On t-reduction and t-integral closure of ideals in integral domains

Salah Kabbaj

Dedicated to David F. Anderson

Abstract Let R be an integral domain and I a nonzero ideal of R. An ideal $J \subseteq I$ is a t-reduction of I if $(JI^n)_t = (I^{n+1})_t$ for some $n \geq 0$. An element x of R is t-integral over I if there is an equation $x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0$ with $a_i \in (I^i)_t$ for $i = 1, ..., n$. The set of all elements that are t-integral over I is called the t-integral closure of I. This paper surveys recent literature which studies t-reductions and t-integral closure of ideals in arbitrary domains as well as in special contexts such as Prüfer v-multiplication domains, Noetherian domains, and pullback constructions.

1 Introduction

Throughout, all rings considered are commutative with identity. Let R be a domain with quotient field K, I a nonzero fractional ideal of R, and let $I^{-1} := (R : I) = \{x \in K \mid xI \subseteq R\}$. The v- and t-closures of I are defined, respectively, by $I_v := (I^{-1})^{-1}$ and $I_t := \cup J_v$, where J ranges over the set of finitely generated subideals of I. The ideal I is a v-ideal (or divisorial) if $I_v = I$ and a t-ideal if $I_t = I$. Under the ideal t-multiplication $(I, J) \mapsto (IJ)_t$ the set $F_t(R)$ of fractional t-ideals of R is a semigroup with unit R. Recall that factorial domains, Krull domains, GCDs, and PvMDs can be regarded as t-analogues of the principal domains, Dedekind domains, Bézout domains, and Prüfer domains, respectively. For instance, a domain is Prüfer (resp., a PvMD) if every nonzero finitely generated ideal is invertible (resp., t-invertible). We also recall the w-closure of I defined by $I_w := \bigcup(I : J)$, where the union is taken over all finitely generated ideals J of R that satisfy $J_v = R$; equivalently, $I_w = \bigcap IR_M$, where M ranges over the maximal t-ideals of R. We always have

S. Kabbaj (*supported by KFUPM under DSR Research Grant # RG1328"*)
Department of Mathematics and Statistics, KFUPM, Dhahran 31261, Saudi Arabia
e-mail: kabbaj@kfupm.edu.sa

$I \subseteq I_w \subseteq I_t \subseteq I_v$. For ample details on the v-, t-, and w-operations, we refer the reader to David Anderson's papers [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16] and also [21, 23, 26, 35, 46, 53, 54, 55, 57, 60, 62, 63, 65, 66, 67].

Let R be a ring and I an ideal of R. An ideal $J \subseteq I$ is a reduction of I if $JI^n = I^{n+1}$ for some positive integer n. An ideal which has no reduction other than itself is called a basic ideal [38, 39, 59]. The notion of reduction was introduced by Northcott and Rees and its usefulness resides mainly in two facts: "First, it defines a relationship between two ideals which is preserved under homomorphisms and ring extensions; secondly, what we may term the reduction process gets rid of superfluous elements of an ideal without disturbing the algebraic multiplicities associated with it" [59]. The main purpose of their paper was to contribute to the analytic theory of ideals in Noetherian (local) rings via minimal reductions. An element $x \in R$ is integral over I if there is an equation $x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0$ with $a_i \in I^i$ for $i = 1, ..., n$. The set of all elements that are integral over I is called the integral closure of I, and is denoted by $\overline{I}$. Reductions happened to be a very useful tool for the theory of integral dependence over ideals. For a full treatment of these topics, we refer the reader to Huneke and Swanson's book "Integral closure of ideals, rings, and modules" [48].

Let R be a domain and I a nonzero ideal. An ideal $J \subseteq I$ is a t-reduction of I if $(JI^n)_t = (I^{n+1})_t$ for some $n \geq 0$; and $x \in R$ is t-integral over I if there is an equation $x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0$ with $a_i \in (I^i)_t$ for $i = 1, ..., n$. The set of all elements that are t-integral over I is called the t-integral closure of I. This paper surveys recent literature which studies t-reductions and t-integral closure of ideals in arbitrary domains as well as in special contexts such as Prüfer v-multiplication domains (PvMDs), Noetherian domains, and pullback constructions. The four papers involved in this survey are [50] (co-authored with A. Kadri), [44] (co-authored with E. Houston and A. Mimouni), and [51, 52] (co-authored with A. Kadri and A. Mimouni). In this survey, we present and discuss the results without proofs and provide most of the examples with full details (from the original papers).

2 The general case of integral domains

This part covers [50] which deals with t-reductions and t-integral closure of ideals in arbitrary domains. The aim is to provide t-analogues of well-known results on the integral closure of ideals and the correlations with reductions. Namely, Section 2.1 identifies basic properties of t-reductions of ideals and give explicit examples discriminating between the notions of reduction and t-reduction. Section 2.2 examines the concept of t-integral closure of ideals as well as its correlation with t-reductions. Section 2.3 studies the persistence and contraction of t-integral closure of ideals under ring homomorphisms. All along this part, the main results are illustrated with original examples.

2.1 t-Reductions of ideals

This section identifies basic ideal-theoretic properties of the notion of t-reduction including its behavior under localizations. We first provide an example (with full details) discriminating between the notions of reduction and t-reduction. Recall that, in a ring R, a subideal J of an ideal I is called a reduction of I if $JI^n = I^{n+1}$ for some positive integer n [59]. An ideal which has no reduction other than itself is called a basic ideal [38, 39].

Definition 2.1. Let R be a domain and $J \subseteq I$ nonzero [fractional] ideals of R.

- J is a *trivial t-reduction* of I if $J_t = I_t$.
- J is a *t-reduction* of I if $(JI^n)_t = (I^{n+1})_t$ for some integer $n \geq 0$.
- I is *t-basic* if it has no t-reduction other than the trivial t-reductions.
- R has the t-basic (resp., finite t-basic) ideal property if every nonzero (resp., finitely generated) [fractional] ideal of R is t-basic.

This is not to be confused with the identically named notion of Epstein [28, 29, 30], which generalizes the original notion of reduction in a different way and was studied in different settings. Namely, let c be a closure operation. An ideal $J \subseteq I$ is a c-reduction of I if $J^c = I^c$. Thus, Epstein's c-reduction coincides with our trivial c-reduction.

Recall a basic property of the t-operation (which, in fact, holds for any arbitrary star operation): for any two nonzero ideals I and J, we have $(IJ)_t = (I_tJ)_t = (IJ_t)_t = (I_tJ_t)_t$. So, for nonzero ideals $J \subseteq I$, J is a t-reduction of I if and only if J is a t-reduction of I_t if and only if J_t is a t-reduction of I_t. Notice also that any reduction is also a t-reduction; and the converse is not true, in general, as shown by the next example which exhibits a domain R with two t-ideals $J \subsetneqq I$ such that J is a t-reduction but not a reduction of I.

Example 2.2 ([50, Example 2.2]). We use a construction from [49]. Let x be an indeterminate over $\mathbb{Z}$ and let $R := \mathbb{Z}[3x, x^2, x^3]$, $I := (3x, x^2, x^3)$, and $J := (3x, 3x^2, x^3, x^4)$. Then $J \subsetneqq I$ are two finitely generated t-ideals of R such that $JI^n \subsetneqq I^{n+1} \forall\, n \in \mathbb{N}$ and $(JI)_t = (I^2)_t$.

Proof. I is a height-one prime ideal and, hence, a t-ideal of R [49]. Next, we prove that J is a t-ideal. We first claim that $J^{-1} = \frac{1}{x}\mathbb{Z}[x]$. Indeed, notice that $\mathbb{Q}(x)$ is the quotient field of R and since $3x \subseteq J$, then $J^{-1} \subseteq \frac{1}{3x}R$. So, let $f := \frac{g}{3x} \in J^{-1}$ where $g = \sum_{i=0}^{m} a_i x^i \in \mathbb{Z}[x]$ with $a_1 \in 3\mathbb{Z}$. Then the fact that $x^3 f \in R$ implies that $a_i \in 3\mathbb{Z}$ for $i = 0, 2, ..., m$; i.e., $g \in 3\mathbb{Z}[x]$. Hence $f \in \frac{1}{x}\mathbb{Z}[x]$, whence $J^{-1} \subseteq \frac{1}{x}\mathbb{Z}[x]$. The reverse inclusion holds since $\frac{1}{x}J\mathbb{Z}[x] = (3, 3x, x^2, x^3)\mathbb{Z}[x] \subseteq R$, proving the claim. Next, let $h \in (R : \mathbb{Z}[x]) \subseteq R$. Then $xh \in R$ forcing $h(0) \in 3\mathbb{Z}$ and thus $h \in (3, 3x, x^2, x^3)$. So, $(R : \mathbb{Z}[x]) \subseteq (3, 3x, x^2, x^3)$, hence $(R : \mathbb{Z}[x]) = \frac{1}{x}J$. It follows that $J_t = J_v = \left(R : \frac{1}{x}\mathbb{Z}[x]\right) = x(R : \mathbb{Z}[x]) = J$, as desired. Next, let $n \in \mathbb{N}$. It is to see that $x^3x^{2n} = x^{2n+3}$ is the monic monomial with the smallest degree in JI^n. Therefore $x^{2(n+1)} = x^{2n+2} \in I^{n+1} \setminus JI^n$. That is, J is not a reduction of I. It remains to prove $(JI)_t = (I^2)_t$. We first claim that $(JI)^{-1} = \frac{1}{x^2}\mathbb{Z}[x]$. Indeed, $(JI)^{-1} \subseteq (J^{-1})^2 = \frac{1}{x^2}\mathbb{Z}[x]$ and the reverse inclusion holds since $\frac{1}{x^2}JI\mathbb{Z}[x] = (3, 3x, x^2, x^3)(3, x, x^2)\mathbb{Z}[x] \subseteq R$, proving the claim. Now, observe that $I^2 = (9x^2, 3x^3, x^4, x^5)$. It follows that $(IJ)_t = (IJ)_v = \left(R : \frac{1}{x^2}\mathbb{Z}[x]\right) = x^2(R : \mathbb{Z}[x]) = xJ \supseteq I^2$. Thus $(IJ)_t \supseteq (I^2)_t$, as desired.

