Theorem 73 (Natanson [12], p. 133)

$$\left.\begin{array}{l} Df \ \textit{exists at every } t \in [a,b] \\ \\ Df \ \textit{bounded on } [a,b] \end{array}\right\} \quad \Longrightarrow \quad \left\{\begin{array}{ll} Df \in L^1(a,b) & \& \\ \\ f(t) = I_a Df(t) + f(a), & t \in [a,b]. \\ \quad\quad \Longleftrightarrow f \in AC[a,b] & \end{array}\right.$$

Note: Df exists at every $t \in [a,b] \quad \Longrightarrow \quad f \in C[a,b]$.

Theorem 74

$$\left.\begin{array}{l} f : [a,b] \to R \\ \\ Df \ \textit{exists at every } t \in [a,b] \\ \\ Df \in L^1(a,b) \end{array}\right\} \quad \Longrightarrow \quad \left\{\begin{array}{c} f(t) = I_a Df(t) + f(a), \quad a \le t \le b, \\ \\ \Updownarrow \\ \\ f \in AC[a,b]. \end{array}\right.$$

See Rudin [15], Theorem 7.21 p. 149, and Natanson [12], p. 266.

Example 75

Find $f : [0,1] \to \mathbb{R}$ *which is differentiable at every point of* $[0,1]$ *with*

$$Df(t) = \begin{cases} \frac{3}{2} t^{\frac{1}{2}} \sin \frac{1}{t} - t^{-\frac{1}{2}} \cos \frac{1}{t}, & t \in (0,1], \\ \\ d, & t = 0. \end{cases}$$

Solution.

Since f *is differentiable at every point,* $f \in C[0,1]$*. Moreover,*

$$\begin{cases} Df \ \textit{exists everywhere on } [0,1], \\ \\ Df \in L^1(0,1), \ \textit{unbounded}\ , \end{cases}$$

since $|Df(t)| \le \frac{3}{2} + \frac{1}{\sqrt{t}}$.

Therefore, by Theorem 74,

$$f(t) - f(0) = I_0 Df(t) = t^{\frac{3}{2}} \sin \frac{1}{t}, \qquad t \in [0.1].$$

Since,

$$Df(0) = \lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \sqrt{h} \sin \frac{1}{h} = 0,$$

It is necessary to let $d = 0$*, otherwise there is no solution.*

Accordingly, $f \in AC[0,1]$ *is of the form*

$$f(t) - f(0) = \begin{cases} t^{\frac{3}{2}} \sin \frac{1}{t}, & t \in (0,1], \\ \\ 0, & t = 0. \end{cases}$$

1.12.2 Sufficient condition: $\boldsymbol{Df \in C[a,b]}$

Theorem 76 (FTC)

$$\boxed{\begin{array}{ccccc} Df \in C[a,b] & \Longrightarrow & \underset{Df\in C[a,b]}{f \in C[a,b]} & \& & f = \underbrace{I_a Df}_{\in AC} + f(a) \ \textit{in} \ [a,b] \\ & & \Updownarrow & & \Updownarrow \\ & & f \in C^1[a,b] & \subset H^1[a,b] & \subset AC[a,b].\ \textit{(Lemma 55).} \end{array}}$$

Proof. Follows as a special case from Theorem 74. ■

Corollary 77

$$\boxed{D^n f \in C[a,b] \quad \underset{induction}{\overset{thm\ 76}{\Longleftrightarrow}} \quad f \in C^n[a,b].}$$

1.12.3 Sufficient condition: $\boldsymbol{Df \in CL^1(a,b)}$

Lemma 78

$$Df \in CL^1(a,b) \quad \Longrightarrow \quad \begin{cases} f(a^+) := \lim_{t\to a^+} f(t) \text{ exists.} \\ f(t) = I_a Df(t) + f(a^+), \quad t \in (a,b) \\ \text{thus} \quad f \in C(a,b) \text{ and bounded in } (a,b). \end{cases}$$

Proof.

For any sufficiently small $\epsilon > 0$, and $a + \epsilon < b_- < b$, we have $Df \in C[a+\epsilon, b_-]$. Thus

$$f(a+\epsilon) \quad \underset{thm\,76}{\overset{FTC}{=}} \quad \int_{a+\epsilon}^{b_-} Df(s)\,ds - f(b_-).$$

Since $Df \in L^1(a,b)$, the limit of the RHS exists as $\epsilon \to 0$.

Now, Given sufficiently small $\epsilon > 0$, $Df \in C[a+\epsilon, b]$ and thus

$$f(t) \quad \underset{thm\,76}{\overset{FTC}{=}} \quad I_{a+\epsilon}Df(t) + f(a+\epsilon), \quad t > a + \epsilon.$$

By taking $\epsilon \longrightarrow 0$ we obtain

$$f(t) \quad \overset{Df\in L^1}{=} \quad \underset{Cor\,41}{I_a Df(t)} + f(a^+), \qquad t \in (a,b).$$

Clearly $f \in C(a,b)$ since $I_a Df \in AC[a,b]$ and f is bounded since $f(a^+)$ exists. ■

Remark 9 *Note that the right-hand side in Lemma 78 satisfies:*

$$g(t) = \begin{cases} I_a Df(t) + f(a^+), & t \in (a,b), \\ f(a^+), & t = a. \end{cases}$$

and thus $g \in AC[a,b]$.

