A Continuous Differentiable Function with Discontinuous Derivative

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Is it possible to have a function which is differentiable everywhere and the derivative not be continuous? It turns out that there is a fairly simple example of such a creature.

First, a general comment. To show that f' is continuous, we need

$$f'(a) = \lim_{x \to a} f'(x).$$

On the left hand side,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

so that

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} f'(x) = \lim_{x \to a} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

To claim equality for this string of limits is to claim that it does not matter which order we compute the $\lim\limits_{x \to a}$ and $\lim\limits_{h \to 0}$, but it does matter as we shall see. This may seem like an intuitive thing to do, but re-ordering limits is, in fact, one of the most error-prone aspects of a mathematician's life.

But we can do better than this philosophical discussion about limits. Here is an explicit example of a function which is continuous and differentiable, but does NOT have a continuous derivative. Let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

The function f is clearly continuous for $x \neq 0$, but at $x = 0$, we need the squeeze theorem to see that

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$$

so that f is continuous for all $\mathbb{R}$.

Furthermore, f is differentiable when $x \neq 0$ simply by use of the product and chain rules (try it!) and

$$f'(x) = -\cos \frac{1}{x} + 2x \sin \frac{1}{x}.$$

However, we need to use the definition of derivative to compute $f'(0)$. Compute

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h} - 0}{h} \\ &= \lim_{h \to 0} h \sin \frac{1}{h}. \end{aligned}$$

Once again, the squeeze theorem (probably considering one-sided limits) comes to our rescue to compute this final limit so that $f'(0) = 0$.

So f is continuous and f' exists for all x, but lets consider the continuity of $f'(x)$. For $x \neq 0$, clearly f' is continuous, but it is not continuous at $x = 0$. For this continuity we would need

$$f'(0) = \lim_{x \to 0} f'(x) = \lim_{x \to 0} -\cos\frac{1}{x} + 2x\sin\frac{1}{x}$$

but this last limit fails to exist at 0 because of the oscillation of $\cos\frac{1}{x}$.