In the above example, the domain R is not integrally closed. In fact, there is a class of integrally closed domains where the notions of reduction and t-reduction are always distinct. Indeed, in [50, Example 2.3], we show that if R is any integrally closed Mori domain that is not completely integrally closed, then there always exist nonzero ideals $J \subsetneqq I$ in R such that J is a t-reduction but not a reduction of I. Another crucial fact concerns reductions of t-ideals. That is, if J is a reduction of a t-ideal, then so is J_t; and the converse is not true, in general, as shown by [50, Example 2.4] which features a domain R with a t-ideal I and an ideal $J \subseteq I$ such that J_t is a reduction but J is not a reduction of I.

In the rest of this section, we provide basic ideal-theoretic properties of t-reduction. Let R be an arbitrary domain. Recall that, for any nonzero ideals I, J, H of R, the equality $(IJ + H)_t = (I_tJ + H)_t$ always holds. This property allowed us to prove the next basic result which examines the t-reduction of the sum and product of ideals.

Lemma 2.3. *Let $J \subseteq I$ and $J' \subseteq I'$ be nonzero ideals of R. If J and J' are t-reductions of I and I', respectively, then $J+J'$ and JJ' are t-reductions of $I+I'$ and II', respectively.*

The next basic result examines the transitivity for t-reduction.

Lemma 2.4. *Let $K \subseteq J \subseteq I$ be nonzero ideals of R. Then:*

(1) *If K is a t-reduction of J and J is a t-reduction of I, then K is a t-reduction of I.*
(2) *If K is a t-reduction of I, then J is a t-reduction of I.*

The next basic result examines the t-reduction of the power of an ideal.

Lemma 2.5. *Let $J \subseteq I$ be nonzero ideals of R and let n be a positive integer. Then:*

(1) *J is a t-reduction of $I \Leftrightarrow J^n$ is a t-reduction of I^n.*
(2) *If $J = (a_1, ..., a_k)$, then: J is a t-reduction of $I \Leftrightarrow (a_1^n, ..., a_k^n)$ is a t-reduction of I^n.*

The next basic result examines the t-reduction of localizations.

Lemma 2.6. *Let $J \subseteq I$ be nonzero ideals of R and let S be a multiplicatively closed subset of R. If J is a t-reduction of I, then $S^{-1}J$ is a t-reduction of $S^{-1}I$.*

Note that, in a PvMD, J is a t-reduction of I if and only if J is t-locally a reduction of I (Lemma 3.9).

2.2 t-Integral closure of ideals

This section investigates the concept of t-integral closure of ideals and its correlation with t-reductions. Our objective is to establish satisfactory t-analogues of (and in some cases generalize) well-known results, in the literature, on the integral closure of ideals and its correlation with reductions.

Definition 2.7. Let R be a domain and I a nonzero ideal of R. An element $x \in R$ is t-integral over I if there is an equation

$$x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0 \text{ with } a_i \in (I^i)_t \ \forall i = 1, ..., n.$$

The set of all elements that are t-integral over I is called the t-integral closure of I, and is denoted by $\widetilde{I}$. If $I = \widetilde{I}$, then I is called t-integrally closed.

The t-integral closure of the ideal R is always R, whereas the t-integral closure of the ring R (also called pseudo-integral closure) may be larger than R. Also, we have $J \subseteq I \Rightarrow \widetilde{J} \subseteq \widetilde{I}$. More properties are listed in Remark 2.14. It is well-known that the integral closure of an ideal is an ideal which is integrally closed. The next theorem provides a t-analogue for this result.

Theorem 2.8. *The t-integral closure of an ideal is an integrally closed ideal. In general, it is not t-closed and, a fortiori, not t-integrally closed.*

The proof of the first statement of this theorem relied on the following lemma which sets a t-analogue for the notion of Rees algebra of an ideal [48, Chapter 5]. The Rees algebra of an ideal I (in a ring R) is the graded subring of $R[x]$ given by $R[Ix] := \bigoplus_{n\geq 0} I^n x^n$ [48, Definition 5.1.1] and whose integral closure in $R[x]$ is the graded ring $\bigoplus_{n\geq 0} \overline{I^n}\, x^n$ [48, Proposition 5.2.1].

Lemma 2.9. *Let R be a domain, I a t-ideal of R, and x an indeterminate over R. The t-Rees algebra of I is given by $R_t[Ix] := \bigoplus_{n\geq 0}(I^n)_t x^n$, and it is a graded subring of $R[x]$ and its integral closure in $R[x]$ is the graded ring $\bigoplus_{n\geq 0} \widetilde{I^n} x^n$.*

The proof of the last statement of the above theorem is handled by the next example, which provides a domain with an ideal I such that $\widetilde{I}$ is not a t-ideal and, hence, not t-integrally closed since $(\widetilde{I})_t \subseteq \widetilde{\widetilde{I}}$ always holds.

Example 2.10 ([50, Example 3.10]). Let $R := \mathbb{Z} + x\mathbb{Q}(\sqrt{2})[x]$, $I := (\frac{x}{\sqrt{2}})$, and $a := \frac{x}{2}$, where x is an indeterminate over $\mathbb{Q}$. Then:

(1) I is a t-reduction of $I + aR$ and $a \notin \widetilde{I}$.

(2) $\widetilde{I} \subsetneqq (\widetilde{I})_t$ and hence $\widetilde{I} \subsetneqq \widetilde{\widetilde{I}}$.

Proof. (1) First, we prove that $(I(I+aR))_t = ((I+aR)^2)_t$. It suffices to show that $a^2 \in (I(I+aR))_t$. For this purpose, let $f \in (I(I+aR))^{-1} = (\frac{x^2}{2}, \frac{x^2}{2\sqrt{2}})^{-1} \subseteq (\frac{x^2}{2})^{-1} = \frac{2}{x^2}R$. Then, $f = \frac{2}{x^2}(a_0 + a_1x + \ldots + a_nx^n)$, for some $n \geq 0$, $a_0 \in \mathbb{Z}$, and $a_i \in \mathbb{Q}(\sqrt{2})$ for $i \geq 1$. Since $\frac{x^2}{2\sqrt{2}} f \in R$, $a_0 = 0$. It follows that $(I(I+aR))^{-1} \subseteq \frac{1}{x}\mathbb{Q}(\sqrt{2})[x]$. On the other hand, $(I(I+aR))(\frac{1}{x}\mathbb{Q}(\sqrt{2})[x]) \subseteq R$. So, we have

$$\Big(I(I+aR)\Big)^{-1} = \left(\frac{x^2}{2}, \frac{x^2}{2\sqrt{2}}\right)^{-1} = \frac{1}{x}\mathbb{Q}(\sqrt{2})[x] \tag{1}$$

Now, clearly, $a^2(I(I+aR))^{-1} \subseteq R$. Therefore, $a^2 \in (I(I+aR))_v = (I(I+aR))_t$, as desired. Next, we prove that $a \notin \widetilde{I} = \overline{I}$. By [48, Corollary 1.2.2], it suffices to show that I is not a reduction of $I + aR$. Deny and suppose that $I(I+aR)^n = (I+aR)^{n+1}$, for some positive integer n. Then $a^{n+1} = (\frac{x}{2})^{n+1} \in I(I+aR)^n = \frac{x}{\sqrt{2}}(\frac{x}{\sqrt{2}}, \frac{x}{2})^n$. One can check that this yields $1 \in \sqrt{2}(\sqrt{2}, 1)^n \subseteq (\sqrt{2})$ in $\mathbb{Z}[\sqrt{2}]$, the desired contradiction.

(2) We claim that $a \in (\widetilde{I})_t$. Notice first that $x \in \widetilde{I}$ as $x^2 \in I^2 = (I^2)_t$. Therefore, $A := (x, \frac{x}{\sqrt{2}}) \subseteq \widetilde{I}$. Clearly, $A = \frac{2}{x}(\frac{x^2}{2}, \frac{x^2}{2\sqrt{2}})$. Hence, by (1), $A^{-1} = \mathbb{Q}(\sqrt{2})[x]$. However, $aA^{-1} \subseteq R$. Whence, $a \in A_v = A_t \subseteq (\widetilde{I})_t$. Consequently, $a \in (\widetilde{I})_t \setminus \widetilde{I}$.

The next result shows that the t-integral closure coincides with the t-closure in the class of integrally closed domains. It also completes two existing results in the literature on the integral closure of ideals (Gilmer [37] and Mimouni [57]).

Theorem 2.11. *Let R be a domain. The following assertions are equivalent:*

(1) *R is integrally closed;*
(2) *Every principal ideal of R is integrally closed;*
(3) *Every t-ideal of R is integrally closed;*
(4) $\overline{I} \subseteq I_t$ *for each nonzero ideal I of R;*
(5) *Every principal ideal of R is t-integrally closed;*
(6) *Every t-ideal of R is t-integrally closed;*
(7) $\widetilde{I} = I_t$ *for each nonzero ideal I of R.*

If all ideals of a domain are t-integrally closed, then it must be Prüfer. This is a well-known result in the literature:

Corollary 2.12 ([37, Theorem 24.7]). *A domain R is Prüfer if and only if every ideal of R is (t-)integrally closed.*

Now, we examine the correlation between the t-integral closure and t-reductions of ideals. In this vein, recall that, for the trivial operation, two crucial results assert that $x \in \overline{I} \Leftrightarrow$ *I is a reduction of* $I + Rx$ [48, Corollary 1.2.2] and *if I is finitely generated and* $J \subseteq I$, *then:* $I \subseteq \overline{J} \Leftrightarrow$ *J is a reduction of I* [48, Corollary 1.2.5]. Here are the t-analogues of these two results:

Proposition 2.13. *Let R be a domain and let $J \subseteq I$ be nonzero ideals of R.*

(1) *If $x \in \widetilde{I}$, then I is a t-reduction of $I + Rx$.*
(2) *If I is finitely generated with $I \subseteq \widetilde{J}$, then J is a t-reduction of I.*

Moreover, both implications are irreversible in general.

