King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 102- Calculus II
Exam II
2008-2009 (081)

Tuesday, December 30, 2008 **Allowed Time: 2 hours**

Name: Solution Key

ID Number: ____________

Section Number:___________ **Serial Number:**___________

Instructions:

1. Write neatly and eligibly. You may lose points for messy work.
2. **Show all your work.** No points for answers without justification.
3. **Calculators and Mobiles are not allowed.**
4. Make sure that you have 7 different problems (6 pages + cover page)

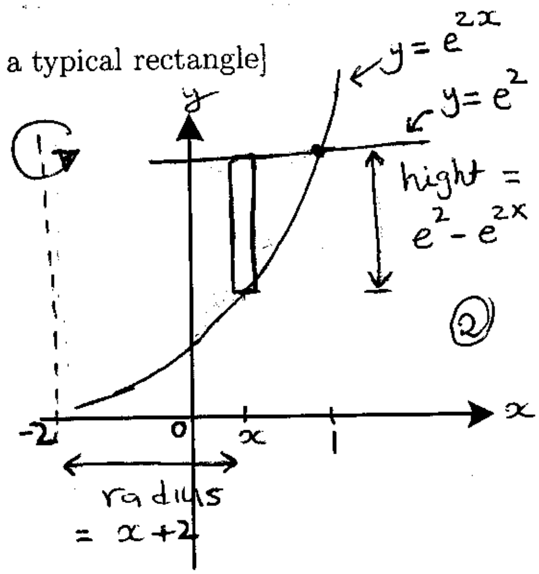
Problem No	Grade	Maximum Points
1		12
2		7
3(a, b, c)		20
3(d, e)		16
4		12
5		6
6		9
7		18
Total		100

1. **[6 + 6 points]** Using the method of Cylindrical Shells, **set up**, but **do not evaluate**, an integral for the volume of the solid obtained by rotating the region bounded by the curves $y = e^{2x}, y = e^2$, and $x = 0$

(a) about the line $x = -2$. [Sketch the region and a typical rectangle]

$$V = 2\pi \int_0^1 (x+2)\,(e^2 - e^{2x})\,dx$$

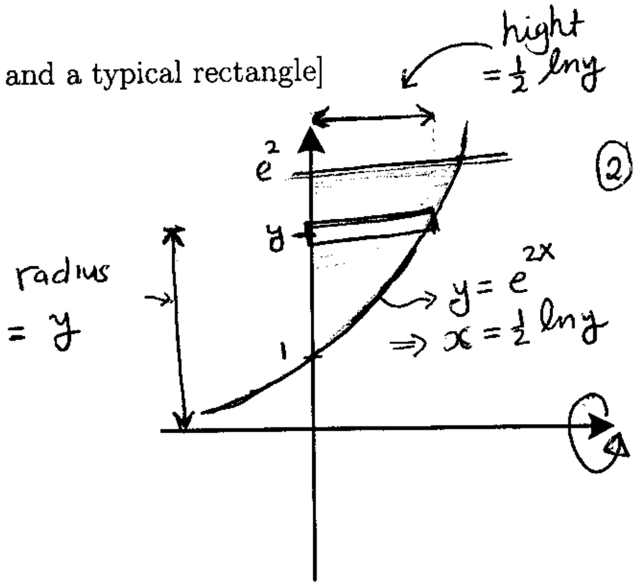
1 1 2



(b) about the x-axis. [Sketch the region and a typical rectangle]

$$V = 2\pi \int_1^{e^2} y \cdot \tfrac{1}{2}\ln y \cdot dy$$

1 1 2



2. **[7 points]** Find the average value of the function $f(x) = x\sec^2(2x)$ on the interval $\left[0, \frac{\pi}{8}\right]$.

