rectangles. Adopting the right-end point approach, we use the formula: $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$ and $x_i = 3 + \frac{7i}{n}$.

So, $A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{ln(3+\frac{7i}{n})}{(3+\frac{7i}{n})} \frac{7i}{n}$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} 7i \frac{ln(3+\frac{7i}{n})}{(3n+7i)}$$

Example: Determine a region whose area is equal to the given limit.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}(5+\frac{2i}{n})^{10}$$

Solution:

Width of interval is $\Delta x = \frac{2}{n}$. So the gap is 2.

$x_i = 5 + (\Delta x)i \Rightarrow x \in [5,7]$ and $f(x) = x^{10}$.

In distance problems, we use the fact that distance = velocity $\times$ time. This means that the area under a velocity-time curve is the distance.

Example: Speedometer readings for a motorcycle at 12-second intervals are given in the table.
(a) Estimate the distance travelled by the motorcycle during this time period using the velocities at the beginning of the time intervals.
(b) Give another estimate of the velocities at the end of the time periods.
(c) Are your estimates in (a) and (b) upper and lower estimates?

$t(s)$	$v(ft/s)$
0	30
12	28
24	25
36	22
48	24
60	27

Solution:

(a) use left-end point approach and hence $A = \sum_{i=1}^{5} f(x_{i-1})\Delta x$

$\Delta x = 12$, $A = [30+28+25+22+24] \times 12 = 1,548$ ft.

(b) use right-end point approach and hence $A = \sum_{i=1}^{5} f(x_i)\Delta x$

$\Delta x = 12$, $A = [28+25+22+24+27] \times 12 = 1,512$ ft.

(c) Yes, (a) is upper estimate and (b) is lower estimate of distance covered. This is because the area using sample points between the left-end and right-end points fall within these two area configurations.