The next remark collects some basic properties of the t-integral closure.

Remark 2.14. Let R be a domain and let I, J be nonzero ideals of R. Then:

(1) $\forall\, x \in R$, $x\widetilde{I} \subseteq \widetilde{xI}$.
(2) $\widetilde{I \cap J} \subseteq \widetilde{I} \cap \widetilde{J}$. The inclusion can be strict, see Example 2.15(3).
(3) $I \subseteq \overline{I} \subseteq \widetilde{I} \subseteq \sqrt{I_t}$. These inclusions can be strict, see Example 2.15(1).
(4) $\forall\, n \geq 1$, $(\widetilde{I})^n \subseteq \widetilde{I^n}$. The inclusion can be strict, see Example 2.15(2).
(5) $\widetilde{I} + \widetilde{J} \subseteq \widetilde{I + J}$. The inclusion can be strict. For instance, in $\mathbb{Z}[x]$, we have $\widetilde{(2)} + \widetilde{(x)} = (2, x)$ and $\widetilde{(2, x)} = (2, x)_t = \mathbb{Z}[x]$ (via Theorem 2.11).

Example 2.15 ([50, Example 3.9]). Let $R := \mathbb{Z}[\sqrt{-3}][2x, x^2, x^3]$. Let $J := (x^3)$ and $I := (2x^2, 2x^3, x^4, x^5)$, where x is an indeterminate over $\mathbb{Z}$. Then I is a t-ideal such that

(1) $I \subsetneqq \overline{I} \subsetneqq \widetilde{I} \subsetneqq \sqrt{I}$.
(2) $(\widetilde{I})^2 \subsetneqq \widetilde{I^2}$.
(3) $\widetilde{J \cap I} \subsetneqq \widetilde{J} \cap \widetilde{I}$.

Proof. We first show that I is a t-ideal. Clearly, $\frac{1}{x^2}\mathbb{Z}[\sqrt{-3}][x] \subseteq I^{-1}$. For the reverse inclusion, let $f \in I^{-1} \subseteq x^{-4}R$. Then $f = \frac{1}{x^4}(a_0 + a_1x + \cdots + a_nx^n)$ for some $n \in \mathbb{N}$, $a_0 \in \mathbb{Z}[\sqrt{-3}]$, $a_1 \in 2\mathbb{Z}[\sqrt{-3}]$, and $a_i \in \mathbb{Z}[\sqrt{-3}]$ for $i \geq 2$. Since $2x^2f \in R$, then $a_0 = a_1 = 0$. It follows that $f \in \frac{1}{x^2}\mathbb{Z}[\sqrt{-3}][x]$. Therefore $I^{-1} = \frac{1}{x^2}\mathbb{Z}[\sqrt{-3}][x]$. Next, let $g \in (R : \mathbb{Z}[\sqrt{-3}][x]) \subseteq R$. Then $xg \in R$, forcing $g(0) \in 2\mathbb{Z}[\sqrt{-3}]$ and hence $g \in (2, 2x, x^2, x^3)$. So $(R : \mathbb{Z}[\sqrt{-3}][x]) \subseteq (2, 2x, x^2, x^3)$. The reverse inclusion is obvious. Thus, $(R : \mathbb{Z}[\sqrt{-3}][x]) = (2, 2x, x^2, x^3)$. Consequently, we obtain $I_t = I_v = (R : \frac{1}{x^2}\mathbb{Z}[\sqrt{-3}][x]) = x^2(R : \mathbb{Z}[\sqrt{-3}][x]) = I$.

(1) Next, we prove the strict inclusions $I \subsetneqq \overline{I} \subsetneqq \widetilde{I} \subsetneqq \sqrt{I}$. For $I \subsetneqq \overline{I}$, notice that $(1 + \sqrt{-3})x^2 \in \overline{I} \setminus I$ as $\left((1 + \sqrt{-3})x^2\right)^3 = -8x^6 \in I^3$ and $1 + \sqrt{-3} \notin 2\mathbb{Z}[\sqrt{-3}]$.

For $\overline{I} \subsetneqq \widetilde{I}$, we claim that $x^3 \in \widetilde{I} \setminus \overline{I}$. Indeed, let $f \in (I^2)^{-1} \subseteq x^{-8}R$. Then there are $n \in \mathbb{N}$, $a_i \in \mathbb{Z}[\sqrt{-3}]$ for $i \in \{0, 2, \ldots, n\}$, and $a_1 \in 2\mathbb{Z}[\sqrt{-3}]$ such that $f = \frac{1}{x^8}(a_0 + a_1x + \cdots + a_nx^n)$. Since $4x^4f \in R$, then $a_0 = a_1 = a_2 = a_3 = 0$. Therefore, $(I^2)^{-1} \subseteq \frac{1}{x^4}\mathbb{Z}[\sqrt{-3}][x]$. The reverse inclusion is obvious. Hence, $(I^2)^{-1} = \frac{1}{x^4}\mathbb{Z}[\sqrt{-3}][x]$. It follows that $(I^2)_t = (I^2)_v = (R : \frac{1}{x^4}\mathbb{Z}[\sqrt{-3}][x]) = x^4(R : \mathbb{Z}[\sqrt{-3}][x]) = x^2I$. Hence $x^6 \in (I^2)_t$ and thus $x^3 \in \widetilde{I}$. It remains to show that $x^3 \notin \overline{I}$. By [48, Corollary 1.2.2], it suffices to show that I is not a reduction of $I + (x^3)$. Let $n \in \mathbb{N}$. It is easy to see that x^4x^{3n} is the monic monomial with the smallest degree in $I\Big(I+(x^3)\Big)^n$. Therefore, $x^{3(n+1)} = x^{3n+3} \in \Big(I+(x^3)\Big)^{n+1} \setminus I\Big(I+(x^3)\Big)^n$. Hence, I is not a reduction of $I+(x^3)$, as desired.

For $\widetilde{I} \subsetneqq \sqrt{I}$, we claim that $x^2 \in \sqrt{I} \setminus \widetilde{I}$. Obviously, $x^2 \in \sqrt{I}$. In order to prove that $x^2 \notin \widetilde{I}$, it suffices by Proposition 2.13 to show that I is not a t-reduction of $I+(x^2)$. To this purpose, notice that $I+(x^2) = (x^2)$. Suppose by way of contradiction that $(I(I+(x^2))^n)_t = ((I+(x^2))^{n+1})_t$ for some $n \in \mathbb{N}$. Then $(x^2)^{n+1} = x^{2n+2} \in (I(I+(x^2))^n)_t = x^{2n}I$. Consequently, $x^2 \in I$, absurd.

(2) We first prove that $\widetilde{I} = (2x^2, (1+\sqrt{-3})x^2, x^3, x^4)$. In view of (1) and its proof, we have $(2x^2, (1+\sqrt{-3})x^2, x^3, x^4) \subseteq \widetilde{I}$. Next, let $\alpha := (a+b\sqrt{-3})x^2 \in \widetilde{I}$ where $a, b \in \mathbb{Z}$. If $b = 0$, then $a \neq 1$ as $x^2 \notin \widetilde{I}$. Moreover, since $2x^2 \in \widetilde{I}$, a must be even; that is, $\alpha \in (2x^2)$. Now assume $b \neq 0$. If $a = 0$, then $b \neq 1$ as $\sqrt{-3}x^2 \notin \widetilde{I}$. Moreover, since $2\sqrt{-3}x^2 \in \widetilde{I}$, b must be even; that is, $\alpha \in (2x^2)$. So suppose $a \neq 0$. Then similar arguments force a and b to be of the same parity. Further, if a and b are even, then $\alpha \in (2x^2)$; and if a and b are odd, then $\alpha \in (2x^2, (1+\sqrt{-3})x^2)$. Finally, we claim that $\widetilde{I}$ contains no monomials of degree 1. Deny and let $ax \in \widetilde{I}$, for some nonzero $a \in 2\mathbb{Z}[\sqrt{-3}]$. Then, by [48, Remark 1.1.3(7)], $ax \in \widetilde{I} \subseteq \widetilde{(x^2)} = \overline{(x^2)} \subseteq \overline{x^2\mathbb{Z}[\sqrt{-3}]}[x]$. By [48, Corollary 1.2.2], (x^2) is a reduction of (ax, x^2) in $\mathbb{Z}[\sqrt{-3}][x]$, absurd. Consequently, $\widetilde{I} = (2x^2, (1+\sqrt{-3})x^2, x^3, x^4)$. Now, we are ready to check that $(\widetilde{I})^2 \subsetneqq \widetilde{I^2}$. For this purpose, recall that $(I^2)_t = x^2 I$. So, $2x^4 \in \widetilde{I^2}$. We claim that $2x^4 \notin (\widetilde{I})^2$. Deny. Then, $2x^4 \in (4x^4, 2(1+\sqrt{-3})x^4)$. So $x^2 \in (2x^2, (1+\sqrt{-3})x^2) \subseteq \widetilde{I}$, absurd.

(3) We claim that $x^3 \in \widetilde{I} \cap \widetilde{J} \setminus \widetilde{I \cap J}$. We proved in (1) that $x^3 \in \widetilde{I}$. So, $x^3 \in \widetilde{I} \cap \widetilde{J}$. Now, observe that $I \cap J = xI$ and assume, by way of contradiction, that $x^3 \in \widetilde{I \cap J} = \widetilde{xI}$. Then x^3 satisfies an equation of the form $(x^3)^n + a_1(x^3)^{n-1} + \cdots + a_n = 0$ with $a_i \in ((xI)^i)_t = x^i(I^i)_t$, $i = 1, \ldots, n$. For each i, let $a_i = x^i b_i$, for some $b_i \in (I^i)_t$. Therefore $(x^2)^n + b_1(x^2)^{n-1} + \cdots + b_n = 0$. It follows that $x^2 \in \widetilde{I}$, the desired contradiction.

