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ZERO-DIVISOR GRAPHS OF AMALGAMATIONS (⋆)

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Abstract

Let $f : A \to B$ be a homomorphism of commutative rings and let J be an ideal of B. The *amalgamation* of A with B along J with respect to f is the subring of $A \times B$ given by

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}.$$

This paper investigates the zero-divisor graph of amalgamations. Our aim is to characterize when the graph is complete and compute its diameter and girth for various contexts of amalgamations. The new results recover well-known results on duplications, and yield new and original examples issued from amalgamations.

1. Introduction

Throughout, all rings considered are commutative with identity. Let R be a ring and let $\mathrm{Z}(R)$ denote the set of zero-divisors of R and $\mathrm{Z}(R)^\star := \mathrm{Z}(R) \setminus \{0\}$. The zero-divisor graph of R, denoted $\Gamma(R)$, is the graph whose vertices are the elements of $\mathrm{Z}(R)^\star$ and, for distinct $x, y \in \mathrm{Z}(R)^\star$, there is an edge connecting x and y if and only if $xy = 0$. For two distinct vertices a and b in the graph $\Gamma(R)$, the distance between a and b, denoted $\mathrm{d}(a, b)$, is the length of the shortest path connecting a and b, if such a path exists; otherwise, $\mathrm{d}(a, b) = \infty$. The diameter of the graph $\Gamma(R)$ is given by $\mathrm{diam}(\Gamma(R)) = \sup\{\mathrm{d}(a, b) \mid a \text{ and } b \text{ are distinct vertices of } \Gamma(R)\}$. The girth of the graph $\Gamma(R)$, denoted $\mathrm{gr}(\Gamma(R))$, is the length of a shortest cycle in $\Gamma(R)$, provided $\Gamma(R)$ contains a cycle; otherwise, $\mathrm{gr}(\Gamma(R)) = \infty$. A graph is connected if there exists a path between any two distinct vertices, and it is complete if it is connected with diameter less than or equal to one. A singleton graph is connected and of diameter zero. Also, $\Gamma(R)$ is empty if and only if R is a domain.

The concept of a zero-divisor graph was first introduced by Beck in 1988 for his study of the coloring of a (commutative) ring [7]. In his work, all elements of the ring were vertices of the graph. In 1993, D. D. Anderson and Naseer used this same concept in [2]. In 1999, D. F. Anderson and Livingston considered only

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nonzero zero-divisors as vertices of the graph and proved that the zero-divisor graph of a (commutative) ring is always connected with diameter less than or equal to three. Later, in 2002, Mulay [17] and DeMeyer and Schneider [13] examined, among other properties, the diameter and girth of the zero-divisor graph of a ring. The following paragraphs collect background and main contributions on the zero-divisor graph of some relevant ring extensions.

In [5], Axtell, Coykendall and Stickles examined the preservation, or lack thereof, of the diameter and girth of the graph of a ring under extensions to polynomial and power series rings. One of the difficulties in dealing with $R[[x]]$, when R is not reduced, is that the zero-divisors of $R[[x]]$ can be rather strange. For example, they cited an example in [5] of a non-reduced ring R with $\mathrm{diam}(\Gamma(R)) = \mathrm{diam}(\Gamma(R[x])) = 2$ while $\mathrm{diam}(\Gamma(R[[x]])) = 3$. But they left open the existence of a reduced ring with the same sequence of diameters.

In [15], Lucas characterized the diameter of $\Gamma(R)$, $\Gamma((R[x])$, and $\Gamma(R[[x]])$ strictly in terms of properties of the ring R. For reduced rings, he gave complete characterizations for all three graphs; and for non-reduced rings, he succeeded in characterizing the diameters of $\Gamma(R)$ and $\Gamma(R[x])$. He constructed a reduced ring R for which $\mathrm{diam}(\Gamma(R)) = \mathrm{diam}(\Gamma(R[x])) = 2$ and $\mathrm{diam}(\Gamma(R[[x]])) = 3$. He also provided examples of both reduced and non-reduced rings R where $\mathrm{diam}(\Gamma(R)) = 2$ and $\mathrm{diam}(\Gamma(R[x])) = \mathrm{diam}(\Gamma(R[[x]])) = 3$.

For a ring A and an A-module E, the trivial ring extension of A by E is the ring $A \ltimes E$ where the underlying group is $A \times E$ and the multiplication is defined by $(a, e)(b, f) = (ab, af + be)$. It is also called the (Nagata) idealization of E over A and is denoted by $A(+)E$. This construction was first introduced and studied by Nagata [18]. In [6], Axtell and Stickles investigated the preservation of the diameter and girth under trivial ring extensions. Specifically, they completely characterized the girth of the zero-divisor graph of a trivial ring extension and when it is complete. They also provided conditions for the zero-divisor graph to have diameter 2.

