King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

Math 102 –Calculus II
Exam II
Semester 072

May 5, 2008 **Time: 8:30 -10:20 PM**

Section Number:______

ID Number: ____________. Key Solution

Name: ____

Instructions :

- Write neatly and eligibly. You may lose points for messy work.
- Show all your work. No credits for answers without justification.
- All types of calculators and mobiles are not allowed.
- Make sure that you have 13 different problems. (7 pages +cover)

Page Number	Student Grade	Maximum Points
2		14
3		12
4		17
5		15
6		12
7		16
8		14
Total		100

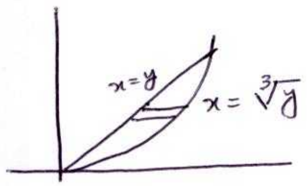
1. (4 points) Write out the form of partial fraction decomposition of $\dfrac{x+2}{(x-2)^2(x^2+4)^2}$.

   Do not determine the numerical values of the coefficients.

$$\frac{x+2}{(x-2)^2(x^2+4)^2} = \underbrace{\frac{A}{x-2} + \frac{B}{(x-2)^2}}_{2} + \underbrace{\frac{Cx+D}{x^2+4} + \frac{Ex+F}{(x^2+4)^2}}_{2}$$

2. (10 points) Use cylindrical shells to set up the integral for finding the volume of the solid obtained by rotating the region in the first quadrant bounded by $y = x$ and $y = x^3$: (DO NOT EVALUATE THE INTEGRALS)

   (a) about the x-axis.

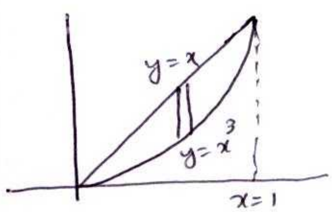


- $x = x^3 \Rightarrow x=0, x=1 \Rightarrow y=0, y=1$

- $V = 2\pi \displaystyle\int_0^1 \underbrace{y}_{1} \cdot \underbrace{(\sqrt[3]{y} - y)}_{2}\, dy$

limits, figure and points of intersection = 2

   (b) about the line $x = 1$.



$x = x^3 \Rightarrow x=0, x=1$

- $V = 2\pi \displaystyle\int_0^1 \underbrace{(1-x)}_{1} \cdot \underbrace{(x-x^3)}_{2}\, dx$

limits, figure, points of intersection = 2

3. (6 points) Find the average value of the function $f(x) = x\tan^{-1}x$ on the interval $[-1, 1]$.

- $f_{av} = \frac{1}{2}\int_{-1}^{1} x\tan^{-1}x\,dx$ 2

- Integrating by parts

$$\int_{-1}^{1} x\tan^{-1}x\,dx = \frac{x^2}{2}\tan^{-1}x\Big|_{-1}^{1} - \int_{-1}^{1}\frac{x^2}{2(1+x^2)}dx$$ 1

$$= \frac{\pi}{4} - \frac{1}{2}\int_{-1}^{1}\left(1 - \frac{1}{1+x^2}\right)dx$$

$$= \frac{\pi}{4} - \frac{1}{2}\left[x\Big|_{-1}^{1} - \tan^{-1}x\Big|_{-1}^{1}\right] = \frac{\pi}{4} - \frac{1}{2}\left[2 - \frac{\pi}{2}\right] = \frac{\pi}{2} - 1$$ 3

- $f_{av} = \frac{\pi}{4} - \frac{1}{2}$

4. (6 points) Evaluate $\int \frac{\sqrt{x^2-4x}}{x-2}\,dx$.

- $x^2 - 4x = (x-2)^2 - 4 \Rightarrow I = \int \frac{\sqrt{x^2-4x}}{x-2}dx = \int \frac{\sqrt{(x-2)^2-4}}{x-2}dx$ 2

- $y = x-2 \Rightarrow dy = dx$ and $I = \int \frac{\sqrt{y^2-4}}{y}dy$

- $\| y = 2\sec\theta \Rightarrow \sqrt{y^2-4} = 2\tan\theta,\ dy = 2\sec\theta\tan\theta\,d\theta \|$ 2

and

$$I = 2\int \tan^2\theta\,d\theta$$

$$= 2\int(\sec^2\theta - 1)\,d\theta = 2\tan\theta - 2\theta + C$$ 1

$$= \sqrt{y^2-4} - 2\sec^{-1}\left(\frac{y}{2}\right) + C$$ 1

$$= \sqrt{(x-2)^2-4} - 2\sec^{-1}\left(\frac{x-2}{2}\right) + C$$

$$= \sqrt{x^2-4x} - 2\sec^{-1}\left(\frac{x-2}{2}\right) + C$$

5. Evaluate the following integrals

(a) (5 points) $\int \sin^2 3x \, dx$.

