Section 11.10 - Taylor and Maclaurin Series

Taylor series of the function f centered at a:

If $f(x) = \sum_{n=1}^{\infty} c_n(x-a)^n$ where $|x-a| < R$ then its coefficients are given by the formula

$c_n = \frac{f^n(a)}{n!}$

Maclaurin series of the function f:

$f(x) = \sum_{n=1}^{\infty} c_n(x)^n$ where $|x| < R$ then its coefficients are given by the formula

$c_n = \frac{f^n(0)}{n!}$

Theorem: If $f(x) = T_n(x) + R_n(x)$, where T_n is the n-th degree Taylor polynomial of f at a and R_n is the remainder of the Taylor series where $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$, then

f is equal to the sum of its Taylor series on the interval $|x - a| < R$.

Taylor's inequality: If $|f^{n+1}(x)| \leq M$ for $|x - a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$|R_n(x)| \leq \frac{M}{(n+1)!}|x - a|^{n+1}$ for $|x - a| \leq d$

Example: Find the Taylor series for $f(x) = ln(x)$ centered at $a = 2$. Assume that $f(x)$ has a power series expansion.

After working through for the first three derivatives, you can derive this relation: $f^{(n)}(x) = (-1)^{n-1}(n-1)!x^{-n}$.

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(2)(x-2)^n$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n2^n}(x-2)^n$$

Example: Obtain the Maclaurin series for the function $xcos(2x)$.

$$cos(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2x)^{2n}$$

$$xcos(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(2n)!}(x)^{2n+1}$$

Example: Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!}$

Recall that this series represents the expansion for $cos(x)$. So, the sum of this series is $cos(\frac{\pi}{6}) = \frac{1}{2}$.