Section 11.9 - Representations of Functions as Power Series

Theorem: If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, the function f defined by

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable and thus continuous on the interval $(a-R, a+R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$.

(ii) $\int f(x)dx = C + c_0(x-a) + c_1\frac{(x-a)}{2} + \cdots = C + \sum_{n=1}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$.

Example: Find a power series representation for the function $f(x) = \frac{x}{9+x^2}$ and determine the interval of convergence.

Here, we need to use the geometric series idea.

$\frac{x}{9+x^2} = \frac{x}{9} \cdot \frac{1}{1-\frac{-x^2}{3}}$

$$= \frac{x}{9} \sum_{n=0}^{\infty} (\frac{-x^2}{9})^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{9^{n+1}}.$$

Let $a_n = \frac{x^{2n+1}}{9^{n+1}}$.

This series converges when $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L < 1$.

This works out to $x^2 < 9$. Now, we need to test the convergence of the series at $x = \pm 3$. At $x = -3$, $a_n = \frac{(-1)^{n+1}}{3}$ and the resulting series clearly diverges by the alternating series test. At $x = 3$, $a_n = (-1)^n$ and the resulting series diverges by the alternating series test.

Hence, the interval is $(-3, 3)$.