Math 590/690 (101)

Applied Fractional Calculus

HW#1

(1) Express $(m-n+1)_n$ in terms of the factorial function, where $m = 1, 2, \ldots$.

(2) Show that

$$\int_a^t (t-s)^{\alpha-1}\,(s-a)^{\beta-1}\,ds = B(\alpha,\beta)\,(t-a)^{\alpha+\beta-1}, \qquad \alpha>0, \beta>0.$$

(3) Show that

$$E_{m,\beta-m}(t^m) = t^m E_{m,\beta}(t^m), \qquad \beta = 0, 1, 2, \ldots, m.$$

HW#2

(1) Using the definitions, prove that

$$C^1[a,b] \subset H[a,b] \subset AC[a,b].$$

(2) Let $f(t) = \sqrt{t}$. Does $f \in H[0,1]$? Does $f \in AC[0,1]$? Justify.

(3) Let $f(t) = |t|$. Does $f \in H[-1,1]$? Does $f \in AC[-1,1]$? Justify.

(4) Find a function f such that $f \in AC[a,b] \setminus H[a,b]$.

HW#3

(1) Find a function which is continuous everywhere on an interval but not differentiable at every point of that interval.

(2) Show that

$$AC^{n+1}[a,b] \subset C^n[a,b] \subset AC^n[a,b].$$

(3) Let $f(t) = t^\nu$, $\nu > 0$. Find the largest n such that $f \in AC^n[0,1]$.

(4) Show that

$$\lim_{h\to 0} \Delta_h^{-3} f(t) = I_a^3 f(t).$$

(5) Show that

$$\mathbb{D}_a^p f(t) = D^p f(t), \qquad t > a, \qquad p = 1, 2, \ldots.$$

where $\mathbb{D}_a^p$ is the Grünwald-Letnikov fractional derivative of order p.

HW#4

(1) Show that if $f, g \in L^1(a,b)$ then

$$\int_a^b f(t)\, I_{a+}^p g(t)\, dt = \int_a^b g(t)\, I_{b-}^p f(t)\, dt.$$

(2) Show that if $f, g \in L^1(a,b)$ then

$$I_a^p I_a^q f \;=\; I_a^{p+q} f, \qquad \text{a.e. on } (a,b).$$

What happens if $p + q \geq 1$? Justify.

(3) As in Podlubny, p. 77, consider the following class of functions:

$$\tilde{C}^n = \{\, f \in C^{n-1}[a,b] : D^n f \in L^1(a,b) \,\}.$$

Then clearly $C^n[a,b] \subset AC^n[a,b] \subset \tilde{C}^n$. Let $f \in \tilde{C}^n[a,b]$. Prove or disprove the following statement:

$$D_a^p f(t)|_{t=a} = 0 \qquad \Longleftrightarrow \qquad D^k f(a) = 0,\ k = 0, 1, \ldots, [p].$$

HW#5

(1) If $f \in C[a,b]$ and $I_a^{\sqrt{2}} f$ is constant on (a,b), find f. Is it unique?

(2) Find the general solution of $D_a^p u(t) = 0$, $t \in (a,b)$, that satisfies:

- (a) $u \in C[a,b]$, $p > 0$.
- (b) $u \in C[a,b]$, $0 < p < 1$.
- (c) $u \in C(a,b)$ and $u(a^+) = 1$, $0 < p < 1$.

(3) Are the following steps justifiable? Justify your answer.

$$f \in L^1(a,b) \text{ with } D_a^p f \in L^1(a,b) \quad \Longrightarrow \quad D_a^p[I_a^p D_a^p f - f] \overset{a.e.}{=} 0 \quad \Longrightarrow$$

$$I_a^p D_a^p f - f = D^{m-p} T_a^{m-1}, \qquad m = [p] + 1,$$

where

$$D_a^{m-p} T_a^{m-1}(t) = \sum_{k=1}^{m} c_k \, (t-a)^{p-k}.$$

HW#6

Use fractional differencing (eqn 7.10) to approximate the fractional derivatives $D_0^\alpha f(t)$ of orders

$$\alpha = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1.$$

Plot the curves of all these orders for each function in one figure. Compare and comment on the behavior near $t = 0$.

(1) $f(t) = \sin t$, $0 \le t \le 2\pi$.

(2) $f(t) = \cos t$, $0 \le t \le 2\pi$.

(3) $f(t) = \sin(1/t)$, $0 < t \le 0.1$.

(4) $f(t) = e^t$, $0 \le t \le 1$.