2.3 *Persistence and contraction of t-integral closure*

For any ring homomorphism, $\varphi : R \to T$, the persistence of integral closure describes the fact $\varphi(\overline{I}) \subseteq \overline{\varphi(I)T}$ for every ideal I of R; and the contraction of integral closure describes the fact $\overline{\varphi^{-1}(J)} = \varphi^{-1}(J)$ for every integrally closed ideal J of T. This section deals with the persistence and contraction of t-integral closure. For this purpose, we first need to introduce the concept of t-compatible homomorphism (which extends the well-known notion of t-compatible extension [13]). Throughout, t (resp., t_1) and v (resp., v_1) denote the t- and v- closures in R (resp., T).

Lemma 2.16. *Let $\varphi : R \longrightarrow T$ be a homomorphism of domains. Then, the following statements are equivalent:*

(1) $\varphi(I_v)T \subseteq \big(\varphi(I)T\big)_{v_1}$*, for each nonzero finitely generated ideal I of R;*
(2) $\varphi(I_t)T \subseteq \big(\varphi(I)T\big)_{t_1}$*, for each nonzero ideal I of R;*
(3) $\varphi^{-1}(J)$ *is a t-ideal of R for each t_1-ideal J of T such that $\varphi^{-1}(J) \neq 0$.*

Definition 2.17. A homomorphism of domains $\varphi : R \longrightarrow T$ is called t-compatible if it satisfies the equivalent conditions of Lemma 2.16.

Under the embedding $R \subseteq T$, this definition matches the notion of t-compatible extension (i.e., $I_tT \subseteq (IT)_{t_1}$) well studied in the literature (cf. [13, 18, 27, 31]). Next, the main result of this section establishes persistence and contraction of t-integral closure under t-compatible homomorphisms.

Proposition 2.18. *Let $\varphi : R \longrightarrow T$ be a t-compatible homomorphism of domains. Let I be an ideal of R and J an ideal of T. Then:*

(1) $\varphi(\widetilde{I})T \subseteq \widetilde{\varphi(I)T}$.
(2) $\widetilde{\varphi^{-1}(J)} \subseteq \varphi^{-1}(\widetilde{J})$*; and if J is t-integrally closed, then* $\widetilde{\varphi^{-1}(J)} = \varphi^{-1}(J)$.

If both R and T are integrally closed, then persistence of t-integral closure coincides with t-compatibility by Theorem 2.11. So the t-compatibility assumption in Proposition 2.18 is imperative.

Corollary 2.19. *Let $R \subseteq T$ be a t-compatible extension of domains and let I be an ideal of R. Then:*

(1) $\widetilde{I}T \subseteq \widetilde{IT}$.
(2) $\widetilde{I} \subseteq \widetilde{IT \cap R} \subseteq \widetilde{IT} \cap R$.

Moreover, the above inclusions are strict in general.

Corollary 2.20. *Let R be a domain, I an ideal of R, and S a multiplicatively closed subset of R. Then* $S^{-1}\widetilde{I} \subseteq \widetilde{S^{-1}I}$.

Recall that, for the integral closure, we have $S^{-1}\overline{I} = \overline{S^{-1}I}$ [48, Proposition 1.1.4], whereas in the above corollary the inclusion can be strict, as shown by the following example.

Example 2.21. We use a construction due to Zafrullah [65]. Let E be the ring of entire functions and x an indeterminate over E. Let S denote the set generated by the principal primes of E. Then, we claim that $R := E + xS^{-1}E[x]$ contains a prime ideal P such that $S^{-1}\widetilde{P} \subsetneq \widetilde{S^{-1}P}$. Indeed, R is a P-domain that is not a PvMD [65, Example 2.6]. By [66, Proposition 3.3], there exists a prime t-ideal P in R such that PR_P is not a t-ideal of R_P. By Theorem 2.11, we have $\widetilde{P}R_P = PR_P \subsetneq R_P = (PR_P)_t = \widetilde{PR_P}$ since R is integrally closed. Also notice that $P = \widetilde{PR_P \cap R} \subsetneq \widetilde{PR_P} \cap R = R$.

Corollary 2.22. *Let R be a domain and I a t-ideal that is t-locally t-integrally closed (i.e., I_M is t-integrally closed in R_M for every maximal t-ideal M of R). Then I is t-integrally closed.*

3 The case of Prüfer v-multiplication domains

In [38, 39], Hays investigated reductions of ideals in commutative rings with a particular focus on Prüfer domains. He studied the notion of basic ideal and examined domains subject to the basic ideal property. He showed that this class of domains is strictly contained in the class of Prüfer domains; namely, a domain is Prüfer if and only if it has the finite basic ideal property [38, Theorem 6.5]. The second main result of these two papers characterizes domains with the basic ideal property as one-dimensional Prüfer domains ([38, Theorem 6.1] and [39, Theorem 10]).

This part covers [44] which deals with the extension of Hays' aforementioned results on Prüfer domains to Prüfer v-multiplication domains (PvMDs). In Section 3.1 we first extend the definition of t-reduction to $\star$-reduction, for any arbitrary $\star$-operation, and then discuss the notion of $\star$-basic ideals and prove that a domain with the finite $\star$-basic ideal property (resp., $\star$-basic ideal property) must be integrally closed (resp., completely integrally closed). We also observe that a domain has the v-basic ideal property if and only if it is completely integrally closed. Section 3.2 is devoted to generalizing Hays' results; we show that a domain has the finite w-basic ideal property (resp., w-basic ideal property) if and only if it is a PvMD (resp., PvMD of t-dimension one). In Section 3.3, we present a diagram of implications among domains having various $\star$-basic properties and provide examples showing that most of the implications are not reversible.

3.1 $\star$-basic ideals

Let R be a domain with quotient field K and let $F(R)$ denote the set of nonzero fractional ideals of R. A map $\star : F(R) \to F(R), I \mapsto I^\star$, is called a *star operation on R* if the following conditions hold for every $0 \neq a \in K$ and $I, J \in F(R)$:

- $R^\star = R$ and $(aI)^\star = aI^\star$,
- $I \subseteq J \Rightarrow I^\star \subseteq J^\star$,
- $I \subseteq I^\star$ and $I^{\star\star} = I^\star$.

The next definition extends the notion of t-reduction and related concepts to an arbitrary star operation $\star$ on R.

Definition 3.1. Let $J \subseteq I$ be nonzero [fractional] ideals of R.

- J is a *trivial $\star$-reduction* of I if $J^\star = I^\star$.
- J is a *$\star$-reduction* of I if $(JI^n)^\star = (I^{n+1})^\star$ for some integer $n \geq 0$.
- I is *$\star$-basic* if it has no $\star$-reduction other than the trivial $\star$-reductions.
- R has the $\star$-basic (resp., finite $\star$-basic) ideal property if every nonzero (resp., finitely generated) [fractional] ideal of R is $\star$-basic.

If $\star_1$ and $\star_2$ are two star operations on R with $I^{\star_1} \subseteq I^{\star_2}$ for each ideal I, then any $\star_1$ reduction is also a $\star_2$-reduction, and the converse is not true in general; since a t-reduction may not be a reduction (see also Example 2.2).

The next results provide elementary properties and natural examples of $\star$-basic ideals and domains with the (finite) $\star$-basic ideal property.

Lemma 3.2. *$\star$-invertible ideals and $\star$-idempotent ideals are $\star$-basic.*

Recall that R is *completely integrally closed* (resp., a *v-domain*) if every nonzero ideal (resp., finitely generated ideal) of R is v-invertible.

Proposition 3.3. *The following assertions always hold:*

(1) *If R has the finite $\star$-basic ideal property, then R is integrally closed.*
(2) *If R has the $\star$-basic ideal property, then R is completely integrally closed.*
(3) *R has the v-basic ideal property if and only if R is completely integrally closed.*
(4) *If R is a v-domain, then R has the finite v-basic ideal property.*

The next example features a Noetherian domain with two t-ideals I, J such that J is a t-reduction, but not a reduction, of I. Since the v- and t-operations coincide under Noetherianess, such domain is not (completely) integrally closed by Proposition 3.3.

Example 3.4. Let k be a field, x, y two indeterminates over k, and $T := k[x, y]$. Consider the Noetherian domain $R = k + M^2$, where $M := (x, y)T$ (cf. [22]). As an ideal of T, M is basic [38, Theorem 2.3]. In particular, M^2 is not a reduction of M in T, and hence it is not a reduction of M as a fractional ideal of R. However, M^2 is a nontrivial t-reduction of M in R. Indeed, we have $(T : M) = T$. It follows that $M \subseteq M^{-1}$ $(= (R : M)) \subseteq T$. On the other hand, if $f \in T$ satisfies $fM \subseteq R$, then, writing $f = a + m$ with $a \in k$ and $m \in M$, we immediately obtain that $aM \subseteq R$, whence $a = 0$, i.e., $f \in M$. Thus $M^{-1} = M$, whence also $M_t = M_v = M$. However, $(R : T) = M^2$, whence $(M^2)^{-1} = ((R : M) : M) = (M : M) = T$ and then $(M^2)_t = (M^2)_v = (R : T) = M^2$, where the t- and v-operations are taken in R. A similar argument yields $(M^n)_t = M^2$ for $n \geq 2$. Hence $M^2 = (M^3)_t = (M^2M)_t$. Consequently, $J := xM^2 \subseteq I := xM$ are two (integral) t-ideals of R, where J is a non-trivial t-reduction, but not a reduction, of I.

Recall that, to the star operation $\star$, we may define an associated star operation $\star_f$ by setting, for each $I \in F(R)$, $I^{\star_f} = \bigcup J^\star$, where J ranges over all finitely generated subideals of I; and then $\star$ is of *finite type* if $\star = \star_f$. In this case, minimal primes of $\star$-ideals are necessarily $\star$-ideals and each $\star$-ideal is contained in a maximal $\star$-ideal. For instance, $v_f = t$ and $t_f = t$.

Lemma 3.5. *Assume that $\star$ is of finite type. If I is a finitely generated ideal of R and J is a $\star$-reduction of I, then there is a finitely generated ideal $K \subseteq J$ such that K is a $\star$-reduction of I.*

This lemma allows to prove the following result.