2 $\quad f_{ave} = \dfrac{1}{\frac{\pi}{8} - 0} \displaystyle\int_0^{\pi/8} x \sec^2(2x)\,dx$, by Parts: $u = x$, $du = dx$, $dv = \sec^2(2x)\,dx$, $v = \frac{1}{2}\tan(2x)$

2 $\quad = \dfrac{8}{\pi}\left[\tfrac{1}{2}x\tan(2x)\Big|_0^{\pi/8} - \tfrac{1}{2}\displaystyle\int_0^{\pi/8} \tan(2x)\,dx \right]$

2 $\quad = \dfrac{8}{\pi}\left[\dfrac{\pi}{16} - \dfrac{1}{4}\ln|\sec(2x)|\Big|_0^{\pi/8} \right]$

1 $\quad = \dfrac{1}{2} - \dfrac{\ln 2}{\pi}$

3. Evaluate each of the following integrals:

(a) **[6 points]** $\int \sin^5(3t)\cos^4(3t)\, dt$.

1 $= \int \sin^4(3t)\cos^4(3t) \cdot \sin(3t)\, dt$

1 $= \int [1-\cos^2(3t)]^2 \cos^4(3t) \cdot \sin(3t)\, dt$

1 Let $u = \cos(3t)$. Then $du = -3\sin(3t)\, dt$

1 $= \frac{-1}{3}\int (1-u^2)^2 u^4\, du = -\frac{1}{3}\int (u^4 - 2u^6 + u^8)\, du$

1 $= -\frac{1}{3}\left[\frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9\right] + C$

1 $= -\frac{1}{15}\cos^5(3t) + \frac{2}{21}\cos^7(3t) - \frac{1}{27}\cos^9(3t) + C$

(b) **[6 points]** $\int \frac{\sqrt{x}}{x+4}\, dx$. Let $u = \sqrt{x}$. 2

Then $x = u^2$ & $dx = 2u\, du$.

$\int \frac{\sqrt{x}}{x+4}\, dx = \int \frac{u}{u^2+4} \cdot 2u\, du = 2\int \frac{u^2}{u^2+4}\, du$ 1

$= 2\int \left[1 - \frac{4}{u^2+4}\right] du$ (Long Division) 1

$= 2\left[u - 2\tan^{-1} u/2\right] + C$ $\boxed{1+1}$

$= 2\sqrt{x} - 4\tan^{-1}\frac{\sqrt{x}}{2} + C$

(c) **[8 points]** $\int \frac{1}{1+2\sin x}\, dx$. Hint: Use the substitution $t = \tan\left(\frac{x}{2}\right)$.

Let $t = \tan(\frac{x}{2})$, $-\pi < x < \pi$

$\boxed{1+1}$ $\left\{ \begin{array}{l} \text{Then } \sin x = \frac{2t}{1+t^2}, \\ dx = \frac{2}{1+t^2}\, dt \end{array} \right.$

Thus

$\int \frac{1}{1+2\sin x}\, dx = \int \frac{1}{1+\frac{4t}{1+t^2}} \cdot \frac{2}{1+t^2}\, dt$

2 $= \int \frac{2}{t^2+4t+1}\, dt$

$= \int \frac{2}{(t+2)^2-3}\, dt$ 1

$= \frac{1}{\sqrt{3}} \ln\left|\frac{t+2-\sqrt{3}}{t+2+\sqrt{3}}\right| + C$ 2

$= \frac{1}{\sqrt{3}} \ln\left|\frac{\tan(\frac{x}{2})+2-\sqrt{3}}{\tan(\frac{x}{2})+2+\sqrt{3}}\right| + C$ 1