In [4], D. F. Anderson and Mulay characterized when either $\mathrm{diam}(\Gamma(R)) \leq 2$ or $\mathrm{gr}(\Gamma(R)) \geq 4$ for a ring R, and used their results to investigate the diameter and girth for the zero-divisor graphs of polynomial rings, power series rings, and trivial ring extensions. They answered some open questions and gave alternative proofs to previous results in [5, 6, 15]. Their new approach consisted in working in the total quotient ring of R.

For a ring A and an ideal I of A, the amalgamated duplication of A along I is the subring of $A \times A$ given by $A \bowtie I := \big\{(a, a + i) \mid a \in A, i \in I\big\}$. If $I^2 = 0$, then $A \bowtie I$ coincides with the trivial ring extension $A \ltimes I$. This construction was introduced and its basic properties were studied by D'Anna and Fontana in [8, 11, 12]. It was motivated by a construction of D. D. Anderson [1] related to a classical construction due to Dorroh [14] on endowing a ring (without unity) with a unity. In [16], Maimani and Yassemi studied the diameter and girth of the graph of $A \bowtie I$. More precisely, two of their main results assert that "$\Gamma(A \bowtie I)$ *is complete if and only if* $(\mathrm{Z}(A))^2 = 0$ *and* $I \subseteq \mathrm{Z}(A)$ *if and only if* $(\mathrm{Z}(A \bowtie I))^2 = 0$" [16, Theorem 4.8] and "*if* $\mathrm{Z}(A \bowtie I)$ *is not an ideal, then* $\mathrm{diam}(\Gamma(A \bowtie I)) = 3$" [16, Theorem 4.12]. Also, they showed that $\mathrm{gr}(\Gamma(A \bowtie I))$ is equal to 3 if A is not a domain, or 4 if A is a domain with $0 \subsetneq I \subsetneq A$.

In [9, 10], D'Anna, Finocchiaro, and Fontana introduced the more general context of amalgamations. They have studied these constructions in the frame of pullbacks which allowed them to establish numerous results on the transfer of various ideal and ring-theoretic properties from A and $f(A)+J$ to $A \bowtie^f J$. The interest of amalgamations resides in their ability to cover basic constructions in commutative algebra, including classical pullbacks and trivial ring extensions. Recall that the latter are never reduced, whereas amalgamations can be domains or reduced [9, Propositions 5.2 & 5.3].

This paper investigates the zero-divisor graph of amalgamations. Our aim is to characterize when the graph is complete and compute its diameter and girth for various contexts of amalgamations. Precisely, in view of Anderson-Livingston's aforementioned result, Sections 2, 3, and 4 will handle, respectively, the cases when $\Gamma(A \bowtie^f J)$ is complete, $\mathrm{diam}\big(\Gamma(A \bowtie^f J)\big) = 2$, and $\mathrm{diam}\big(\Gamma(A \bowtie^f J)\big) = 3$. Then, Section 5 will be devoted to the girth of $\Gamma(A \bowtie^f J)$. The new results recover and generalize well-known results on amalgamated duplications [16], as well as yield original examples issued from amalgamations.

2. When is $\Gamma(A \bowtie^f J)$ complete?

To avoid unnecessary repetition, let us fix notation for the rest of the paper. Let $f : A \to B$ be a homomorphism of rings and J a **nonzero proper** ideal of B. Let R denote the amalgamation of A with B along J with respect to f; that is,

$$R := A \bowtie^f J = \big\{(a, f(a)+j) \mid a \in A, j \in J\big\}.$$

Note that if $J = 0$, then $R \cong A$; and if $J = B$, then $R = A \times B$. Also, recall that $f^{-1}(J) = 0$ if and only if R and $f(A)+J$ are isomorphic [10, Proposition 2.1] and hence have the same zero-divisor graph. Moreover, R is a domain if and only if $f(A)+J$ is a domain and $f^{-1}(J) = 0$ [11, Proposition 5.2]. Therefore, all along the paper, the assumption $f^{-1}(J) \neq 0$ ensures that $\Gamma(R) \neq \emptyset$ and rules out the trivial case $R \cong f(A)+J$.

The main result of this section establishes necessary and sufficient conditions for $\Gamma(R)$ to be complete and computes its girth. To this purpose, it is worthwhile recalling Anderson-Livingston's result that "*the graph of a ring D is complete if* *and only if* $D=\mathbb{Z}_2\times\mathbb{Z}_2$ *or* $(\mathrm{Z}(D))^2=0$" [3, Theorem 2.8]. Throughout, let

$$\mathrm{Z}(A)\bowtie^f J:=\big\{(a,f(a)+j)\mid a\in\mathrm{Z}(A),\ j\in J\big\}$$

and for any $j\in J$, let

$$(0:j):=\big\{x\in f(A)+J\mid jx=0\big\}.$$

THEOREM 2.1. *Assume that* $f^{-1}(J)\neq 0$. *Then, the following conditions are equivalent:*

(1) $\Gamma(R)$ *is complete;*
(2) $(\mathrm{Z}(A))^2=0$, $J^2=0$, *and* $\mathrm{Z}(A)=f^{-1}(0:j)$, $\forall\ 0\neq j\in J$;
(3) $(\mathrm{Z}(R))^2=0$.