• Using half angle formula

$$\sin^2 3x = \frac{1-\cos 6x}{2}$$ 2

$$\bullet \Rightarrow I = \frac{1}{2}\left[\int dx - \int \cos 6x \, dx\right]$$

$$= \frac{x}{2} - \frac{1}{12}\sin 6x + C$$ 3

(b) (5 points) $\int \sin^{\frac{3}{2}} x \, \cos^3 x \, dx$.

$$\bullet \; I = \int \sin^{3/2} x \cos^2 x \cdot \cos x \, dx$$

$$= \int \sin^{3/2} x \, (1 - \sin^2 x) \cdot \cos x \, dx$$ 2

$\| \; u = \sin x \Rightarrow du = \cos x \, dx \; \|$

and

$$I = \int u^{3/2}(1-u^2)\, du = \frac{2}{5}u^{5/2} - \frac{2}{9}u^{9/2} + C$$

$$= \frac{2}{5}\sin^{5/2} x - \frac{2}{9}\sin^{9/2} x + C$$ 3

(c) (7 points) $\int \frac{dx}{1 - \sin x + \cos x}$ [Hint: you may use the substitution $t = \tan \frac{x}{2}$]

$\bullet \; t = \tan x/2 \Rightarrow \sin x = \frac{2t}{1+t^2}, \; \cos x = \frac{1-t^2}{1+t^2}, \; dx = \frac{2dt}{1+t^2}$ 3

$$\Rightarrow I = \int \frac{\frac{2dt}{1+t^2}}{1 - \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} = \int \frac{dt}{1-t}$$ 2

$$= -\int \frac{dt}{t-1} = -\ln|t-1|$$

$$= -\ln\left|\tan\frac{x}{2} - 1\right|$$ 2

6. (8 points) Evaluate $I = \displaystyle\int \frac{x^2+2x-1}{x^3-x}dx$

- $\dfrac{x^2+2x-1}{x^3-x} = \dfrac{x^2+2x-1}{x(x+1)(x-1)} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x-1}$) 2

- Finding coefficients by standard procedure

$\Rightarrow$

$A = 1$ 1

$B = -1$ 1

$C = 1$ 1

- $\Rightarrow I = \displaystyle\int \left(\frac{1}{x} - \frac{1}{x+1} + \frac{1}{x-1} \right) dx$

$= \ln|x| - \ln|x+1| + \ln|x-1| + C$ } 3

7. (7 points) Compute $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\, dx$.

- $I = \displaystyle\int_1^\infty \frac{\ln x}{x^2}\, dx = \lim_{a\to\infty} \int_1^a \frac{\ln x}{x^2}\, dx$ } 2

- Integrating by parts gives (Taking $u = \ln x$)

$I = \displaystyle\lim_{a\to\infty} \left[-\frac{\ln x}{x}\Big|_1^a + \int_1^a \frac{dx}{x^2} \right]$ } 2

$= \displaystyle\lim_{a\to\infty} \left[-\frac{\ln a}{a} + \frac{\ln 1}{1} + \left(-\frac{1}{x}\Big|_1^a \right) \right]$

$= \displaystyle\lim_{a\to\infty} \left[-\frac{\ln a}{a} - \frac{1}{a} + 1 \right]$ } 2

$= 1$ } 1

8. (6 points) Determine whether the integral is convergent or divergent. If it converges, evaluate it. $\int_0^1 \frac{dx}{3x-2}$

$$I = \int_0^1 \frac{dx}{3x-2} = \frac{1}{3}\int_0^1 \frac{dx}{x-2/3}$$

$$= \frac{1}{3}\int_0^{2/3} \frac{dx}{x-2/3} + \frac{1}{3}\int_{2/3}^1 \frac{dx}{x-2/3} \quad \textcircled{2}$$

where

$$\int_0^{2/3} \frac{dx}{x-2/3} = \lim_{t\to\frac{2}{3}^-} \int_0^t \frac{dx}{x-2/3} = \lim_{t\to\frac{2}{3}^-} \ln\left|x-\frac{2}{3}\right|\Big|_0^t$$

$$= \lim_{t\to\frac{2}{3}^-} \left\{\ln\left|t-\frac{2}{3}\right| - \ln\left|-\frac{2}{3}\right|\right\} = -\infty \quad \textcircled{2}$$

$\Rightarrow \int_0^{2/3} \frac{dx}{x-2/3}$ is divergent (1)

Hence $\int_0^1 \frac{dx}{3x-2}$ is divergent (1)