Proposition 3.6. *If R has the finite $\star$-basic ideal property, then R also has the finite $\star_f$-basic ideal property. In particular, if R has the finite v-basic ideal property, then R also has the finite t-basic ideal property.*

Corollary 3.7. *A v-domain has the finite t-basic ideal property.*

3.2 Characterizations

At this point, we recall Kang's result [55, Theorem 3.5] that a PvMD is an integrally closed domain in which the t- and w-operations coincide. The next theorem features an analogue of Hays' first result that "*a domain is Prüfer if and only if it has the finite basic ideal property*" [38, Theorem 6.5].

Theorem 3.8. *A domain is a PvMD if and only if it has the finite w-basic ideal property.*

Hays proved that, in a Prüfer domain, the definition of a reduction can be restricted; namely, $J \subseteq I$ is a reduction if and only if $JI = I^2$ [39, Proposition 1]. The next lemma establishes a similar property for t-reductions and shows that this notion is local in the class of PvMDs.

Lemma 3.9. *Let R be a PvMD and $J \subseteq I$ nonzero ideals of R. Then, the following assertions are equivalent:*

(1) $(JI)_t = (I^2)_t$;
(2) *J is a t-reduction of I;*
(3) $JR_M IR_M = (IR_M)^2$ *for each maximal t-ideal M of R.*

It is useful to note if J is a t-reduction of an ideal I, then a prime t-ideal of R contains I if and only if it contains J. We also recall that if I is a nonzero ideal of a domain R and S is a multiplicatively closed subset of R, then $(I_t R_S)_{t_{R_S}} = (IR_S)_{t_{R_S}}$ (this fact follows from [64, Lemma 4] and is stated explicitly in [55, Lemma 3.4]).

Lemma 3.10. *Let R be a PvMD and let $0 \neq x \in R$. Let P be a minimal prime of xR and set $I := xR_P \cap R$. Then*

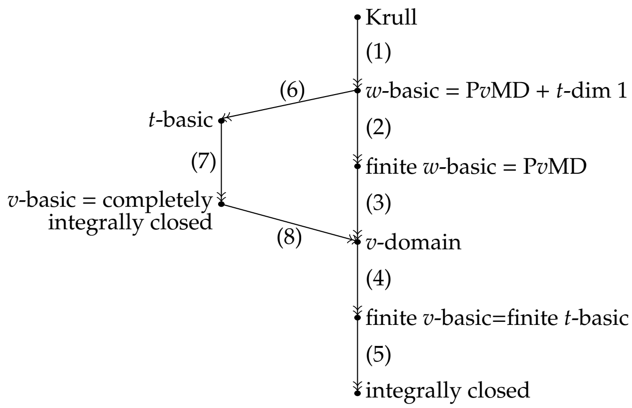
(1) *I is a w-ideal of R.*
(2) *$xR + I^2$ is a w-reduction of I.*
(3) *If I is w-basic, then P is a maximal t-ideal of R.*

The above two lemmas allowed us to prove the next theorem, which features an analogue of Hays' second result that "*a domain has the basic ideal property if and only if it is a Prüfer domain of dimension 1*" [39, Theorem 10].

Theorem 3.11. *A domain has the w-basic ideal property if and only if it is a PvMD of t-dimension 1.*

3.3 Examples

Consider the following diagram of implications putting in perspective the (finite) v-, t-, and w-basic ideal properties.



Notice that the implications (1)-(3) and (8) are well known, and (4)-(7) follow from Proposition 3.3, Proposition 3.6, Theorem 3.11, and the fact that the w- and t-operations coincide in a PvMD. Also, it is well-known that (1)-(3) and (8) are irreversible in general. Moreover, the finite v-basic ideal property obviously implies the finite t-basic ideal property, and in Section 5.2 we will see that in fact they are equivalent (Theorem 5.5).

Next, we provide examples with full details, from [44], proving that the remaining implications in the diagram are, too, irreversible in general.

Example 3.12 ([44, Example 3.1]). Implication (4) is irreversible.

Proof. Let k be a field and X, Y, Z indeterminates over k. Let $T := k((X)) + M$ and $R := k[[X]] + M$, where $M := (Y, Z)k((X))[[Y, Z]]$. Let A be an ideal of R. Then A is comparable to M. Suppose $A \subseteq M$ and A is not invertible. If $AA^{-1} \supsetneq M$, then AA^{-1} is principal, and hence A is invertible, contrary to assumption. Hence $AA^{-1} \subseteq M$. We claim that $(AA^{-1})_v = M$. To verify this, first recall that M is divisorial in R. Then, since AA^{-1} is a trace ideal, that is,

$(AA^{-1})^{-1} = (AA^{-1} : AA^{-1})$, we have $(AA^{-1})^{-1} \subseteq (AA^{-1}T : AA^{-1}T) = T = M^{-1}$ (the first equality holding since T is Noetherian and integrally closed). This forces $(AA^{-1})^{-1} = M^{-1}$, whence $(AA^{-1})_v = M_v = M$, as claimed. Now let I be a finitely generated ideal of R and J a v-reduction of I, so that $(JI^n)_v = (I^{n+1})_v$ for some positive integer n. We shall show that $J^{-1} = I^{-1}$ (and hence that $J_v = I_v$), and for this we may assume that I is not invertible. Suppose, by way of contradiction, that $IT(T : IT) = T$, i.e., that IT is invertible in T. Then, since T is local, IT is principal and, in fact, $IT = aT$ for some $a \in I$. We then have $R \subseteq a^{-1}I \subseteq T$. Then $k[[X]] \cong R/M \subseteq a^{-1}I/M \subseteq T/M \cong k((X))$, from which it follows that $a^{-1}I/M$ must be a cyclic $k[[X]]$-module. However, this is easily seen to imply that $a^{-1}I$, hence I, is principal, the desired contradiction. We therefore have $(T : IT)I \subseteq M$, whence $(IM)^{-1} = (R : IM) = ((R : M) : I) = (T : I) = (M : I) \subseteq I^{-1}$. This immediately yields $I^{-1} = (IM)^{-1}$.

Now set $Q = I^n(I^n)^{-1}$. From above (setting $A = I^n$), we have $Q_v = M$. Therefore, $I^{-1} \subseteq J^{-1} \subseteq (JM)^{-1} = (JQ)^{-1} = (IQ)^{-1} = (IM)^{-1} = I^{-1}$, which yields $J^{-1} = I^{-1}$, as desired. Hence R has the finite v-basic property. Finally, again from above, we have $((y,z)(y,z)^{-1})_v = M$, so that R is not a v-domain.

Example 3.13 ([44, Example 3.2]). Implication (5) is irreversible.

Proof. Let k be a field and X, Y indeterminates over k. Let $V = k(X)[[Y]]$ and $R = k + M$, where $M = Yk(X)[[Y]]$. Clearly, R is an integrally closed domain. Of course, M is divisorial in R. Also, $(M^2)^{-1} = ((R : M) : M) = (V : M) = Y^{-1}V$, and so $(M^2)_v = (R : Y^{-1}V) = Y(R : V) = YM = M^2$, i.e., M^2 is also divisorial. We claim that R does not have the finite t-basic ideal property. Indeed, let $W := k + Xk$ and consider the finitely generated ideal I of R given by $I = Y(W + M)$. We have $(k : W) = (0)$; otherwise, we have $0 \neq f \in (k : W)$, and both f and $fX \in k$, whence $X \in k$, a contradiction. Therefore, $I^{-1} = Y^{-1}M$ and thus $I_t = I_v = YM^{-1} = M$. Now, let $J = YR$. Then $J_t = YR \subsetneq M = I_t$. However, $(JI)_t = (YI)_t = YI_t = YM = M^2 = ((I_t)^2)_t = (I^2)_t$, and so R does not have the finite t-basic ideal property.

Example 3.14 ([44, Example 3.3]). Implication (6) is irreversible.

Proof. In [42] Heinzer and Ohm give an example of an essential domain that is not a PvMD. In that example, k is a field, y, z, and $\{x_i\}_{i=1}^{\infty}$ are indeterminates over k, and $D = R \cap (\bigcap_{i=1}^{\infty} V_i)$, where $R = k(\{x_i\})[y,z]_{(y,z)k(\{x_i\})[y,z]}$ and V_i is the rank-one discrete valuation ring on $k(\{x_j\}_{j=1}^{\infty}, y, z)$ with x_i, y, z all having value 1 and x_j having value 0 for $j \neq i$ (using the "infimum" valuation). As further described in [58, Example 2.1], we have $\mathrm{Max}(D) = \{M\} \cup \{P_i\}$, where M is the contraction of $(y,z)R$ to D and the P_i are the centers of the maximal ideals of the V_i; moreover, $D_M = R$ and $V_i = D_{P_i}$.

It was pointed out in [35, Example 1.7] that each finitely generated ideal of D is contained in almost all of the V_i. If fact, one can say more. Let a be an element of D. We may represent a as a quotient f/g with $f, g \in T := k[\{x_i\}, y, z]_{(y,z)k[\{x_i\},y,z]}$ and $g \notin (y,z)T$ (and hence $g \notin M$). Since f and g involve only finitely many x_j and $g \notin M$, the sequence $\{v_i(a)\}$ must be eventually constant, where v_i is the valuation corresponding to V_i. We denote this constant value by $w(a)$. A similar statement holds for finitely generated ideals of D.

Let K be a nonzero ideal of D. Then $K_tD_{P_i} \supseteq KD_{P_i} = (KD_{P_i})_{t_{D_{P_i}}} = (K_tD_{P_i})_{t_{D_{P_i}}} \supseteq K_tD_{P_i}$, whence $K_tD_{P_i} = KD_{P_i}$. Now suppose that we have nonzero ideals $J \subseteq I$ of D with $(JI^n)_t = (I^{n+1})_t$. Let $a \in I$, and choose $a_0 \in I$ so that $w(a_0)$ is minimal. Then $aa_0^n \in I^{n+1} \subseteq (JI^n)_t$, and so $aa_0^n \in (BA^n)_v$ for finitely generated ideals $B \subseteq J$ and $A \subseteq I$. With the observation in the preceding paragraph, we then have $aa_0^n \in BA^nD_{P_i}$ for each i. However, since $w(a_0) \leq w(A)$, it must be the case that $w(a) \geq w(B)$; i.e., for some integer k, $a \in BD_{P_i}$ for all $i > k$. Since the equality $(JI^n)_t = (I^{n+1})_t$ yields $JD_{P_i} = ID_{P_i}$ for each i, we may choose elements $b_j \in J$ for which $v_j(a) = v_j(b_j)$, $j = 1, \dots, k$. With $B' = (B, b_1, \dots, b_k)$, we then have $a \in B'D_{P_i}$ for each i. This yields $a(B')^{-1} \subseteq \bigcap D_{P_i}$.