Moreover, if any one condition holds, then $\mathrm{Z}(R)=\mathrm{Z}(A)\bowtie^f J$ *and* $\mathrm{gr}(\Gamma(R))=3$.

PROOF. (1) $\Rightarrow$ (2) Assume that $\Gamma(R)$ is complete. Then, so is $\Gamma(A)$. We claim that $(\mathrm{Z}(A))^2=0$. Otherwise, suppose that $A=\mathbb{Z}_2\times\mathbb{Z}_2$. Then, $\mathrm{Z}(A)^\star$ consists of two elements $a:=(0,1)$ and $b:=(1,0)$. Let $0\neq j\in J$. Then, the fact $a+b=1$ yields $j=jf(a)+jf(b)$. So, either $jf(a)\neq 0$ or $jf(b)\neq 0$. Say, $jf(a)\neq 0$. Also the fact $1=a^2+b^2$ yields $j=jf(a^2)+jf(b^2)$. Now, since $(b,f(b))(0,jf(a))=0$, then $(0,jf(a))\in\mathrm{Z}(R)^\star$ and, as $\Gamma(R)$ is complete, $(a,f(a))(0,jf(a))=0$. Hence $jf(a^2)=0$. It follows that $j=jf(b^2)$ and therefore $jf(a)=jf(b^2)f(a)=0$, the desired contradiction. So, $(\mathrm{Z}(A))^2=0$.

Next, let $0\neq c\in f^{-1}(J)$. Then, for every $0\neq j\in J$, $(c,0)(0,j)=0$ and hence $(0,j)\in\mathrm{Z}(R)^\star$. If $|J|\geq 3$, then for every nonzero $i\neq j\in J$, $(0,i)(0,j)=0$, and so $ij=0$. Also, since $(c,i)(0,j)=0$, $(c,i)\in\mathrm{Z}(R)^\star$. Thus, $(c,i)(0,i)=0$ and therefore $i^2=0$. If $|J|=2$, then let $0\neq i\in J$. If $i^2=i$, then $i(i-1)=0$. So $(1,1-i)(0,i)=0$ and then $(1,1-i)\in\mathrm{Z}(R)^\star$. But, J is proper, so that $1-i\neq 0$ and hence $(1,1-i)(c,0)=0$. Whence $c=0$, absurd. Thus, $i^2=0$. In both cases, we showed that $J^2=0$. Next, let $0\neq j\in J$ and let $0\neq a\in A$. If $a\in\mathrm{Z}(A)$, then $(a,f(a))\in\mathrm{Z}(R)^\star$. So, $(a,f(a))(0,j)=0$; that is, $jf(a)=0$. Hence, $\mathrm{Z}(A)\subseteq f^{-1}(0:j)$. If $jf(a)=0$, then $(a,f(a))(0,j)=0$. Hence $(a,f(a))\in\mathrm{Z}(R)^\star$. It follows that $(c,0)(a,f(a))=0$ if $f(a)\neq 0$, and $(c,j)(a,0)=0$ if $f(a)=0$. Both cases yield $ac=0$. Thus, $a\in\mathrm{Z}(A)$. Consequently, $\mathrm{Z}(A)=f^{-1}(0:j)$, as desired.

(2) $\Rightarrow$ (3) Assume $(\mathrm{Z}(A))^2=0$, $J^2=0$, and $\mathrm{Z}(A)=f^{-1}(0:j)$ for each $0\neq j\in J$. Clearly, $(0,j)\in\mathrm{Z}(R)$, $\forall\ j\in J$. Further, if $0\neq a\in\mathrm{Z}(A)$ and $j\in J$, then $\exists\ 0\neq b\in\mathrm{Z}(A)$ with $ab=0$ and hence $(a,f(a)+j)(b,f(b))=(0,jf(b))=0$; i.e., $(a,f(a)+j)\in\mathrm{Z}(R)$. So, $\mathrm{Z}(A)\bowtie^f J\subseteq\mathrm{Z}(R)$. For the reverse inclusion, let $x:=(a,f(a)+i)\in\mathrm{Z}(R)$ and $y:=(b,f(b)+j)\in R\setminus\{0\}$ such that $xy=0$. If $a=0$, then $x=(0,i)\in\mathrm{Z}(A)\bowtie^f J$. Assume that $a\neq 0$. If $b\neq 0$, then $a\in\mathrm{Z}(A)$. If $b=0$, then $0=jf(a)+ij=jf(a)$ and hence $a\in\mathrm{Z}(A)$. In both

cases, $x \in \mathrm{Z}(A) \bowtie^f J$. Consequently, we get

$$\mathrm{Z}(R) = \mathrm{Z}(A) \bowtie^f J.$$

Now, one can easily check that $xy = 0$ for every $x, y \in \mathrm{Z}(A) \bowtie^f J$. It follows that $(\mathrm{Z}(R))^2 = 0$, as desired.

