31.3 Generalized Exponential Function

Definition

$$E_t(\nu, a) = t^\nu \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(\nu + k + 1)}.$$

Properties

- $I_0^\nu e^{at} = E_t(\nu, a)$.
- $D_0^\nu e^{at} = E_t(-\nu, a)$.
- $D_0^\mu E_t(w, c) = E_t(w - \mu, c)$.
- $E_t(0, a) = e^{at}$.
- $E_t(\nu, a) = t^\nu e^{at}\, \gamma^*(\nu, at)$, where $\gamma^*(\nu, at)$ is the incomplete gamma function.
- Relation to Miettag-Leffler function.

  $$E_{1/q,1}(ct^{1/q}) = \sum_{k=0}^{q-1} c^k E_t(k/q, c^q), \qquad q \in \mathbb{N},$$

  where

  $$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}, \qquad t, \beta \in \mathbb{C}, Re\, \alpha > 0.$$

  Also,

  $$E_t(\nu, a) = t^\nu E_{1,\nu+1}(at).$$

- Laplace transform

  $$\mathcal{L}\Big\{E_t(\nu, a)\Big\} = \frac{1}{s^\nu (s - a)}, \qquad Re\, \nu > -1.$$

Theorem 379 (Initial value theorem of Laplace transform)

$$\left.\begin{array}{l} D^{\nu-r-1}y(0)=0 \\ \\ r=0,\dots,m-1 \\ \\ m=-[-\nu] \end{array}\right\} \quad \Longrightarrow \quad D^{\nu}y(0)=\lim_{s\to\infty}\left[\,s^{\nu+1}\,Y(s)\right].$$

Proof. The result follows from the initial value theorem of the Laplace transform

$$f(0^{+})=\lim_{s\to\infty}\,[sF(s)].$$

To see this we have

$$\begin{aligned} D^{\nu}y(0) &= \lim_{s\to\infty}\left[\,s\,\mathcal{L}\{D^{\nu}y(t)\}\right]=\lim_{s\to\infty}\,s\left[s^{\nu}\,Y(s)-\sum_{r=0}^{m-1}\,s^{r}\,D^{\nu-r-1}y(0)\right] \\ &= \lim_{s\to\infty}\left[\,s^{\nu+1}\,Y(s)\right] \end{aligned}$$

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32 Linear FDE of Order (n, q)

32.1 Definition

Notation 380 *Let*

$$P(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n.$$

Then

$$P(D^\nu) = D^{n\nu} + a_1 D^{(n-1)\nu} + \cdots + a_n, = P(D^\nu) = \sum_{k=0}^{n} a_{n-k}\, D^{k\nu}, \qquad a_0 = 1.$$

Definition 381 *The equation*

$$P(D^\nu)\, y(t) = 0, \qquad \nu = \frac{1}{q}, \quad q \in \mathbb{N}$$

is called homogeneous fractional differential equation of order (n, q)*.*

The polynomial $P(x)$ *is call the corresponding indicial polynomial.*

Lemma 382

$$w(t) = \mathcal{L}^{-1}\left\{\frac{1}{P(s^\nu)}\right\} \quad \Longrightarrow \quad D^{\nu-r-1} w(0) = 0, \quad r = 0, \ldots, m-1, \quad m = -[-\nu].$$

Proof. Since $w(t)$ is a linear combination of the function $E_t(r\nu, \cdot)$.

32.2 Laplace transform of the solution of the IVP

From Laplace transformation formula we have

$$\mathcal{L}\Big\{D^{k\nu}\ y(t)\Big\} = s^{k\nu}\ Y(s) - \sum_{r=0}^{m_k-1} s^r\ D^{k\nu-r-1}y(0),$$

$$k\nu > 0, \qquad m_k - 1 < k\nu \leq m_k, \qquad m_k \in \mathbb{N}.$$

Thus the transformation of the equation gives

$$\begin{aligned}
\mathcal{L}\Big\{P(D^\nu)\ y(t)\Big\} &= \mathcal{L}\left\{\sum_{k=0}^{n} a_{n-k}D^{k\nu}y(t)\right\} = \sum_{k=0}^{n} a_{n-k}\ \mathcal{L}\Big\{D^{k\nu}y(t)\Big\} \\
&= Y(s) + \sum_{k=1}^{n} a_{n-k}\ \left\{s^{k\nu}\ Y(s) - \sum_{r=0}^{m_k-1} s^r\ D^{k\nu-r-1}y(0)\right\} \\
&= P(s^\nu)\ Y(s) - \sum_{k=1}^{n} a_{n-k} \sum_{r=0}^{m_k-1} s^r\ D^{k\nu-r-1}y(0) \\
&= P(s^\nu)\ Y(s) - \sum_{r=0}^{N-1} B_r(y)\ s^r,
\end{aligned}$$

where $B_r(y)$ is a linear combination of the terms

$$D^{k\nu-r-1}y(0), \qquad k = 1, \dots, n, \quad r = 0, \dots, N-1, \qquad N = -[-n\nu].$$