(d) **[7 points]** $\int_{2/5}^{4/5} \frac{\sqrt{25x^2-4}}{x}\,dx.$

Let $5x = 2\sec\theta$, $0 \le \theta < \frac{\pi}{2}$. $\leftarrow$ 2

Then
- $dx = \frac{2}{5}\sec\theta\tan\theta$
- $\sqrt{25x^2-4} = \sqrt{4\sec^2\theta - 4} = 2\tan\theta$
- $x = \frac{2}{5} \Rightarrow \theta = 0$
- $x = \frac{4}{5} \Rightarrow \theta = \frac{\pi}{3}$

$$\int_{2/5}^{4/5} \frac{\sqrt{25x^2-4}}{x}\,dx = \int_0^{\pi/3} \frac{2\tan\theta}{\frac{2}{5}\sec\theta} \cdot \frac{2}{5}\sec\theta\tan\theta\,d\theta$$

$$= \int_0^{\pi/3} 2\tan^2\theta\,d\theta \quad \leftarrow 3$$

$$= 2\int_0^{\pi/3} (\sec^2\theta - 1)\,d\theta$$

$$= 2\left(\tan\theta - \theta\right)\Big|_0^{\pi/3} \quad \leftarrow 1$$

$$= 2\left(\sqrt{3} - \frac{\pi}{3}\right). \quad \leftarrow 1$$

(e) **[9 points]** $\int \frac{4x}{(x+1)(x-1)^2}\,dx.$

$$\frac{4x}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \quad 3$$

$$\Rightarrow \quad 4x = A(x-1)^2 + B(x-1)(x+1) + C(x+1)$$

$$\Rightarrow \quad A = -1, \quad B = 1, \quad C = 2. \quad 1+1+1$$

Thus
$$\int \frac{4x}{(x+1)(x-1)^2}\,dx = \int \frac{-1}{x+1} + \frac{1}{x-1} + \frac{2}{(x-1)^2}\,dx$$

$$= -\ln|x+1| + \ln|x-1| - \frac{2}{x-1} + C \quad 1+1+1$$

$$= \ln\left|\frac{x-1}{x+1}\right| - \frac{2}{x-1} + C$$

4. (a) Determine if the integral is improper or not. Justify.

(i) **[3 points]** $\int_2^5 \ln(x-1)\,dx$.

(2) Since the Integrand $f(x) = \ln(x-1)$ is continuous on the Interval $[2,5]$,

(1) then the Integral is <u>not</u> Improper.

(ii) **[3 points]** $\int_1^5 \frac{1}{x^2-x}\,dx$.

(2) Since the Integrand $f(x) = \frac{1}{x(x-1)}$ has an infinite discontinuity at $x=1$,

(1) then the integral is an Improper Integral.

(b) **[6 points]** Determine if the integral $\int_{-\infty}^{0} \frac{1}{\sqrt{3-x}}\,dx$ is convergent or divergent. If it is convergent, find its value.

$$\int_{-\infty}^{0} \frac{1}{\sqrt{3-x}}\,dx = \lim_{t\to-\infty} \int_t^0 \frac{1}{\sqrt{3-x}}\,dx \quad (2)$$

$$= \lim_{t\to-\infty} -2\sqrt{3-x}\,\Big|_t^0 \quad (1)$$

$$= \lim_{t\to-\infty} \left[-2\sqrt{3} + 2\sqrt{3-t}\right] \quad (1)$$

$$(1)$$

$$= +\infty$$

Hence $\int_{-\infty}^{0} \frac{1}{\sqrt{3-x}}\,dx$ is <u>Divergent</u>. (1)

5. **[6 points]** Determine whether the sequence $\left\{\frac{\sin(3n)}{3n+1}\right\}_{n=1}^{+\infty}$ converges or diverges. If it converges, find its limit.

Since $-1 \le \sin(3n) \le 1$, ①

then $\frac{-1}{3n+1} \le \frac{\sin(3n)}{3n+1} \le \frac{1}{3n+1}$. ① $(3n+1>0)$

Since $\lim_{n\to+\infty} \frac{-1}{3n+1} = 0$ & $\lim_{n\to+\infty} \frac{1}{3n+1} = 0$, ①

then, by the Squeeze Theorem, ①

$$\lim_{n\to+\infty} \frac{\sin(3n)}{3n+1} = 0. \quad ①$$

Thus the sequence converges & its limit is 0. ①

6. Consider the series $\sum_{n=1}^{+\infty} \frac{\ln(n+2)-\ln(n+1)}{\ln(n+1)\cdot\ln(n+2)}$.

(a) **[5 points]** Find a formula for S_n, the sequence of partial sums.