1.12.4 Sufficient Condition: $f \in C[a,b]$ and $Df \in CL^1(a,b)$ / Df defined *n.e.*

Definition 81 *A statement is true nearly everywhere (n.e.) in $S \subset R$ if it is true in S except for a countable subset of S.*

Theorem 82

$$\left.\begin{array}{l} f \in C[a,b] \\ \\ Df \;\; \textit{exists n.e. in } [a,b] \\ \\ Df \in L^1(a,b) \end{array}\right\} \underset{Koliha[9]}{\Longrightarrow} \quad f \in AC[a,b].$$

Corollary 83

$$\left.\begin{array}{l} f \in C[a,b] \\ \\ Df \in CL^1(a,b) \end{array}\right\} \Longrightarrow \qquad f \in AC[a,b]$$

Proof. Follows from Theorem 76. Also it follows from Lemma 78, since

$$f(t) = I_a Df(t) + f(a), \qquad t \in [a,b]. \quad \blacksquare$$

3.4 Zero Derivative, $\boldsymbol{Df = 0}$

Theorem 104

$$\textit{jump function} \quad \overset{Kolm,\, prob\, 8}{\underset{p332}{\Longrightarrow}} \quad \textit{has a zero derivative a.e.}$$

Theorem 105

$$\left.\begin{array}{l} f \in AC[a,b] \\ \\ Df \overset{a.e.}{=} 0 \quad in\ [a,b] \end{array}\right\} \quad \overset{Kolm,\, p339}{\underset{Royden\ p110}{\Longrightarrow}} \quad f \textit{ is constant} \quad \forall\, t \in [a,b].$$

Proof.

$$f \in AC[a,b] \quad \Longrightarrow \quad f(t) = I_a Df(t) + c = c, \quad \forall\, t \in [a,b]. \quad \blacksquare$$

Theorem 106

$$\left.\begin{array}{l} f \in C[a,b] \\ \\ Df \overset{n.e.}{=} 0 \ in\ [a,b] \end{array}\right\} \quad \Longrightarrow \quad f \textit{ is constant on } [a,b].$$

Proof.

$$f \in C[a,b] \ \&\ Df \in L^1(a,b) \text{ exists n.e. in } [a,b] \quad \underset{thm\, 82}{\Longrightarrow} \quad f \in AC[a,b]. \quad \blacksquare$$

Theorem 107

$$Df(t) = 0 \quad \forall t \in (a,b) \quad \Longrightarrow \quad \begin{cases} f(t) = c, \quad \forall t \in (a,b). \\ \\ f(a^+) := \lim_{t \to a^+} f(t) = c \quad exists \end{cases}$$

Proof. *The result follows from MVT, Theorem 67 or from Lemma 78 since* $Df \in CL^1(a,b)$.

Remark 12

$$Df \overset{a.e.}{=} 0 \ in\ [a,b] \quad \not\Longrightarrow \quad f \textit{ is constant on } [a,b],$$

since Cantor function is monotone on $[0,1]$ *with* $Df = 0$ *a.e. on* $[0,1]$.

$$\left.\begin{array}{l} f \in C[a,b] \\ \\ Df \overset{a.e.}{=} 0 \ in\ [a,b] \end{array}\right\} \quad \not\Longrightarrow \quad f \textit{ is constant on } [a,b],$$

since Cantor function is continuous and non constant on $[0,1]$ *with* $Df \overset{a.e.}{=} 0$ *but not n.e.*

3.5 Zero nth Derivative, $D^n f = 0$

Lemma 108

$$\left.\begin{array}{lcll} f \in AC^n[a,b] & & & \\ D^n f & \overset{a.e.}{=} & 0 & in\ [a,b] \end{array}\right\} \Longrightarrow f(t) = \sum_{k=0}^{n-1} c_k\,(t-a)^k, \quad \forall\, t \in [a,b].$$

where c_k are arbitraries.

Proof.

$$f \in AC^n[a,b] \quad \Longrightarrow \quad f(t) = I_a^n D^n f(t) + T_a^{n-1}(t) = T_a^{n-1}(t), \quad \forall\, t \in [a,b]. \quad \blacksquare$$

Lemma 109

$$\left.\begin{array}{lcll} f \in C^{n-1}[a,b] & & & \\ D^n f & \overset{n.e.}{=} & 0 & in\ [a,b] \end{array}\right\} \Longrightarrow f(t) = \sum_{k=0}^{n-1} c_k\,(t-a)^k, \quad \forall t \in [a,b].$$

Proof.

$$D^{n-1} f \in C[a,b] \quad \& \quad D^n f = DD^{n-1} f \in L^1(a,b) \text{ exists n.e. in } [a,b] \underset{thm\,82}{\Longrightarrow}$$

$$D^{n-1} f \in AC[a,b] \quad \Longrightarrow \quad f \in AC^n[a,b] \underset{lem\,108}{\Longrightarrow} \quad \text{result.} \quad \blacksquare$$

Lemma 110

$$D^n f = 0 \quad \forall t \in (a,b) \quad \Longrightarrow \quad \begin{cases} f(t) = \sum_{k=0}^{n-1} c_k\,(t-a)^k, & \forall t \in (a,b), \\ f(a^+) := \lim_{t\to a^+} f(t) = c_0. & \end{cases}$$

Consequently,

$$D^n f = 0 \quad \forall t \in [a,b] \quad \Longrightarrow \quad f(t) = \sum_{k=0}^{n-1} c_k\,(t-a)^k, \quad \forall t \in [a,b].$$

Proof.
By Lemma 78, for $t \in (a,b)$,

$$D^{n-1} f(t) = I_a D^n f(t) + c_{n-1}.$$

$$D^{n-2} f(t) = I_a D^{n-1} f(t) + c_{n-2} = c_{n-1}(t-a) + c_{n-2}.$$

The result follows by induction. $\blacksquare$

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