(3) $\Rightarrow$ (1) Trivial.

Finally, assume that any one of the conditions (1)–(3) holds. Then, as stated above, $\mathrm{Z}(R) = \mathrm{Z}(A) \bowtie^f J$. Further, let $0 \neq c \in f^{-1}(J)$ and $0 \neq j \in J$. Then, clearly, $(c, 0)-(0, j)-(c, j)-(c, 0)$ is a cycle in $\Gamma(R)$ and therefore $\mathrm{gr}(\Gamma(R)) = 3$.

Theorem 2.1 recovers the special case of duplications, as recorded in the next corollary. Recall that a duplication can be viewed as a special amalgamation $R := A \bowtie^f I$ with $A = B$ and $f := 1_A$.

Corollary 2.2 ([16, Theorem 4.8]). *Let D be a ring and I a nonzero proper ideal of D. Then, the following conditions are equivalent:*

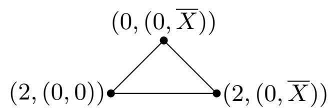
(1) $\Gamma(D \bowtie I)$ *is complete;*
(2) $(\mathrm{Z}(D))^2 = 0$ *and* $I \subseteq \mathrm{Z}(D)$*;*
(3) $(\mathrm{Z}(D \bowtie I))^2 = 0$.

Moreover, if any one condition holds, then $\mathrm{Z}(D \bowtie I) = \mathrm{Z}(D) \bowtie I$ *and* $\mathrm{gr}(\Gamma(D \bowtie I)) = 3$.

Proof. The proof is straightforward via Theorem 2.1. Notice that the condition "$\mathrm{Z}(D) = (0 : i),\ \forall\ 0 \neq i \in I$" becomes redundant with "$(\mathrm{Z}(D))^2 = 0$ and $I \subseteq \mathrm{Z}(D)$."

Unlike duplications, the graph of an amalgamation R (with $J \neq 0$) can be complete under $(\mathrm{Z}(R))^2 = 0$ or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, as shown by the following examples.

Example 2.3. Let $A := \mathbb{Z}_4$, $B := \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$, and $J = \langle(0, \overline{X})\rangle$. Consider the ring homomorphism $f : A \to B$, defined by $f(0) = f(2) = 0$ and $f(1) = f(3) = 1$; and let $R := A \bowtie^f J$. Notice that $f^{-1}(J) \neq 0$, $J^2 = 0$, and $\mathrm{Ker}(f) = \mathrm{Z}(A) = \{0, 2\}$. Hence, $\mathrm{Z}(A)^2 = 0$ and $f^{-1}\big(0 : (0, \overline{X})\big) = \mathrm{Z}(A)$. By Theorem 2.1, $\Gamma(R)$ is complete with $(\mathrm{Z}(R))^2 = 0$. Moreover, $\mathrm{Z}(R)^\star = \big\{(0, (0, \overline{X})), (2, (0, 0)), (2, (0, \overline{X}))\big\}$ and $\mathrm{gr}(\Gamma(R)) = 3$, as illustrated below:



Example 2.4. Let $A := \mathbb{Z}_2$, $B := \mathbb{Z}_2 \times \mathbb{Z}_2$, and $J = \langle (0,1) \rangle$. Consider the ring homomorphism $f : A \to B$, defined by $f(0) = 0$ and $f(1) = 1$; and let $R := A \bowtie^f J$. Notice that $f^{-1}(J) = 0$. Hence $R \cong f(A) + J = \mathbb{Z}_2 \times \mathbb{Z}_2$ and thus $\mathrm{Z}(R)^\star = \big\{(0,(0,1)), (1,(1,0))\big\}$. So, $\Gamma(R)$ is complete with $(\mathrm{Z}(R))^2 \neq 0$, as desired.

3. When is $\operatorname{diam}(\Gamma(R)) = 2$ or 3?

Throughout, let $\mathrm{Z}(f^{-1}(J))$ denote the set of zero-divisors on the ideal $f^{-1}(J)$ of A and let $\mathrm{Z}(J)$ denote the set of zero-divisors on J *as an ideal of* $f(A)+J$; that is, the elements of $f(A)+J$ that annihilate some nonzero element of J, and let

$$\mathrm{Z}^\star(J) := \big\{f(a) + j \in \mathrm{Z}(J) \mid a \neq 0\big\}.$$

Consider the following conditions:

($\mathcal{C}_1$) $\forall x \in \mathrm{Z}^\star(J)$ and $\forall j \in J$: $jx \neq 0 \Rightarrow ij = ix = 0$ for some $0 \neq i \in J$.
($\mathcal{C}_2$) $\forall x, y \in \mathrm{Z}^\star(J)$: $x \neq y \Rightarrow ix = iy = 0$ for some $0 \neq i \in J$.
($\mathcal{C}_3$) $\forall a \in \mathrm{Z}(f^{-1}(J))$, $\forall b \in f^{-1}(J)$: $ab \neq 0 \Rightarrow ac = bc = 0$ for some $0 \neq c \in f^{-1}(J)$.
($\mathcal{C}_4$) $\forall a, b \in \mathrm{Z}(f^{-1}(J))$: $a \neq b \Rightarrow ac = bc = 0$ for some $0 \neq c \in f^{-1}(J)$.