① $S_n = \sum_{i=1}^{n} \frac{\ln(i+2)-\ln(i+1)}{\ln(i+1)\cdot\ln(i+2)}$

① $= \sum_{i=1}^{n} \left(\frac{1}{\ln(i+1)} - \frac{1}{\ln(i+2)}\right)$

① $= \left(\frac{1}{\ln 2} - \frac{1}{\ln 3}\right) + \left(\frac{1}{\ln 3} - \frac{1}{\ln 4}\right) + \left(\frac{1}{\ln 4} - \frac{1}{\ln 5}\right) + \cdots + \left(\frac{1}{\ln(n+1)} - \frac{1}{\ln(n+2)}\right)$

② $= \frac{1}{\ln 2} - \frac{1}{\ln(n+2)}$

(b) **[4 points]** Determine whether the given series converges or diverges. If it converges, find its sum.

Since $\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \frac{1}{\ln 2} - \frac{1}{\ln(n+2)}$ ①

$= \frac{1}{\ln 2} - 0 = \frac{1}{\ln 2}$, ①

then the given series converges & ①

its sum is $\frac{1}{\ldots}$. ①

7. Determine whether the series converges or diverges. Justify your answer.

(a) **[5 points]** $\sum_{n=1}^{+\infty} \sin^{-1}\left(\frac{n}{n+1}\right)$.

We use the Test for Divergence: 1

$\lim_{n\to+\infty} \sin^{-1}\left(\frac{n}{n+1}\right) = \sin^{-1} 1 = \frac{\pi}{2}$. 1

Since $\lim_{n\to+\infty} \sin^{-1}\left(\frac{n}{n+1}\right) \neq 0$, 2

then $\sum_{n=1}^{+\infty} \sin^{-1}\left(\frac{n}{n+1}\right)$ diverges. 1

(b) **[5 points]** $\sum_{n=0}^{+\infty} \frac{(2e)^{n+1}}{3^{2n-1}}$. $= \sum_{n=0}^{+\infty} \frac{(2e)(2e)^n}{3^{-1} \cdot 3^{2n}} = \sum_{n=0}^{+\infty} 6e\left(\frac{2e}{9}\right)^n$

The Series is a Geometric Series 1

with ratio $r = \frac{2e}{9}$. 1

Since $|r| = \frac{2e}{9} < 1$, 2

the the Series Converges. 1

(c) **[8 points]** $\sum_{n=1}^{+\infty} n e^{-2n}$.

We use the Integral Test:

Let $f(x) = x e^{-2x}$, $[1, +\infty)$.

i. f is continuous on $[1, +\infty)$ 1

ii. f is positive on $[1, +\infty)$ 1

iii. $f'(x) = x e^{-2x}(-2) + e^{-2x}$
$= (-2x+1)e^{-2x} < 0$ for $x \geq 1$ 1

So f is decreasing on $[1, +\infty)$

Thus the Integral Test is applicable.

$\int_1^{+\infty} x e^{-2x}\,dx = \lim_{t\to+\infty} \int_1^t x e^{-2x}\,dx$ 1

$= \lim_{t\to+\infty} \left(-\frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2x}\right)\Big|_1^t$ 1

$= \lim_{t\to+\infty} \left(-\frac{1}{2} t e^{-2t} - \frac{1}{4} e^{-2t} + \frac{3}{4} e^{-2}\right)$

$= 0 - 0 + \frac{3}{4} e^{-2}$

$= \frac{3}{4} e^{-2}$ 1

$\left.\begin{array}{l}\text{Since } \int_1^{+\infty} x e^{-2x}\,dx \text{ Converges,} \\ \text{then } \sum_{n=1}^{+\infty} n e^{-2n} \text{ Converges}\end{array}\right\}$ 2