9. (6 points) Given the following sequence:

$$\left(1-\frac{1}{2}\right), \left(\frac{1}{2}-\frac{1}{3}\right), \left(\frac{1}{3}-\frac{1}{4}\right), \left(\frac{1}{4}-\frac{1}{5}\right), \ldots$$

(a) Find the general term of the sequence.

$$a_n = \frac{1}{n} - \frac{1}{n+1} \quad (2)$$

$$= \frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)} \quad (1)$$

(b) Show that the sequence is convergent.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n(n+1)} = 0 \quad (2)$$

$\Rightarrow$ the sequence is convergent. (1)

10. Determine whether the series is convergent or divergent. If it is convergent, find its sum:

(a) (4 points) $\sum_{n=1}^{\infty} n \sin \frac{1}{n}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n \sin \frac{1}{n} \quad \Big\} \ 2$$

$$= \lim_{n\to\infty} \frac{\sin 1/n}{1/n} = 1 \neq 0 \quad \Big\} \ 1$$

$\Rightarrow$ series diverges by divergence test. $\Big\}$ 1

(b) (6 points) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} + 2^n}{3^n}$

- Series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^n} = \frac{1}{3}\sum_{n=1}^{\infty}\left(-\frac{1}{3}\right)^{n-1}$ is convergent geometric series with $r = -1/3$ and has sum $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^n} = \frac{1}{3} \cdot \frac{1}{1-(-1/3)} = \frac{1}{4}$ $\Big\}$ 2
- Series $\sum_{n=1}^{\infty} \frac{2^n}{3^n} = \sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$ is convergent geometric series with $r = \frac{2}{3}$ and has sum $\sum_{n=1}^{\infty} \frac{2^n}{3^n} = \frac{2/3}{1-2/3} = 2$ $\Big\}$ 2
- $\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n-1} + 2^n}{3^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^n} + \sum_{n=1}^{\infty} \frac{2^n}{3^n} = \frac{1}{4} + 2 = \frac{9}{4}$. $\Big\}$ 2

(c) (6 points) $\sum_{n=1}^{\infty} \frac{2}{(n+1)(n+3)}$.

$$S_n = \sum_{i=1}^{n} \frac{2}{(i+1)(i+3)} = \sum_{i=1}^{n}\left(\frac{1}{i+1} - \frac{1}{i+3}\right)$$

$$= \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\frac{1}{3} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{6}\right) + \left(\cancel{\frac{1}{5}} - \frac{1}{7}\right) + \cdots + \left(\frac{1}{n+1} - \frac{1}{n+3}\right) \quad \Big\} \ 3$$

$$\Rightarrow S = \lim_{n\to\infty} S_n = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

Therefore, the series converges to $\frac{5}{6}$ $\Big\}$ 3

11. (6 points) Determine whether the series $\sum_{n=1}^{\infty} \frac{5-\sqrt{n}}{n^3}$ is convergent or divergent.

- $\sum_{n=1}^{\infty} \frac{5-\sqrt{n}}{n^3} = \sum_{n=1}^{\infty} \left(\frac{5}{n^3} - \frac{1}{n^{5/2}} \right)$ (1)

- Since $\sum \frac{5}{n^3} = 5 \sum \frac{1}{n^3}$ is convergent (P-series with $P = 3 > 1$) (2)

and $\sum \frac{1}{n^{5/2}}$ is convergent (P- series with $P = 5/2 > 1$). (2)

Therefore $\sum \frac{5-\sqrt{n}}{n^3}$ is convergent. (1)

12. (8 points) Consider the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

(a) Use the integral test to show that the series converges.

It is not required to show that the integral test is applicable, because the question only asks to use it to test convergence

- Consider $\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{t \to \infty} \int_2^t \frac{dx}{x(\ln x)^2}$ } (2)

$= \lim_{t\to\infty} \left[-\frac{1}{\ln x} \right]_2^t = \lim_{t\to\infty} \left[-\frac{1}{\ln t} + \frac{1}{\ln 2} \right] = \frac{1}{\ln 2}$ } (1)

- Thus $\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx$ converges,

hence by integral test $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges. } (1)

(b) How many terms of the series would you need to add so that the error in the sum is within 0.01? (Write the answer in the form $n > ...$)

- Since the error $R_n \leq \int_n^{\infty} \frac{dx}{x(\ln x)^2}$, it is sufficient to find 'n' so that $\int_n^{\infty} \frac{dx}{x(\ln x)^2} < 0.01$ } (2)

- $\Rightarrow$ To find 'n' so that $\frac{1}{\ln n} < 0.01$

using part (a) $\Rightarrow$ $n > e^{100}$ } (2)