King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

MS Thesis Proposal

Reductions and Cores of Ideals

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1 Summary

In 1954, Northcott and Rees introduced the notions of reduction of an ideal and of basic ideal in a Noetherian local rings with infinite residue field. In 1973-75, Hays investigated these concepts in more general settings; namely, Noetherian rings and Prüfer domains. The core of an ideal is the intersection of all its reductions. This notion was initially introduced by Northcott and Rees (1954) and later systematically studied by Sally and Rees (1988). It also arose in the context of Briancon-Skoda theorem which asserts that the core of any ideal in a regular ring contains a certain fixed power of its integral closure. In 1995, Huneke and Swanson determined explicitly the core of integrally closed ideals in 2-dimensional regular Noetherian local rings. The goal of this MS thesis is to study in detail the above concepts.

2 Introduction

Let R be a commutative ring with identity and I a nonzero ideal of R. A subideal J is a reduction of I if $J \subseteq I$ and $JI^n = I^{n+1}$ for some positive integer n, and I is called a basic ideal if it has no proper reductions. In a Noetherian local ring (R, m) with infinite residue field, an ideal I is basic if and only if the elements of a minimal base of I are analytically independent. Moreover, if I is an m-primary ideal then I is basic if and only if it can be generated by a system of parameters. Also, if J is contained in a regular ideal I, then J is a reduction of I if and only if $JC = IC$ for some regular ideal C [21, 22].

In Noetherian rings, this notion local to the effect that an ideal I is basic if and only if so is I_M for every maximal ideal M. A very interesting result is that a domain is Prüfer if and only if every finitely generated ideal is basic. Moreover, a domain has all its ideals basic if and only if it is a one-dimensional Prüfer domain [9, 10].

The core of I, denoted $core(I)$, is the intersection of all reductions of I. This notion was introduced by Northcott and Rees and studied by Sally and Rees. It has since then played a crucial role in the study of Rees algebras. It also naturally appears in Briancon-Skoda theorem. Recently, the notion of core gained some prominence when Hyry and Smith showed how it is closely related to Kawamata conjecture on the existence of sections for numerically effective line bundles which are adjoint to an ample line bundle over a complex smooth algebraic variety [15]. The core notion has been intensively studied by many authors in Noetherian settings

by the use of invariant factors such as multiplicity, minimal number of generators, analytic spread [2, 3, 13, 14, 15].

3 Literature Review

In 1954, Northcott and Rees introduced the notions of reduction of an ideal and of basic ideal in a Noetherian local ring with infinite residue field [21]. They provided a nice description of basic ideals in this setting. Namely, an ideal is basic if and only if it can be generated by analytically independent elements. They also showed that an ideal B which does not consist entirely of zero divisors is a reduction of A if and only if the elements of A are analytically dependent on B. This condition was also shown to be equivalent to the existence of an ideal C with $BC = AC$.

In 1973, Hays investigated reductions of ideals in arbitrary commutative rings (i.e., beyond the Noetherian local context) [9]. He proved that the concept of basic ideal is a local property in Noetherian rings (i.e., an ideal I is basic if and only if I_M is basic for every maximal ideal M). He also showed that elements of the principal class, a generalization of analytic independence , generate basic ideals but not conversely. Basic properties of C-ideals (those which cannot arise as reductions of larger ideals) were also stated. Finally, he examined the context of Prüfer domains, stating that a domain is Prüfer if and only if every finitely generated ideal is basic.

In 1975, Hays's second paper was devoted to the basic ideal and C-ideal properties in valuation or Prüfer settings [10]. His first main result asserts that the C-ideal and primary ideal properties coincide in valuation rings. The second main theorem states that an integral domain has the basic ideal property for all ideals if and only if it is a one-dimensional Prüfer domain.

In 1994, Craig Huneke and Irena Swanson determined explicitly the core of integrally closed ideals in 2-dimensional regular Noetherian local rings. In a 2-dimensional regular local ring (R, m) with an infinite residue field, if I is an m-primary integrally closed ideal generated minimally by n elements with presenting matrix A, then the adjoint of I, denoted by $adj(I)$, equals $I_{n-2}(A)$, where $I_{n-2}(A)$ is the ideal generated by all $(n-2)\times(n-2)$ minors of A. Their main result establishes the equality $core(I) = Iadj(I)$.

4 Objectives

This MS thesis aims at providing an in-depth study of the following topics:

- The reduction of an ideal in Noetherian local rings.
- Localization of the concept of basic ideal in Noetherian local rings.
- C-ideals (i.e., those which are not reductions of larger ideals).
- Characterization of Prüfer domains via the notion of basic ideal.
- Domains which satisfy the basic ideal property (for all ideals).
- The core of an m-primary integrally closed ideal in 2-dimensional regular Noetherian local rings.

5 Program of Study

The program comprises five tasks:

Task 1: Study of book chapters and research papers on the following basic topics:

1. Valuation and Prüfer domains [8]
2. Macaulay rings and regular rings [16, 19, 28]
3. Integrally closed ideals in 2-dimensional regular local rings [12, 26, 28]
4. Dimension theory [1, 19]

Task 2: Study of the below research papers. This is partly handled in Math 595.

1. Reductions of ideals in Noetherian local rings [21]
2. Reduction of ideals in commutative rings [9]
3. Reduction of ideals in Prüfer domains [10]
4. Core of ideals in 2-dimensional regular local rings [13]

Task 3: Give 12-15 seminars.

Task 4: Develop and set up problems for future research work.

Task 5: Write the MS thesis.

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