19 Geometric and Physical Interpretation

For details see
[14] I. Podlubny. Geometric and physical interpretation of fractional integration and fractional differentiation. Fractional Calculus & Applied Analysis, 5(4):367386, 2002.

Stieltjes Integral Representation

Let

$$g_t(\tau) = \frac{1}{\Gamma(\alpha+1)} \; \{t^\alpha - (t-\tau)^\alpha\}\,.$$

Then we can write

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau$$

as a Stieltjes integral

$$I_0^\alpha f(t) = \int_0^t f(\tau)\, dg_t(\tau).$$

Geometric Interpretation of $I_0^\alpha f$

For a fixed t, consider the three dimensional curve in the space (τ, g, f):

$$C_t : (\ \tau, g_t(\tau), f(\tau)\ )\,, \quad 0 \le \tau \le t.$$

Along C_t build a fence perpendicular to the plane (τ, g) of the varying height $f(\tau)$. Then

- $I_0^1 f(t) = \int_0^t f(\tau)\, d\tau =$ area of the projection of the fence onto the plane (τ, f)
- $I_0^\alpha f(t) = \int_0^t f(\tau)\, d\tau =$ area of the projection of the fence onto the plane (g, f)

Physical Interpretation of $I_0^\alpha f$

If we consider τ as time, then the third dimension $g_t(\tau)$ added to the pair $(\tau, f(\tau))$ can be interpreted as a "transformed" time scale.

Thus we can think of two kinds of time:

1. "Mathematical" time τ: time which is postulated to be homogeneous and equably flowing time. This time is usually depicted as a semi-infinite straight line.
2. "Transformed" time $g(\tau)$: Suppose a clock shows the time τ incorrectly. Suppose the relationship between the measured time τ and the real time T is described by the function $g(\tau)$. Then $T = g(\tau)$ is the correct or "transformed" time.

Thus, the measured time interval is $d\tau$ while the real time interval is given by $dT = dg(\tau)$.

Now, if $v(\tau)$ is the recorded velocity of an object. Then

$$\begin{aligned} I_0^1 v(t) &= \int_0^t v(\tau)\,d\tau &&= \text{wrong passed distance.} \\ I_0^\alpha v(t) &= \int_0^t v(\tau)\,dg(\tau) &&= \text{real passed distance.} \end{aligned}$$