Next, we consider extensions to D_M. From $(JI^n)_t = (I^{n+1})_t$, we obtain $(JI^nD_M)_{t_{D_M}} = (I^{n+1}D_M)_{t_{D_M}}$. Since D_M is a regular local ring, each nonzero ideal of D_M is t-invertible, and we may cancel to obtain $(ID_M)_{t_{D_M}} = (JD_M)_{t_{D_M}}$. There is a finitely generated subideal B_1 of J with $B_1D_M = JD_M$. We then have $IB_1^{-1} \subseteq ID_MB_1^{-1}D_M = ID_M(B_1D_M)^{-1} \subseteq (JD_M(JD_M)^{-1})_{t_{D_M}} \subseteq D_M$. Now let $B_2 = B' + B_1$. Then $a(B_2)^{-1} \subseteq D_M \cap \bigcap D_{P_i} = D$, whence $a \in (B_2)_v \subseteq J_t$. It follows that D has the t-basic property. However, since D is not a PvMD, D cannot have the (finite) w-basic property.

Example 3.15 ([44, Example 3.4]). Implication (7) is irreversible. For instance, the ring of entire functions is a completely integrally closed Prüfer domain with infinite Krull dimension, and hence it does not have the (t-) basic ideal property by [39, Theorem 10].

4 The case of Noetherian domains

This part covers [52], which studies t-reductions and t-integral closure of ideals in Noetherian domains. The main objective is to establish t-analogues for well-known results on reductions and integral closure of ideals in Noetherian rings. Section 4.1 investigates t-reductions of ideals subject to t-invertibility and localization in arbitrary Noetherian domains. Section 4.2 investigates the t-integral closure of ideals and its correlation with t-reductions in Noetherian domains of Krull dimension one.

4.1 *t-reductions subject to t-invertibility and localization*

This section deals with t-reductions of ideals subject to t-invertibility and localization in Noetherian domains. The first main result establishes a t-analogue for Hays' result on the correlation between invertible reductions and the Krull dimension [38, Theorem 4.4]; and the second main result establishes a t-analogue for Hays' global-local result on the basic ideal property [38, Theorem 3.6]. In 1973, Hays proved the following result:

Theorem 4.1 ([38, Theorem 4.4]). *Let R be a Noetherian domain such that R/M is infinite for every maximal ideal M of R. Then, each nonzero ideal has an invertible reduction if and only if* $\dim(R) \leq 1$.

The t-dimension of a domain R, denoted t-$\dim(R)$, is the supremum of the lengths of chains of prime t-ideals in R (here (0) is considered as a prime t-ideal although technically it is not); and the inequality t-$\dim(R) \leq \dim(R)$ always holds [43]. Here is a t-analogue of the above result.

Theorem 4.2. *Let R be a Noetherian domain such that the residue field of each maximal t-ideal is infinite. Then, the following statements are equivalent:*

(1) *Each t-ideal of R has a t-invertible t-reduction;*
(2) *Each maximal t-ideal of R has a t-invertible t-reduction;*
(3) t-$\dim(R) \leq 1$.

The next lemma handles the implication (2) $\Rightarrow$ (3) without the infinite residue field assumption.

Lemma 4.3. *Let R be a Noetherian domain. If every maximal t-ideal of R has a t-invertible t-reduction, then* t-$\dim(R) \leq 1$.

Observe that, in general, the converse of Lemma 4.3 is not true. For instance, colnsider an almost Dedekind domain R which is not Dedekind. Then R is a one-dimensional locally Noetherian Prüfer domain. Hence R has the basic ideal property [38, Theorem 6.1]. Since R is not Dedekind, it has a non-invertible maximal ideal which has no proper reduction.

Next, we move to the global-local transfer of the t-basic ideal property. For this purpose, recall that an ideal I is locally basic (resp., t-locally t-basic) if IR_M is basic (resp., t-basic) for each maximal ideal (resp., maximal t-ideal) M of R containing I. In 1973, Hays proved the following result:

Theorem 4.4 ([38, Theorem 3.6]). *In a Noetherian ring, an ideal is basic if and only if it is locally basic.*

Here is a t-analogue for the "if" assertion of this result.

Theorem 4.5. *In a Noetherian domain, if an ideal is t-locally t-basic, then it is t-basic.*

Now, note that, in his proof of the implication "basic $\Rightarrow$ locally basic," Hays used two basic facts. The first one states that, in an arbitrary ring R, if $J \subseteq I$ and JR_M is a reduction of IR_M, then $(J \cap I) + IM$ is a reduction of I; and here is a t-analogue for this result.

Proposition 4.6. *Let R be a domain, M a maximal t-ideal of R, and $I \subseteq M$ a nonzero ideal of R. If J is an ideal of R such that JR_M is a t-reduction of IR_M, then $(J \cap I) + IM$ is a t-reduction of I.*

However, the second fact was Nakayama's lemma, which ensures that $J \subseteq I \subseteq J + IM$ in a local Noetherian ring (R, M) forces $J = I$; and a t-analogue for this Nakayama property is not true in general. For example, consider the local Noetherian ring $R := k+M^2 \subseteq k[x, y]$, where $M = (x, y)$ and $(M^2)_t = (M^3)_t$ [44, Example 1.5].

4.2 *t-reductions and t-integral closure in one-dimensional Noetherian domains*

This section deals with the t-integral closure of ideals and its correlation with t-reductions in Noetherian domains of Krull dimension one. The objective is to establish t-analogues of well-known results, in the literature, on the integral closure of ideals and its correlation with reductions of ideals in Noetherian settings.

Recall from Section 2.2 that "*$\widetilde{I}$ is an integrally closed ideal which is not t-integrally closed in general*." Several ideal-theoretic properties of $\widetilde{I}$ are collected in Remark 2.14, including the inclusions $I \subseteq \overline{I} \subseteq \widetilde{I} \subseteq \sqrt{I_t}$. Consider the two sets related to the (trivial) d-operation and t-operation, respectively:

$$\widehat{I}^d := \left\{x \in R \mid I \text{ is a reduction of } (I, x)\right\}$$

$$\widehat{I}^t := \left\{x \in R \mid I \text{ is a } t\text{-reduction of } (I, x)\right\}$$

For the trivial operation, it is well-known that the equality $\overline{I} = \widehat{I}^d$ always holds [48, Corollary 1.2.2]. This fact which was used to show that $\overline{I}$ is an ideal [48, Corollary 1.3.1]. However, it is still an open problem of whether $\widehat{I}^t$ is an ideal [51, Question 3.5]. We always have $I_t \subseteq \widetilde{I} \subseteq \widehat{I}^t$ where the second containment is proved by [50, Proposition 3.7] and can be strict as shown by [50, Example 3.10(a)]. Moreover, "*$I_t = \widetilde{I}$ for each nonzero ideal I if and only if R is integrally closed*" [50, Theorem 3.5], and "*$I_t = \widehat{I}^t$ for each nonzero ideal I if and only if R has the finite t-basic ideal property*" [51, Theorem 3.2].

The class of Prüfer domains is the only known class of domains, where the two notions of reduction and t-reduction coincide (since the trivial and

t- operations are the same). The next result shows that the same happens in one-dimensional Noetherian domains (where the trivial and t- operations are not necessarily the same).

Theorem 4.7. *In a one-dimensional Noetherian domain, the notions of reduction and t-reduction coincide. Moreover,* $\overline{I} = \widetilde{I} = \widehat{I}^{t}$ *for any nonzero ideal I.*

As illustrative examples, consider one-dimensional Noetherian domains which are not divisorial (i.e., t-operation is not trivial), as shown below.

Example 4.8. Let $R := \mathbb{Q} + x\mathbb{Q}(\sqrt{2}, \sqrt{3})[[x]]$, where $\mathbb{Q}$ is the field of rational numbers and x is an indeterminate over $\mathbb{Q}$. Then, R is a pseudo-valuation domain (see definition in Section 5.1) issued from the DVR $\mathbb{Q}(\sqrt{2}, \sqrt{3})[[x]]$ and hence is a one-dimensional Noetherian domain. Further, R is not a divisorial domain by [40, Theorem 3.5] or [45, Theorem 2.4] since $[V/M : R/M] \neq 2$.

One wonders whether there exist Noetherian domains of dimension > 1 where the notions of reduction and t-reduction coincide. Next, we show this cannot happen in a large class of Noetherian domains.

Proposition 4.9. *Let R be a Noetherian domain with* $(R : \overline{R}) \neq 0$*. Then, the notions of reduction and t-reduction coincide in R if and only if* $\dim(R) = 1$.

5 The case of pullbacks

This part covers [51], which investigates t-reductions of ideals in pullback constructions (defined in Section 5.3). Section 5.1 examines the correlation between the notions of reduction and t-reduction in pseudo-valuation domains. Section 5.2 solves an open problem raised in [44] on whether the finite t-basic and v-basic ideal properties are distinct. In fact, these two notions coincide in any arbitrary domain (Theorem 5.5). Section 5.3 features the main result, which establishes the transfer of the finite t-basic ideal property to pullbacks in line with Fontana-Gabelli's result on PvMDs [31, Theorem 4.1] and Gabelli-Houston's result on v-domains [34, Theorem 4.15]. This allows us to enrich the literature with new examples, putting the class of domains subject to the finite t-basic ideal property strictly between the two classes of v-domains and integrally closed domains.