Therefore the transform of the solution is

$$Y(s) \quad = \quad \frac{\displaystyle\sum_{r=0}^{N-1} B_r(y)\ s^r}{P(s^\nu)}.$$

Note 383

$$B_0(y) = \sum_{k=1}^{n} a_{n-k}\ D^{k\nu-1}y(0) = \sum_{k=1}^{n} a_{n-k}\ D^{k\nu}Iy(0) = \Big[P(D^\nu) - a_n\Big]\ Iy(0).$$

$$B_1(y) = \sum_{k=1}^{n} a_{n-k}\ s\ D^{k\nu-2}y(0) = s\ \sum_{k=1}^{n} a_{n-k}\ D^{k\nu-2}y(0) = s\Big[P(D^\nu) - a_n\Big]\ I^2y(0) = sB_0(Iy).$$

32.3 Special Solution

Lemma 384

$$\boxed{y_1(t) = \mathcal{L}^{-1}\left\{\frac{1}{P(s^\nu)}\right\} \qquad \Longrightarrow \qquad P(D^\nu)\, y_1(t) = 0.}$$

Proof. We have

$$\mathcal{L}\Big\{P(D^\nu)\; y_1(t)\Big\} \;=\; P(s^\nu)\; Y_1(s) \;-\; \sum_{r=0}^{N-1} B_r(y_1)\; s^r \;=\; 1 \;-\; \sum_{r=0}^{N-1} B_r(y_1)\; s^r.$$

Note that

$$\lim_{s\to\infty}\Big[s^{k\nu}Y_1(s)\Big] = \lim_{s\to\infty}\frac{1}{\displaystyle\sum_{j=0}^{n} a_{n-j}\, s^{(j-k)\nu}} = \begin{cases} 1, & k = n, \\ \\ 0, & k = 0, 1, \ldots, n-1, \end{cases}$$

since for $k < n$ we have a positive power of s in the denominator. This implies that

$$\begin{aligned} B_0(y_1) &= \sum_{k=1}^{n} a_{n-k}\; D^{k\nu-1}y_1(0) \\ &= \lim_{s\to\infty}\sum_{k=1}^{n}\; a_{n-k}\; s^{k\nu}Y_1(s) = \sum_{k=1}^{n}\; a_{n-k}\; \lim_{s\to\infty}\Big[s^{k\nu}Y_1(s)\Big] = 1, \end{aligned}$$

and similarly

$$B_r(y_1) = 0, \qquad r = 1, \ldots, N-1.$$

Consequently,

$$\sum_{r=0}^{N-1} B_r(y_1)\; s^r = 1.$$

Therefore

$$\mathcal{L}\Big\{P(D^\nu)\; y_1(t)\Big\} = 0,$$

and thus $P(D^\nu)\; y_1(t) = 0$. Thus $y_1(t)$ is a solution.

32.4 Linearly Independent Solutions

Theorem 385 *Let*

$$P(D^{\nu})\, y(t) = 0$$

be a fractional differential equation of order (n, q). *If*

$$y_1(t) = \mathcal{L}^{-1}\left\{\frac{1}{P(s^{\nu})}\right\},$$

then

$$y_{j+1}(t) = D^j y_1(t), \qquad j = 0, 1, \ldots, N-1, \quad N = -[-n\nu],$$

are N *linearly independent solutions of the equation.*

Proof. We already proved that $y_1(t)$ is a solution. From initial value theorem,

$$D^k y_1(0) = 0, \qquad k = 0, 1, \ldots, N-2. \tag{8}$$

Thus

$$D^u D^j y_1(t) = D^j D^u y_1(t), \quad j = 0, 1, \ldots, N-1, \quad u \geq 0.$$

Hence

$$P(D^{\nu})\, y_{j+1}(t) = P(D^{\nu}) D^j y_1(t) = D^j P(D^{\nu})\, y_1(t) = 0.$$

Therefore

$$y_{j+1}(t) = D^j y_1(t), \qquad j = 0, 1, \ldots, N-1,$$

are solutions. Next we show they are linearly independent.