The first main result of this section establishes necessary and sufficient conditions for the diameter of the zero-divisor graph of the amalgamation $R := A \bowtie^f J$ to be equal to 2, when A or $f(A)+J$ is a domain.

Theorem 3.1. *Under the above notation, assume $f^{-1}(J) \neq 0$.*
(1) *If A is a domain, then:* $\operatorname{diam}(\Gamma(R)) = 2 \Leftrightarrow (\mathcal{C}_1)$ *and* $(\mathcal{C}_2)$ *hold.*
(2) *If $f(A)+J$ is a domain, then:* $\operatorname{diam}(\Gamma(R)) = 2 \Leftrightarrow (\mathcal{C}_3)$ *and* $(\mathcal{C}_4)$ *hold.*

Proof. First, let

$$\begin{array}{rcl}
\mathbb{E}_1 & := & \big\{(0,j) \mid 0 \neq j \in J\big\} \\
\mathbb{E}_2 & := & \big\{(a,0) \mid 0 \neq a \in f^{-1}(J)\big\} \\
\mathbb{E}_3 & := & \big\{(a, f(a)+j) \mid 0 \neq f(a)+j \in \mathrm{Z}^\star(J)\big\} \\
\mathbb{E}_4 & := & \big\{(a, f(a)+j) \mid f(a)+j \neq 0,\ 0 \neq a \in \mathrm{Z}(f^{-1}(J))\big\}
\end{array}$$

(1) Assume that A is a domain. We claim that

$$\mathrm{Z}(R)^\star = \mathbb{E}_1 \cup \mathbb{E}_2 \cup \mathbb{E}_3.$$

Indeed, let $x = (a, f(a)+i) \in \mathrm{Z}(R)^\star$ and $y = (b, f(b)+j) \in R \setminus \{0\}$ such that $xy = 0$. If $a = 0$, then $x \in \mathbb{E}_1$. If $a \neq 0$, then $b = 0$ and so $j \neq 0$. But then $j(f(a)+i) = 0$ implies that either $f(a)+i = 0$ and so $x \in \mathbb{E}_2$ or $f(a)+i \in \mathrm{Z}^\star(J)$ and so $x \in \mathbb{E}_3$. For the reverse inclusion, note that clearly $\mathbb{E}_3 \subseteq \mathrm{Z}(R)^\star$, and $(0,j)(a,0) = 0$ for any $0 \neq j \in J$ and $0 \neq a \in f^{-1}(J)$, yielding

$\mathbb{E}_1 \cup \mathbb{E}_2 \subseteq \mathrm{Z}(R)^\star$, proving the claim. Notice that the above union is disjoint. For sufficiency, assume that $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$ hold. As mentioned above, every vertex in $\mathbb{E}_1$ is adjacent to every vertex in $\mathbb{E}_2$. Next, let us handle five other possible distinct cases.

- Let $x_1 := (0, i) \neq (0, j) =: y_1 \in \mathbb{E}_1$ and let $0 \neq a \in f^{-1}(J)$. Then, $x_1 - (a, 0) - y_1$ is a path in $\Gamma(R)$ and so $\mathrm{d}(x_1, y_1) \leq 2$.
- Let $x_2 := (a, 0) \neq (b, 0) =: y_2 \in \mathbb{E}_2$ and let $0 \neq j \in J$. Then, $x_2 - (0, j) - y_2$ is a path in $\Gamma(R)$ and so $\mathrm{d}(x_2, y_2) = 2$.
- Let $x_1 := (0, j) \in \mathbb{E}_1$ and $x_3 := (a, f(a) + i) \in \mathbb{E}_3$. If $j(f(a) + i) = 0$, then $x_1x_3 = 0$ and so $\mathrm{d}(x_1, x_3) = 1$. If $j(f(a) + i) \neq 0$, by $(\mathcal{C}_1)$, there is $0 \neq r \in J$ such that $rj = r(f(a) + i) = 0$. Then, $x_1 - (0, r) - x_3$ is a path in $\Gamma(R)$ and so $\mathrm{d}(x_1, x_3) = 2$.
- Let $x_2 := (a, 0) \in \mathbb{E}_2$ and $x_3 := (b, f(b) + j) \in \mathbb{E}_3$. Then, there is $0 \neq i \in J$ such that $i(f(b) + j) = 0$. Hence, $x_2 - (0, r) - x_3$ is a path in $\Gamma(R)$ and so $\mathrm{d}(x_2, x_3) = 2$.
- Let $x_3 := (a, f(a) + i) \neq (b, f(b) + j) =: y_3 \in \mathbb{E}_3$. If $f(a) + i = f(b) + j$, then there is $0 \neq r \in J$ such that $r(f(a) + i) = r(f(b) + j) = 0$ and so $x_3 - (0, r) - y_3$ is a path in $\Gamma(R)$. If $f(a) + i \neq f(b) + j$, by $(\mathcal{C}_2)$, there is $0 \neq r \in J$ such that $r(f(a)+i) = r(f(b)+j) = 0$. Again $x_3 - (0, r) - y_3$ is a path in $\Gamma(R)$ and so $\mathrm{d}(x_3, y_3) = 2$.

