Section 11.3 - The Integral Test and Estimates of Sum

In general, it is difficult to find the exact sum of a series. However, we are able to determine whether the series is convergent or divergent.

Integral Test: Suppose $f(x)$ is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then

$$\int_1^{\infty} f(x) \text{ is convergent} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ is convergent.}$$

$$\int_1^{\infty} f(x) \text{ is divergent} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ is divergent.}$$

p-series test: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Example: Determine if $\displaystyle\sum_{n=2}^{\infty} \frac{1}{nln(n)}$ is convergent.

Solution:
Use the integral test, i.e. determine if $\displaystyle\int_2^{\infty} \frac{dx}{xln(x)}$ is convergent.

Let $u = ln(x)$, then $\displaystyle\int_2^{\infty} \frac{dx}{xln(x)} = \int \frac{du}{u} =$ $ln(u) \to [ln(ln(x))]_2^{\infty}$.

We see that this is not convergent.

As mentioned earlier, it is difficult to find the exact sum of a series but we can sure estimate it.

Remainder estimate for the Integral Test: Suppose $f(k) = a_k$ where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. If R_n is the remainder defined to be $s - s_n$ where $s,\ s_n$ is the sum of infinite series and the sum of n terms in the infinite series respectively, then

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_{n}^{\infty} f(x)dx$$

Example: Estimate $\sum\limits_{n=1}^{\infty} n^{-3/2}$ to within 0.01.

Solution:
From the remainder estimate for the integral test,

$R_n \leq 0.01 \Rightarrow R_n = \int_n^\infty \frac{dx}{x^{3/2}} = \lim_{t\to\infty} [\frac{-2}{\sqrt{x}}]_n^t =$
$\frac{2}{\sqrt{n}} \leq 0.01 \Rightarrow \sqrt{n} \geq 200 \Rightarrow n \geq 40000$

So using a calculator, we have to find $\sum_{n=1}^{40000} n^{-3/2}$.