By (8)

$$\mathcal{L}\left\{y_{j+1}(t)\right\} = \mathcal{L}\left\{D^j y_1(t)\right\} = \frac{s^j}{P(s^{\nu})}, \qquad j = 0, 1, \ldots, N-2,$$

and thus $y_1, y_2, \ldots, y_{N-1}$ are linearly independent (since this shows that non of them is a linear combination of the others.) Moreover,

$$y_1(0) = y_2(0) = \cdots = y_{N-1}(0) = 0.$$

On the other hand,

$$\begin{aligned} y_N(0) &= \lim_{s\to\infty} s\mathcal{L}\{y_N(t)\} = \lim_{s\to\infty} s\mathcal{L}\left\{D^{N-1}y_1(t)\right\} = \lim_{s\to\infty} s^N \mathcal{L}\left\{y_1(t)\right\} \\ &= \lim_{s\to\infty}\left\{\frac{s^N}{P(s^{\nu})}\right\} = \begin{cases} 1, & N = n\nu, \\ \infty, & N > n\nu. \end{cases} \end{aligned}$$

Therefore, $y_N(t)$ is linearly independent of $y_1, \ldots, y_{N-1}$. ■.

Example 386 *Solve the equation of order* $(2, q)$, $q = 2, 3, \ldots,$

$$\left[D^{2\nu} - D^{\nu} - 2\right]\, y(t) = 0, \qquad \nu = 1/q.$$

Solution. In this case $N = -[-2\nu] = 1$ and we have the solution

$$\begin{aligned} y_1(t) &= \mathcal{L}^{-1}\left\{\frac{1}{P(s^{\nu})}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^{2\nu} - s^{\nu} - 2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s^{\nu}+1)(s^{\nu}-2)}\right\} \\ &= \frac{1}{3}\,\mathcal{L}^{-1}\left\{\frac{1}{s^{\nu}-2} - \frac{1}{s^{\nu}+1}\right\} \end{aligned}$$

We have the formula

$$\mathcal{L}^{-1}\left\{\frac{1}{s^{\nu} - \alpha_i}\right\} = \sum_{k=0}^{q-1} \alpha_i^{q-k-1}\, E_t(-k\nu, \alpha_i^q).$$

Therefore,

$$y_1(t) = A\left[\sum_{k=0}^{q-1} 2^{q-k-1}\, E_t(-k\nu, 2^q) + \sum_{k=0}^{q-1} (-1)^{q-k-1}\, E_t(-k\nu, (-1)^q)\right]$$

In particular, if $q = 2$ ($\nu = 1/2$) then

$$\begin{aligned} y_1(t) &= A\left[\sum_{k=0}^{1} 2^{1-k}\, E_t(-k/2, 4) + \sum_{k=0}^{1} (-1)^{2-k}\, E_t(-k/2, 1)\right] \\ &= A\left[2\, E_t(0,4) \;+\; E_t(-1/2,4) \;+\; E_t(0,1) \;-\; E_t(-1/2,1)\right] \end{aligned}$$

is a solution of

$$\left[D - D^{1/2} - 2\right]\, y(t) = 0.$$

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33 Integration by Parts

Lemma 387 (Samko, cor p. 667)

$$\left.\begin{array}{l}\phi \in L^p, \qquad \psi \in L^q \\[2ex] 0 < \alpha < 1 \\[2ex] \frac{1}{p} + \frac{1}{q} \leq 1 + \alpha, \quad p \geq 1, q \geq 1 \\[2ex] p \neq 1,\ q \neq 1 \quad if \quad \frac{1}{p} + \frac{1}{q} = 1 + \alpha \end{array}\right\} \qquad \Longrightarrow$$

$$\boxed{\int_a^b \phi(x)\ I_a^\alpha \psi(x)\ dx \quad = \quad \int_a^b \psi(x)\ I_b^\alpha \phi(x)\ dx.}$$

Proof. Interchange the order of integration by Dirichlet formula. ∎

Lemma 388 (Samko, cor 2, p. 46.)

$$\left.\begin{array}{l} f \in I_b^\alpha(L^p), \qquad g \in I_a^\alpha(L^q) \\[2ex] 0 < \alpha < 1 \\[2ex] \frac{1}{p} + \frac{1}{q} \leq 1 + \alpha \end{array}\right\} \qquad \Longrightarrow$$

$$\boxed{\int_a^b f(x)\ D_a^\alpha g(x)\ dx \quad = \quad \int_a^b g(x)\ D_b^\alpha f(x)\ dx.}$$

Proof.

$$f = I_b^\alpha \phi, \quad g = I_a^\alpha \psi, \qquad \phi \in L^p,\ \psi \in L^q \quad \Longrightarrow$$

$$\int_a^b f(x)\ D_a^\alpha g(x)\ dx = \int_a^b \psi(x)\ I_b^\alpha \phi(x)\ dx \overset{lem387}{=} \int_a^b \phi(x)\ I_a^\alpha \psi(x)\ dx = \int_a^b g(x)\ D_b^\alpha f(x)\ dx.$$