Consequently, $\mathrm{diam}(\Gamma(R)) = 2$. Conversely, let $x := f(a) + i \in \mathrm{Z}^\star(J)$ and let $j \in J$ with $jx \neq 0$. Since $(a, f(a) + i)$ and $(0, j)$ are two non-adjacent vertices, there is a path $(a, f(a)+i) - (b, f(b)+r) - (0, j)$ in $\Gamma(R)$. Necessarily, $b = 0$ and hence $r \neq 0$. But then $r(f(a)+i) = rj = 0$, as desired. Finally, let $x := f(a)+i$ and $y := f(b) + j \in \mathrm{Z}^\star(J)$ with $x \neq y$. Since $(a, f(a) + i)$ and $(b, f(b) + j)$ are two non-adjacent vertices, there is a path $(a, x) - (c, f(c) + r) - (b, y)$ in $\Gamma(R)$. Necessarily, $c = 0$ and hence $r \neq 0$. But then $rx = ry = 0$, completing the proof of (1).

(2) Assume that $f(A) + J$ is a domain. We claim that

$$\mathrm{Z}(R)^\star = \mathbb{E}_1 \cup \mathbb{E}_2 \cup \mathbb{E}_4.$$

Indeed, let $x = (a, f(a) + i) \in \mathrm{Z}(R)^\star$ and $y = (b, f(b) + j) \in R \setminus \{0\}$ such that $xy = 0$. If $a = 0$, then $x \in \mathbb{E}_1$. Assume $a \neq 0$. If $b = 0$, then $j \neq 0$ and hence $f(a) + i = 0$, whence $x \in \mathbb{E}_2$. If $b \neq 0$ and $f(a) + i \neq 0$, then $b \in f^{-1}(J)$ and so $x \in \mathbb{E}_4$. The reverse inclusion is straight. Notice that the above union is disjoint. For sufficiency, assume that $(\mathcal{C}_3)$ and $(\mathcal{C}_4)$ hold. In view of the proof of (1), we envisage only three cases.

- Let $x_1 := (0, i) \in \mathbb{E}_1$ and $x_4 := (b, f(b) + j) \in \mathbb{E}_4$. Let $0 \neq c \in f^{-1}(J)$ such that $bc = 0$. Then, $x_1 - (c, 0) - x_4$ is a path in $\Gamma(R)$ and so $\mathrm{d}(x_1, x_4) = 2$.
- Let $x_2 := (a, 0) \in \mathbb{E}_2$ and $x_4 := (b, f(b) + j) \in \mathbb{E}_4$. If $ab = 0$, then $\mathrm{d}(x_2, x_4) = 1$. If $ab \neq 0$, then by $(\mathcal{C}_3)$, there is $0 \neq c \in f^{-1}(J)$ such that $ac = bc = 0$. Then, $x_2 - (c, 0) - x_4$ is a path in $\Gamma(R)$ and so $\mathrm{d}(x_2, x_4) = 2$.

- Let $x_4 := (a, f(a) + i) \neq (b, f(b) + j) =: y_4 \in \mathbb{E}_4$. If $a = b$, then there is $0 \neq d \in f^{-1}(J)$ such that $ad = bd = 0$ and so $x_4 - (d, 0) - y_4$ is a path in $\Gamma(R)$. If $a \neq b$, by $(\mathcal{C}_4)$, there is $0 \neq c \in f^{-1}(J)$ such that $ac = bc = 0$ and so $x_4 - (c, 0) - y_4$ is a path in $\Gamma(R)$. In both cases, $\mathrm{d}(x_4, y_4) = 2$.