5.1 t-Reductions in pseudo-valuation domains

Recall that a pseudo-valuation domain (PVD) R is a special pullback issued from the following diagram

$$\begin{array}{ccc} R = \varphi^{-1}(k) & \longrightarrow & k \\ \downarrow & & \downarrow \\ V & \xrightarrow{\varphi} & K := V/M \end{array}$$

where (V, M) is a valuation domain with residue field K and k is a subfield of K. We say that R is a PVD issued from (V, M, k). For more details on pseudo-valuation domains, see [40, 41] and also [17, 19, 24, 25, 61].

Note that a reduction is necessarily a t-reduction; and the converse is not true in general. The next result investigates necessary and sufficient conditions for the notions of reduction and t-reduction to coincide in PVDs. This result can be used readily to provide examples discriminating between the two notions of reduction and t-reduction.

Theorem 5.1. *Let R be a PVD issued from (V, M, k) with $K := V/M$. Then, the following statements are equivalent:*

(1) *For every nonzero ideals $J \subseteq I$, J is a t-reduction of $I \Leftrightarrow J$ is a reduction of I.*
(2) *For each k-vector subspace W of K containing k, W^n is a field for some $n \geq 0$.*

Note that Condition (2) in the above result forces K to be algebraic over k, and so this fact can be used to build examples where the two notions of reduction and t-reduction are the same or distinct, as shown below.

Example 5.2 ([51, Example 2.3]). Let R be a PVD issued from (V, M, k) with $K := V/M$.

(1) Assume K is a transcendental over k. Then, the notions of reduction and t-reduction are distinct in R. For example, pick a transcendental element $\lambda \in K$ over k and let $W := k + k\lambda$, $I := a\varphi^{-1}(W)$ and $J =: aR$. Then, J is a proper t-reduction of I, whereas I is basic in R.
(2) Assume $[K : k]$ is finite. Then for every k-submodule W of K with $k \subseteq W \subseteq K$, some power of W is a field, and hence the notions of reduction and t-reduction coincide in R.

Given nonzero ideals $J \subseteq I$, if J_t is a reduction of I_t, then J is a t-reduction of I. The converse is not true in general as shown by Example 2.2. The next result provides a class of (integrally closed) pullbacks where the two assumptions are always equivalent.

Proposition 5.3. *Let R be a PVD and let $J \subseteq I$ be nonzero ideals of R. Then, J is a t-reduction of I if and only if J_t is a reduction of I_t.*

The class of Prüfer domains is, so far, the only known class of domains where the two notions of reduction and t-reduction coincide. We close this section with the next result, which features necessary conditions for such a coincidence. For this purpose, recall that a domain where the trivial and w-operations are the same is said to be a DW-domain [36, 47, 57]. Common examples of DW-domains are pseudo-valuation domains, Prüfer domains, and quasi-Prüfer domains (i.e., domains with Prüfer integral closure) [32, Page 190].

Proposition 5.4. *Let R be a domain where the notions of reduction and t-reduction coincide for all ideals of R. Then:*

(1) *Every nonzero prime ideal of R is a t-ideal. In particular, R is a DW-domain.*
(2) *R is integrally closed if and only if R has the finite t-basic ideal property.*
(3) *R is a PvMD if and only if R is a Prüfer domain.*

5.2 Equivalence of the finite t- and v-basic ideal properties

A domain is called a v-domain if all its nonzero finitely generated ideals are v-invertible; a comprehensive reference for v-domains is Fontana & Zafrullah's survey paper [33]. Also, recall the finite v-basic ideal property obviously implies the finite t-basic ideal property, and the question of whether this implication is reversible was left open in [44, Section 3]. The main result of this section (Theorem 5.5) solves this open question. For this purpose, recall from Section 4.2 the following objects $\widetilde{I} := \left\{x \in R \mid x \text{ is } t\text{-integral over } I\right\}$ and $\widehat{I}^{t} := \left\{x \in R \mid I \text{ is a } t\text{-reduction of } (I, x)\right\}$ along with the basic inclusions $I_t \subseteq \widetilde{I} \subseteq \widehat{I}^{t}$. Finally, in order to put the main result into perspective, recall the important result that "*a domain R is integrally closed if and only if $I_t = \widetilde{I}$ for each nonzero (finitely generated) ideal I of R*" (Theorem 2.11).

Here is the main result of this section.

Theorem 5.5. *For a domain R, the following assertions are equivalent:*

(1) *$I_t = \widehat{I}$ for each nonzero (finitely generated) ideal I of R;*
(2) *R has the finite t-basic ideal property;*
(3) *R has the finite v-basic ideal property.*

The proof of this result required the following two elementary lemmas.

Lemma 5.6 (cf. Lemma 3.5). *Let R be a domain and let I be a finitely generated ideal of R. If $J \subseteq I$ is a t-reduction of I, then there exists a finitely generated ideal $K \subseteq J$ such that K is a t-reduction of I.*

Lemma 5.7. *For a domain R, let $K \subseteq J \subseteq I$ and $J' \subseteq I'$ be nonzero fractional ideals, and let n and k be positive integers.*

(1) *If J and J' are $\star$-reductions of I and I', respectively, then $J + J'$ and JJ' are $\star$-reductions of $I + I'$ and II', respectively.*
(2) *Assume K is a $\star$-reduction of J. If J is a $\star$-reduction of I, then so is K.*
(3) *If K is a $\star$-reduction of I, then J is a $\star$-reduction of I.*
(4) *J is a $\star$-reduction of I if and only if J^n is a $\star$-reduction of I^n.*
(5) *$J = (a_1, ..., a_k)$ is a $\star$-reduction of $I \Leftrightarrow (a_1^n, ..., a_k^n)$ is a $\star$-reduction of I^n.*

New examples of domains subject to the finite t-basic (equiv., v-basic) ideal property will be provided in the next section. We close this section with the following open question:

Question 5.8 ([51, Question 3.5]). Let I be a nonzero ideal, is $\widehat{I}$ always an ideal?

5.3 Transfer of the finite t-basic ideal property to pullbacks

Throughout, R will be the pullback issued from the following diagram of canonical homomorphisms:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & K = T/M. \end{array}$$

where T is a domain, M is a maximal ideal of T with residue field K, $\varphi : T \longrightarrow K$ is the canonical surjection, and D is a proper subring of K with quotient field k. So, $R := \varphi^{-1}(D) \subsetneq T$. First, note that Proposition 3.3 ensures that a domain with the t-basic ideal property is necessarily completely integrally closed, and so, by [37, Lemma 26.5], R never has the t-basic ideal property. This section investigates conditions for R to inherit the finite t-basic (or, equivalently, v-basic) ideal property when T is local.

Recall from Sections 3.3 and 5.2 that the finite t-basic ideal property lies between the two notions of v-domain and integrally closed domain; and that the finite w-basic ideal property coincides with the PvMD notion. Also, at this point, it is worthwhile recalling Fontana & Gabelli's [31] and Gabelli & Houston's [34] well-known results, which establish the transfer of the notions of PvMD and v-domain to pullbacks, respectively, and which summarize as follows:

Theorem 5.9 ([31, Theorem 4.1] & [34, Theorem 4.15]). *R is a PvMD (resp., v-domain) if and only if T and D are PvMDs (resp., v-domains), T_M is a valuation domain, and $k = K$.*

Here is the main result of this section.

Theorem 5.10. *Assume that T is local. Then, R has the finite t-basic ideal property if and only if T and D have the finite t-basic ideal property and $k = K$.*

This result enables the construction of new examples, which put the finite t-basic ideal property strictly between the two notions of integrally closed domain and v-domain. Follow some examples with full details from [51].

Example 5.11 ([51, Example 4.3]). Consider any non-trivial pseudo-valuation domain R issued from (V, M, k) with k algebraically closed in $K := V/M$. Then,

R is an integrally closed domain by [20, Theorem 2.1], which does not have the finite t-basic ideal property by Theorem 5.10. Moreover, the two notions of reduction and t-reduction are distinct in R by Proposition 5.4.

Example 5.12 ([51, Example 4.4]). Consider a pullback R issued from (T, M, D), where T is a non-valuation local v-domain and D is a v-domain with quotient field T/M. Then, R has the finite t-basic ideal property by [44, Proposition 1.6] and Theorem 5.5 and Theorem 5.10, which is not a v-domain by [34, Theorem 4.15]. One can easily build non-valuation local v-domains via pullbacks through [34, Theorem 4.15].

Example 5.13 ([51, Example 4.5]). Let $T := \mathbb{Q}(X)[[Y, Z]] = \mathbb{Q}(X) + M$ and $R := \mathbb{Z}[X] + M$. Clearly, T and $D := \mathbb{Z}[X]$ have the finite t-basic property (since they are Noetherian Krull domains). By Theorem 5.10, R has the finite t-basic property. Also R is not a v-domain since T is a non-valuation local domain. Next, let $0 \neq a \in \mathbb{Z}$ and consider the finitely generated ideal of R given by $I := (a, X)\mathbb{Z}[X] + M = aR + XR$. Clearly $I^{-1} = R$ and so $(I^s)^{-1} = R$, for every positive integer s. In particular, we have $(I^2I)_t = (I^3)_t = (I^3)_v = R = (I^2)_v = (I^2)_t$ and hence I^2 is a t-reduction of I. However, I^2 is not a reduction of I; otherwise, if $I^{n+2} = I^2I^n = I^{n+1}$, for some $n \geq 1$, this would contradict [56, Theorem 76]. It follows that the notions of reduction and t-reduction are distinct in R.

We close this section with the following two open questions from [51].

Question 5.14 ([51, Question 4.6]). Is Theorem 5.10 valid for the classical pullbacks $R = D + M$ issued from $T := K + M$ not necessarily local? The idea here is that (since $k = K$, then) $T = S^{-1}R$ for $S := D \setminus \{0\}$. Clearly, the current proof of the "only if" assertion works for this context.

Question 5.15 ([51, Question 4.7]). Is Theorem 5.10 valid for the non-local case through an additional assumption on T_M? The idea here is that, "($k = K$ and hence) $R_M = T_M$" is a necessity for the finite t-basic property; and for the PvMD and v-domain notions, $R_M = T_M$ is a valuation domain. So, one needs to investigate this localization for the t-basic ideal property in this context.

