18 Caputo's Derivative

Definition 336

$$\boxed{{}^cD_a^\alpha f = I_a^{n-\alpha} D^n f, \qquad n-1<\alpha<n.}$$

Property 337

$$\boxed{{}^cD_a^\alpha [const] = 0.}$$

Lemma 338

$$\boxed{f \in C^{n+1}[a,b] \quad \Longrightarrow \quad \lim_{\alpha\to n} {}^cD_a^\alpha f = D^n f, \qquad n \in \mathbb{N}}$$

Proof.

$$\begin{aligned} {}^cD_a^\alpha f &\overset{def}{=} I_a^{n-\alpha} D^n f = I_a^{n-\alpha}\left[I_a D^{n+1} f + D^n f(a)\right] \\ &= I_a^{n-\alpha+1} D^{n+1} f + \frac{D^n f(a)}{\Gamma(n-\alpha+1)}\,(t-a)^{n-\alpha} \\ &\longrightarrow I_a Df + f(a) = D^n f. \end{aligned}$$

Lemma 339

$$\boxed{\left.\begin{array}{l} f \in L^1(a,b) \\ \\ D^n f \in L^1(a,b) \\ \\ f \overset{a.e.}{=} I_a^n D^n f + T_a^{n-1} \end{array}\right\} \quad \Longrightarrow \quad \left\{\begin{array}{l} {}^cD_a^\alpha f \;\; exists \;\; a.e. \;\; and \\ \\ {}^cD_a^\alpha f \;\overset{a.e.}{=}\; D_a^\alpha\,[\,f \;-\; T_a^{n-1}\,]\,. \end{array}\right.}$$

In particular,

$$\boxed{f \in AC^n[a,b] \quad \Longrightarrow \quad \left\{\begin{array}{l} {}^cD_a^\alpha f \;\; exists \;\; a.e. \;\; and \\ \\ {}^cD_a^\alpha f \;\overset{a.e.}{=}\; D_a^\alpha\,[\,f \;-\; T_a^{n-1} f\,]\,. \end{array}\right.}$$

Proof.

$${}^cD_a^\alpha f \overset{def}{=} I_a^{n-\alpha} D^n f \overset{a.e.}{=} D_a^\alpha I_a^n D^n f = D_a^\alpha\left[f - T_a^{n-1}\right].$$

Remark 46 *Kilbas used the above relation as a definition for Caputo's derivative. i.e.*

$${}^cD_a^\alpha f \;:=\; D_a^\alpha\left[f - T_a^{n-1} f\right].$$

Corollary 340

$$\left.\begin{array}{l} f \in L^1(a,b) \\ \underset{\overset{a.e}{=} D^n I_a^\alpha f}{D_a^{n-\alpha} f} \quad \in L^1(a,b) \\ \underset{\Longleftrightarrow I_a^\alpha f \overset{a.e}{=} I_a^n D^n I_a^\alpha f + T_a^{n-1}}{f \overset{a.e.}{=} I_a^{n-\alpha} D_a^{n-\alpha} f + D_a^\alpha T_a^{n-1}} \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} {}^c D_a^\alpha I_a^\alpha f \text{ exists a.e. and} \\ {}^c D_a^\alpha I_a^\alpha f \overset{a.e.}{=} f - D_a^\alpha T_a^{n-1}. \end{array}\right.$$

Proof. Apply Lemma 339 to $I_a^\alpha f$. ■

Lemma 341

$$D_a^{n-\alpha} f \in CL^1(a,b) \quad \Longrightarrow \left\{\begin{array}{l} D^k I_a^\alpha f \in C(a,b) \text{ and bounded in } (a,b), \\ \text{with } D^k I_a^\alpha f(a^+) \text{ exists, } k = 0, \ldots, n-1. \text{ Also} \\ I_a^\alpha f(t) \overset{a.e.}{=} I_a^n D^n I_a^\alpha f(t) + T_{a^+}^{n-1} I_a^\alpha f(t), \ t \in (a,b). \\ \text{equality holds if } f \in CL^1(a,b). \end{array}\right.$$

In particular,

$$\left.\begin{array}{l} D^{n-\alpha} f \in CL^1(a,b) \\ f \in L^\infty(a,b) \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} T_a^{n-1} I_a^\alpha f \equiv 0 \text{ in } (a,b) \\ \text{and thus} \\ I_a^\alpha f(t) = I_a^n D^n I_a^\alpha f(t), \qquad t \in (a,b). \end{array}\right.$$

Proof.

Since $D_a^{n-\alpha} f = D^n I_a^\alpha f$, the firts part follows from Lemma 98 to $I_a^\alpha f$. For the second part,

$$f \in L^\infty(a,b) \implies D^k I_a^\alpha f(a) = I_a^{\alpha-k} f(a) = 0, \ k = 0, \ldots, n-1 \quad \Longrightarrow \quad T_a^{n-1} I_a^\alpha f = 0.$$

Lemma 342

$$D_a^{n-\alpha} f \in CL^1(a,b) \quad \Longrightarrow \quad \begin{cases} {}^cD_a^\alpha I_a^\alpha f \ \text{exists a.e. and} \\ {}^cD_a^\alpha I_a^\alpha f \overset{a.e.}{=} f \ - \ D_a^\alpha T_{a^+}^{n-1} I_a^\alpha f(t). \end{cases}$$

In particular,

$$\left. \begin{array}{l} D_a^{n-\alpha} f \in CL^1(a,b) \\ f \in L^\infty(a,b) \end{array} \right\} \quad \Longrightarrow \quad \begin{cases} {}^cD_a^\alpha I_a^\alpha f \ \text{exists a.e. and} \\ {}^cD_a^\alpha I_a^\alpha f \overset{a.e.}{=} f. \end{cases}$$