Consequently, $\mathrm{diam}(\Gamma(R)) = 2$. Conversely, let $a \in \mathrm{Z}(f^{-1}(J))$ and $b \in f^{-1}(J)$ such that $ab \neq 0$. Let $i \in J$ with $f(a)+i \neq 0$. Since $(a, f(a)+i)$ and $(b, 0)$ are two vertices that are not adjacent, there is a path $(a, f(a) + i) - (c, f(c) + j) - (b, 0)$. Then $ac = bc = 0$ and $f(c) + j = 0$. So, $0 \neq c \in f^{-1}(J)$, as desired. Finally, let $a, b \in \mathrm{Z}(f^{-1}(J))$ with $a \neq b$. Fix two elements i and j in J such that $x := f(a) + i \neq 0$ and $y := f(b) + j \neq 0$. Then x and y are two non-adjacent vertices. So, there is a path $x - (c, f(c) + r) - y$ in $\Gamma(R)$. Then, $ac = bc = 0$ and $f(c) + r = 0$. Hence $0 \neq c \in f^{-1}(J)$, completing the proof of (2).

The special case where both A and $f(A) + J$ are domains is given below.

Corollary 3.2. *Assume that both A and $f(A)+J$ are domains with $f^{-1}(J) \neq 0$. Then,* $\mathrm{diam}(\Gamma(R)) = 2$.

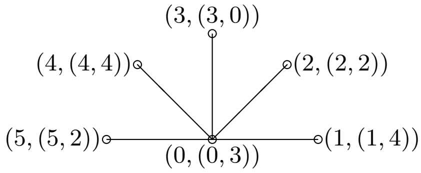
Proof. Here $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$ always hold since $\mathrm{Z}^\star(J) = \emptyset$ and so $\mathrm{diam}(\Gamma(R)) = 2$ by Theorem 3.1.

For the special case of duplications, we obtain the following corollary.

Corollary 3.3. *Let D be a domain and let I be a nonzero proper ideal of D. Then,* $\mathrm{diam}(\Gamma(D \bowtie I)) = 2$. □

Here are illustrative examples for Theorem 3.1. In the first example, A is a domain; and in the subsequent three examples, the diameter of the zero-divisor graph of A is equal to 0, 1, and 2, respectively.

Example 3.4. Let $A := \mathbb{Z}$, $B := \mathbb{Z}_6 \times \mathbb{Z}_6$, and $J := \langle (0, 3) \rangle$. Consider the ring homomorphism $f : A \to B$ defined by $f(n) = (n, n)$, and let $R := A \bowtie^f J$. One can easily check that $\mathrm{Z}^\star(J) = \{(1, 4), (2, 2), (3, 0), (4, 4), (5, 2)\}$ with $x(0, 3) = 0$, $\forall\, x \in \mathrm{Z}^\star(J)$. By Theorem 3.1, $\mathrm{diam}(\Gamma(R)) = 2$ as illustrated below:



and hence, $\mathrm{gr}(\Gamma(R)) = \infty$. Note that

$$f(A) + J := \big\{(0,3),(1,4),(2,5),(3,0),(4,1),(5,2)\big\}$$

and so $\mathrm{diam}(\Gamma(f(A)+J)) = 3$.

Example 3.5. Let $A := \mathbb{Z}_4$ ($\mathrm{diam}(\Gamma(A)) = 0$), $B := \mathbb{Z}_2[X]$, and $J := XB$. Consider the ring homomorphism $f : A \to B$, defined by $f(0) = f(2) = 0$ and $f(1) = f(3) = 1$; and let $R := A \bowtie^f J$. Then it is easy to see that $\mathrm{Z}(f^{-1}(J)) = f^{-1}(J) = \big\{0,2\big\}$. By Theorem 3.1, $\mathrm{diam}(\Gamma(R)) = 2$. Moreover, $\mathrm{Z}(R)^\star = \mathbb{E}_1 \cup \mathbb{E}_2 \cup \mathbb{E}_4 = \big\{(0, Xg) \mid 0 \neq g \in \mathbb{Z}_2[X]\big\} \cup \big\{(2, Xh) \mid h \in \mathbb{Z}_2[X]\big\}$ and hence $\mathrm{gr}(\Gamma(R)) = \infty$.

Example 3.6. Let $A := \mathbb{Z}_2 \times \mathbb{Z}_2$ ($\mathrm{diam}(\Gamma(A)) = 1$), $B := \mathbb{Z}_2[X]$, and $J := XB$. Consider the ring homomorphism $f : A \to B$, defined by $f(n,m) = n$; and let $R := A \bowtie^f J$. It is easy seen that $f^{-1}(J) = \big\{(0,0),(0,1)\big\}$ and $\mathrm{Z}(f^{-1}(J)) = \big\{(0,0),(1,0)\big\}$. By Theorem 3.1, $\mathrm{diam}(\Gamma(R)) = 2$. Moreover, $\mathrm{Z}(R)^\star = \mathbb{E}_1 \cup \mathbb{E}_2 \cup \mathbb{E}_4 = \big\{((0,0), Xg) \mid 0 \neq g \in \mathbb{Z}_2[X]\big\} \cup \big\{((0,1),0)\big\} \cup \big\{((1,0), 1+Xg) \mid g \in \mathbb{Z}_2[X]\big\}$ and hence $\mathrm{gr}(\Gamma(R)) = \infty$.