References

1. Anderson, D.D., Anderson, D.F.: Some remarks on star operations and the class group. J. Pure Appl. Algebra **51**, 27–33 (1988)
2. Anderson, D.D., Anderson, D.F., Costa, D.L., Dobbs, D.E., Mott, J.L., Zafrullah, M.: Some characterizations of v-domains and related properties. Colloq. Math. **58**, 1–9 (1989)
3. Anderson, D.D., Anderson, D.F.: Examples of star operations on integral domains. Comm. Algebra **18**, 1621–1643 (1990)
4. Anderson, D.D., Anderson, D.F., Fontana, M., Zafrullah, M.: On v-domains and star operations. Comm. Algebra **37**, 3018–3043 (2009)

5. Anderson, D.D., Anderson, D.F., Zafrullah, M.: Splitting the t-class group. J. Pure Appl. Algebra **74**, 17–37 (1991)
6. Anderson, D.D., Anderson, D.F., Zafrullah, M.: The ring $D + XD_S[X]$ and t-splitting sets. Arab. J. Sci. Eng. Sect. C **26**, 3–16 (2001)
7. Anderson, D.D., Anderson, D.F., Zafrullah, M.: Completely integrally closed Prüfer v-multiplication domains. Comm. Algebra **45**, 5264–5282 (2017)
8. Anderson, D.F.: Integral v-ideals. Glasgow Math. J. **22**, 167–172 (1981)
9. Anderson, D.F.: A general theory of class groups. Comm. Algebra **16**, 805–847 (1988)
10. Anderson, D.F.: Star operations and the $D+M$ construction. Rend. Circ. Mat. Palermo (2) **41**, 221–230 (1992)
11. Anderson, D.F., Chang, G.W.: The class group of integral domains. J. Algebra **264**, 535–552 (2003)
12. Anderson, D.F., El Baghdadi, S., Kabbaj, S.: On the class group of $A + XB[X]$ domains. Lect. Notes Pure Appl. Math. Dekker **205**, 73–85 (1999)
13. Anderson, D.F., El Baghdadi, S., Zafrullah, M.: The v-operation in extensions of integral domains. J. Algebra Appl. **11**, 1250007, 18 pp. (2012)
14. Anderson, D.F., Fontana, M., Zafrullah, M.: Some remarks on Prüfer $\star$-multiplication domains and class groups. J. Algebra **319**, 272–295 (2008)
15. Anderson, D.F., Houston, E., Zafrullah, M.: Pseudo-integrality. Can. Math. Bull. **34**, 15–22 (1991)
16. Anderson, D.F., Ryckaert, A.: The class group of $D + M$. J. Pure Appl. Algebra **52**, 199–212 (1988)
17. Badawi, A., Houston, E.: Powerful ideals, strongly primary ideals, almost pseudo-valuation domains, and conducive domains. Comm. Algebra **30**, 1591–1606 (2002)
18. Barucci, V., Gabelli, S., Roitman, M.: The class group of a strongly Mori domain. Comm. Algebra **22**, 173-211 (1994)
19. Barucci, V., Houston, E., T. G. Lucas, Papick, I.: m-canonical ideals in integral domains II. Lect. Notes Pure Appl. Math. Dekker **220**, 89–108 (2001)
20. Bastida, E., Gilmer, R.: Overrings and divisorial ideals of rings of the form $D + M$. Michigan Math. J. **20**, 79–95 (1973)
21. Bouvier, A., Zafrullah, M.: On some class groups of an integral domain. Bull. Soc. Math. Grèce (N.S.), **29**, 45–59 (1988)
22. Boynton, J., Sather-Wagstaff, S.: Regular pullbacks. In: Progress in Commutative Algebra 2, pp. 145-169. Walter de Gruyter, Berlin (2012)
23. Chang, G.M., Zafrullah, M.: The w-integral closure of integral domains. J. Algebra **295**, 195–210 (2006)
24. Chatham, R.D., Dobbs, D.E.: On pseudo-valuation domains whose overrings are going-down domains. Houston J. Math. **28**, 13–19 (2002)
25. Dobbs, D.E.: Coherent pseudo-valuation domains have Noetherian completions. Houston J. Math. **14**, 481–486 (1988)
26. El Baghdadi, S., Gabelli, S.: w-divisorial domains. J. Algebra **285**, 335–355 (2005)
27. Elliott, J.: Functorial properties of star operations. Comm. Algebra **38**, 1466–1490 (2010)
28. Epstein, N.: A tight closure analogue of analytic spread. Math. Proc. Cambridge Philos. Soc. **139**, 371–383 (2005)
29. Epstein, N.: Reductions and special parts of closures. J. Algebra **323**, 2209–2225 (2010)
30. Epstein, N.: A guide to closure operations in commutative algebra. In: Progress in Commutative Algebra 2, pp. 1-37. Walter de Gruyter, Berlin (2012)
31. Fontana, M., Gabelli, S.: On the class group and the local class group of a pullback. J. Algebra **181**, 803–835 (1996)
32. Fontana, M., Picozza, G.: On some classes of integral domains defined by Krull's a.b. operations. J. Algebra **341**, 179–197 (2011)
33. Fontana, M., Zafrullah, M.: On v-domains: a survey. In: Commutative Algebra. Noetherian and non-Noetherian Perspectives, pp. 145-179. Springer, New York (2011)
34. Gabelli, S., Houston, E.: Coherentlike conditions in pullbacks. Michigan Math. J. **44**, 99–123 (1997)

35. Gabelli, S., Houston, E., Lucas, T.G.: The t#-property for integral domains. J. Pure Appl. Algebra **194**, 281–298 (2004)
36. Gabelli, S., Picozza, G.: w-stability and Clifford w-regularity of polynomial rings. Comm. Algebra **43**, 59–75 (2015)
37. Gilmer, R.: Multiplicative Ideal Theory. Dekker, New York (1972)
38. Hays, J.: Reductions of ideals in commutative rings. Trans. Amer. Math. Soc. **177**, 51–63 (1973)
39. Hays, J.: Reductions of ideals in Prüfer domains. Proc. Amer. Math. Soc. **52**, 81–84 (1975)
40. Hedstrom, J., Houston, E.: Pseudo-valuation domains. Pacific J. Math. **75**, 137–147 (1978)
41. Hedstrom, J., Houston, E.: Pseudo-valuation domains II. Houston J. Math. **4**, 199–207 (1978)
42. Heinzer, W., Ohm, J.: An essential ring which is not a v-multiplication ring. Can. J. Math. **25**, 856–861 (1973)
43. Houston, E.: Prime t-ideals in $R[X]$. Lect. Notes Pure Appl. Math. Dekker **153**, 163–170 (1994)
44. Houston, E., Kabbaj, S., Mimouni, M.: $\star$-reductions of ideals and Prüfer v-multiplication domains. J. Commut. Algebra **9**, 491–505 (2017)
45. Houston, E., Mimouni, M., Park, M.H.: Noetherian domains which admit only finitely many star operations. J. Algebra **366**, 78–93 (2012)
46. Houston, E., Zafrullah, M.: Integral domains in which each t-ideal is divisorial. Michigan Math. J. **35**, 291–300 (1988)
47. Houston, E., Zafrullah, M.: Integral domains in which any two v-coprime elements are comaximal. J. Algebra **423**, 93–113 (2015)
48. Huneke, C., Swanson, I.: Integral Closure of Ideals, Rings, and Modules. Cambridge University Press, Cambridge (2012)
49. Hutchins, H.C.: Examples of Commutative Rings. Polygonal Publishing House, Washington (1981)
50. Kabbaj, S., Kadri, A.: t-Reductions and t-integral closure of ideals. Rocky Mountain J. Math. **47**, 1875–1899 (2017)
51. Kabbaj, S., Kadri, A., Mimouni, M.: On t-reductions of ideals in pullbacks. New York J. Math. **22**, 875–889 (2016)
52. Kabbaj, S., Kadri, A., Mimouni, M.: t-Reductions and t-integral closure of ideals in Noetherian domains, New York J. Math., to appear.
53. Kabbaj, S., Mimouni, M.: t-Class semigroups of integral domains. J. Reine Angew. Math. **612**, 213–229 (2007)
54. Kabbaj, S., Mimouni, M.: Constituent groups of Clifford semigroups arising from t-closure. J. Algebra **321**, 1443–1452 (2009)
55. Kang, B.G.: Prüfer v-multiplication domains and the ring $R[X]_{N_v}$. J. Algebra **123**, 151–170 (1989)
56. Kaplansky, I.: Commutative Rings. University of Chicago Press, Chicago (1974)
57. Mimouni, A.: Integral domains in which each ideal is a w-ideal. Comm. Algebra **33**, 1345–1355 (2005)
58. Mott, J.L., Zafrullah, M.: On Prüfer v-multiplication domains. Manuscripta Math. **35**, 1–26 (1981)
59. Northcott, D.G., Rees, D.: Reductions of ideals in local rings. Proc. Cambridge Philos. Soc. **50**, 145–158 (1954)
60. Park, M.H.: Prime t-ideals in power series rings over a discrete valuation domain. J. Algebra **324**, 3401–3407 (2010)
61. Park, M.H.: Krull dimension of power series rings over a globalized pseudo-valuation domain. J. Pure Appl. Algebra **214**, 862–866 (2010)
62. Wang, F.: w-dimension of domains. Comm. Algebra **27**, 2267–2276 (1999)
63. Wang, F.: w-dimension of domains II. Comm. Algebra **29**, 2419–2428 (2001)

64. Zafrullah, M.: On finite-conductor domains. Manuscripta Math. **24**, 191–204 (1978)
65. Zafrullah, M.: The $D+XD_S[X]$ construction from GCD-domains. J. Pure Appl. Algebra **50**, 93–107 (1988)
66. Zafrullah, M.: Well behaved prime t-ideals. J. Pure Appl. Algebra **65**, 199–207 (1990)
67. Zafrullah, M.: t-invertibility and Bazzoni-like statements. J. Pure Appl. Algebra **214**, 654–657 (2010)