34 Inequalities

34.1 Triangle Inequalities

$$|a+b| \leq |a| + |b|$$

Inverse triangle inequality

$$|a| - |b| \leq |a-b| \qquad \Longrightarrow \qquad |\,|a| - |b|\,| \leq |a-b|$$

34.2 Convex/Concave Functions

Definition 389 *A real valued function f defined on an interval Ω is convex if*

$$f(tx + (1-t)y) \leq tf(t) + (1-t)f(y), \qquad \forall x, y \in \Omega, t \in [0,1].$$

f is concave if $-f$ is convex. i.e.

$$f(tx + (1-t)y) > tf(t) + (1-t)f(y), \qquad \forall x, y \in \Omega, t \in [0,1].$$

Remark 51 *The definition states that the graph is above the secant line.*

Property 390

- *A differentiable function of one variable is convex on an interval if and only if its derivative is monotonically non-decreasing on that interval.*
- *A twice differentiable function of one variable is convex on an interval if and only if its second derivative is non-negative there.*
- $$\phi(x) = |x|^p \quad is \quad \begin{cases} convex, & p \geq 1. \\ concave, & 0 < p < 1. \end{cases}$$

Lemma 391

$$\boxed{0 < a < b, \qquad 0 < q < 1 \qquad \Longrightarrow \qquad b^q - a^q \leq (b-a)^q.}$$

Proof. Follows from the concavity of the function $f(x) = x^q$, $0 < q < 1$.

34.3 Jensen's inequality

See `http://en.wikipedia.org/wiki/Jensen_inequality`.

Lemma 392 (Jensen's inequality)

$$\left.\begin{array}{l}\phi \textit{ convex} \\ \\ a_i \textit{ postive weights} \\ \\ x_i \in \textit{Dom}(\phi)\end{array}\right\} \quad \Longrightarrow \quad \phi\left(\frac{\sum_{i=1}^k a_i x_i}{\sum_{i=1}^k a_i}\right) \quad \leq \quad \frac{\sum_{i=1}^k a_i \phi(x_i)}{\sum_{i=1}^k a_i}.$$

Inequality is reversed if ϕ is concave.

Corollary 393

$$\left.\begin{array}{l}\phi \textit{ convex} \\ \\ x_i \in \textit{Dom}(\phi)\end{array}\right\} \quad \Longrightarrow \quad \phi\left(\frac{\sum_{i=1}^k x_i}{k}\right) \quad \leq \quad \frac{\sum_{i=1}^k \phi(x_i)}{k}.$$

Proof. Let $a_i = 1$. ■

Corollary 394 *For $x_i \geq 0$,*

$$(x_1 + x_2 + \cdots + x_k)^p \quad \left\{\begin{array}{lll} \leq & k^{p-1}\,(x_1^p + x_2^p + \cdots + x_k^p), & p \geq 1, \\ \\ > & k^{p-1}\,(x_1^p + x_2^p + \cdots + x_k^p), & 0 < p < 1. \end{array}\right.$$

Proof. *Take $\phi(x) = x^p$.*

1 Application 2

$$D^{\alpha+1}u(t) = F\left(t, \{D^{\beta_i}u(t)\}_{i=1}^{k}\right)$$

$$D\left[D^{\alpha}u(t)\right]^2 = 2D^{\alpha}u(t)\ DD^{\alpha}u(t) = 2D^{\alpha}u(t)\ D^{\alpha+1}u(t) = 2D^{\alpha}u(t)\ F\left(t, \{D^{\beta_i}u(t)\}_{i=1}^{k}\right)$$

$$\left[D^{\alpha}u(t)\right]^2 - \left[D^{\alpha}u(0)\right]^2 = 2\int_0^t D^{\alpha}u(s)F\left(s, \{D^{\beta_i}u(s)\}_{i=1}^{k}\right)$$

2 Future

Repeat the Fractional differential inequality but using Ou-Iang inequlaity and not Bihari-Type inequality

3 Summar of results in [Furati-Tatar 06]

Proposition 395

$$0 < \alpha < 1, \qquad u \in C_{1-\alpha}^{\alpha}[a,b] \quad \Longrightarrow \quad u \in S^{\alpha}(a,b)$$

Theorem 396 *$u \in C_{1-\alpha}^{\alpha}[0,h]$ is a solution of (39), then*

$$u = u_o t^{\alpha-1} + I^{\alpha}D^{\alpha}u$$

Now Suppose $D^{\alpha}u \leq \Theta(t)$. If $t^{1-\alpha}I^{\alpha}\Theta(t)$ is bounded then $u(t) = O(t^{\alpha-1})$ as $t \to 0$

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