Proof.

$$D_a^{n-\alpha} f \in CL^1(a,b) \quad \overset{341}{\Longrightarrow} \quad I_a^\alpha f(t) \overset{a.e.}{=} I_a^n D^n I_a^\alpha f(t) + T_{a^+}^{n-1} I_a^\alpha f(t), \ t \in (a,b) \quad \overset{D_a^\alpha}{\Longrightarrow}$$

$$f \overset{a.e.}{=} I_a^{n-\alpha} D_a^{n-\alpha} f + D_a^\alpha T_a^{n-1} I_a^\alpha f(t) \quad \underset{cor\,340}{\Longrightarrow} \quad \text{first part.}$$

$$f \in L^\infty(a,b) \quad \Longrightarrow \quad T_{a^+}^{n-1} I_a^\alpha f(t) = 0 \quad \Longrightarrow \quad \text{second part.}$$

Lemma 343

$$I_a^\alpha f \in AC^n[a,b] \quad \Longrightarrow \quad \begin{cases} {}^cD_a^\alpha I_a^\alpha f \ \text{exists a.e. and} \\ {}^cD_a^\alpha I_a^\alpha f \overset{a.e.}{=} f \ - \ D_a^\alpha T_a^{n-1} I_a^\alpha f(t). \end{cases}$$

In particular

$$\left. \begin{array}{l} I_a^\alpha f \in AC^n[a,b] \\ f \in L^\infty(a,b) \end{array} \right\} \quad \Longrightarrow \quad \begin{cases} {}^cD_a^\alpha I_a^\alpha f \ \text{exists a.e. and} \\ {}^cD_a^\alpha I_a^\alpha f \overset{a.e.}{=} f. \end{cases}$$

Proof.

$$I_a^\alpha f \in AC^n[a,b] \quad \Longrightarrow \quad D_a^{n-\alpha} f = D^n I_a^\alpha f \in L^1(a,b) \quad \& \quad I_a^\alpha f = I_a^n D^n I^\alpha f + T_a^{n-1} I_a^\alpha f.$$

The first part follows from Corollary 340.

Property 344 *Let $\alpha > 0$ and $n = -[-\alpha]$. Then*

$${}^cD_a^\alpha (t-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} \, (t-a)^{\beta-\alpha}, \qquad \beta > n-1.$$

and

$${}^cD_a^\alpha (t-a)^k = 0, \qquad k = 0, \ldots, n-1$$

$${}^cD_a^\alpha T_a^m \equiv 0, \qquad m = 0, \ldots, n-1$$

Note that ${}^cD_a^\alpha (t-a)^\beta = D_a^\alpha (t-a)^\beta$ however it is valid only for $\beta > n-1$ while D_a^α is valid for $\beta > -1$.

Lemma 345

$$\boxed{f \in AC^n[a,b] \quad \Longrightarrow \quad I_a^\alpha\, {}^cD_a^\alpha f = I_a^n D^n f = f - T_a^{n-1} f.}$$

Proof.

$$I_a^\alpha\, {}^cD_a^\alpha f \overset{lem\,339}{=} I_a^\alpha D_a^\alpha \left[f - T_a^{n-1} f\right] = I_a^\alpha D_a^\alpha I_a^n D^n f = I_a^\alpha I_a^{n-\alpha} D^n f = I_a^n D^n f.$$

Lemma 346

$$\boxed{\left.\begin{array}{l} f \in L^1(a,b) \\ {}^cD_a^{\alpha+m} f \ \text{exists} \end{array}\right\} \quad \Longrightarrow \quad {}^cD_a^\alpha D^m f \quad \overset{a.e.}{=} \quad {}^cD_a^{\alpha+m} f.}$$

Proof.

$${}^cD_a^\alpha D^m \overset{def}{=} I_a^{n-\alpha} D^n D^m = I_a^{n-\alpha} D^{m+n} = I_a^{n+m-(\alpha+m)} D^{m+n} \overset{def}{=} {}^cD_a^{\alpha+m}. \quad \blacksquare$$

Lemma 347

$$\boxed{\left.\begin{array}{l} f \in L^1(a,b) \\ {}^cD_a^{\alpha+m} f \ \text{exists} \\ f \overset{a.e.}{=} I_a^n D^n f + T_a^{n-1} \end{array}\right\} \quad \Longrightarrow \quad D^m\, {}^cD_a^\alpha f \ \overset{a.e.}{=} \ {}^cD_a^{\alpha+m} f + R_a^n}$$

where

$$R_a^n = \sum_{k=n}^{m+n-1} \frac{D^k f(a)}{\Gamma(k-\alpha-m+1)} (t-a)^{k-\alpha-m+1}.$$

Proof.

$$\begin{aligned} D^m\, {}^cD_a^\alpha f \quad &\overset{def}{=} \quad D^m I_a^{n-\alpha} D^n f = D_a^{\alpha+m-n} D^n f \overset{lem\,327}{=} D_a^{\alpha+m} f - D_a^{\alpha+m} T_a^{n-1} \\ &= \quad D_a^{\alpha+m} f - D_a^{\alpha+m} \left[T_a^{m+n-1} - \sum_{k=n}^{m+n-1} \frac{D^k f(a)}{k!} (t-a)^k\right] \\ &\overset{lem\,339}{=} \quad {}^cD_a^{\alpha+m} f + D_a^{\alpha+m} \sum_{k=n}^{m+n-1} \frac{D^k f(a)}{k!} (t-a)^k. \quad \blacksquare \end{aligned}$$
