Section 11.8 - Power Series  A power series is a series of the form: \[ \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots \]

More generally, a series of the form: \[ \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots \] is called a power series centered at a or power series about a or power series in (x-a).

Theorem: For a given power series \[ \sum_{n=0}^{\infty} c_n (x-a)^n \] there are only three possibilities:

(i) The series converges only when \( x = a \).

(ii) The series converges for all \( x \).

(iii) There is a positive number \( R \) such that the series converges if \( |x-a| < R \) and diverges if \( |x-a| > R \). \( R \) is known as the radius of convergence of the power series. The interval
of convergence of a power series is the interval that consists of all values of \( x \) in which the series converges.

How do we find the radius of convergence?

Example: Find the radius of convergence and the interval of convergence of the series
\[ \sum_{n=1}^{\infty} (-1)^n \frac{(x+2)^n}{n2^n}. \]

Let \( a_n = (-1)^n \frac{(x+2)^n}{n2^n} \). Convergence occurs when \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1 \).

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|
\]

\[ \Rightarrow \lim_{n \to \infty} \left| \frac{(x+2)^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n2^n}{(x+2)^n} \right| \]

\[ \Rightarrow \lim_{n \to \infty} \left| \frac{(x+2)}{2} \cdot \frac{n}{n+1} \right| = \left| \frac{x+2}{2} \right| < 1. \]
So the interval of convergence is $-4 < x < 0$. We need to check at the end points, i.e. at $x = 0$ and $x = -4$.

At $x = 0$, the series converges by the alternating series test. At $x = -4$, the resulting series diverges by the p-test. Hence, the interval of convergence is $(-4, 0]$. 