Example 3.7. Let $A := \mathbb{Z}_6$ ($\mathrm{diam}(\Gamma(A)) = 2$), $B := \mathbb{Z}_2[X]$, and $J := XB$. Consider the ring homomorphism $f : A \to B$, defined by $f(0) = f(2) = f(4) = 0$ and $f(1) = f(3) = f(5) = 1$; and let $R := A \bowtie^f J$. It is easy seen that $f^{-1}(J) = \big\{0,2,4\big\}$ and $\mathrm{Z}(f^{-1}(J)) = \big\{0,3\big\}$. By Theorem 3.1, $\mathrm{diam}(\Gamma(R)) = 2$. Moreover, $\mathrm{Z}(R)^\star = \mathbb{E}_1 \cup \mathbb{E}_2 \cup \mathbb{E}_4 = \big\{(0, Xg) \mid 0 \neq g \in \mathbb{Z}_2[X]\big\} \cup \big\{(2,0),(4,0)\big\} \cup \big\{(3, 1+Xh) \mid h \in \mathbb{Z}_2[X]\big\}$ and hence $\mathrm{gr}(\Gamma(R)) = 4$.

Next, we investigate conditions under which $\mathrm{diam}(\Gamma(R))$ is equal to 3. To this purpose, let us negate the aforementioned conditions ($\mathcal{C}_1$), ($\mathcal{C}_2$), ($\mathcal{C}_3$), ($\mathcal{C}_4$) to get the following:

($\overline{\mathcal{C}_1}$) $\exists x \in \mathrm{Z}^\star(J)$ and $\exists j \in J$ with $jx \neq 0$ and, for any $0 \neq i \in J$, $ij \neq 0$ or $ix \neq 0$.
($\overline{\mathcal{C}_2}$) $\exists x, y \in \mathrm{Z}^\star(J)$ with $x \neq y$ and, for any $0 \neq i \in J$, $ix \neq 0$ or $iy \neq 0$.
($\overline{\mathcal{C}_3}$) $\exists a \in \mathrm{Z}(f^{-1}(J))$, $\exists b \in f^{-1}(J)$ with $ab \neq 0$ and, $\forall 0 \neq c \in f^{-1}(J)$, $ac \neq 0$ or $bc \neq 0$.
($\overline{\mathcal{C}_4}$) $\exists a, b \in \mathrm{Z}(f^{-1}(J))$ with $a \neq b$ and, $\forall 0 \neq c \in f^{-1}(J)$, $ac \neq 0$ or $bc \neq 0$.

As a straightforward application of Theorem 3.1, we get the next "dual" result.

Proposition 3.8. *Under the above notation, assume* $f^{-1}(J) \neq 0$.

(1) *If A is a domain, then:* $\operatorname{diam}(\Gamma(R)) = 3 \Leftrightarrow (\overline{\mathcal{C}_1})$ *or* $(\overline{\mathcal{C}_2})$ *holds.*
(2) *If $f(A)+J$ is a domain, then:* $\operatorname{diam}(\Gamma(R)) = 3 \Leftrightarrow (\overline{\mathcal{C}_3})$ *or* $(\overline{\mathcal{C}_4})$ *holds.* $\square$

Next, we show how one may use Proposition 3.8 to construct original examples of amalgamations R with $\operatorname{diam}(\Gamma(R)) = 3$.

Example 3.9. Let $A := \mathbb{Z}$, $B := \mathbb{Z} \times \mathbb{Z}_6$, and $J := 2\mathbb{Z} \times \mathbb{Z}_6$. Consider the ring homomorphism $f : A \to B$, defined by $f(a) = (a, a)$; and let $R := A \bowtie^f J$. Notice, first, that $f^{-1}(J) = 2A$ and $f(A) + J = B$. Now, let $x := f(1) + (0, 1) = (1, 2) \in \mathrm{Z}^\star(J)$ and $j := (0, 1) \in J$. Obviously, $jx \neq 0$. Moreover, we have $j(0, 3) \neq 0$; $x(0, m) \neq 0$ for all $m \in \mathbb{Z}_6 \setminus \{0, 3\}$; and $x(2n, m) \neq 0$ for all $0 \neq n \in \mathbb{Z}$ and $m \in \mathbb{Z}_6$. Therefore, $(\overline{\mathcal{C}_1})$ holds and, by Proposition 3.8, $\operatorname{diam}(\Gamma(R)) = 3$.

The second main result of this section establishes conditions for the diameter of the zero-divisor graph of $R := A \bowtie^f J$ to be equal to 3, beyond the domain settings. In particular, it generalizes Maimani-Yassemi's result on duplications that "$\operatorname{diam}(\Gamma(D \bowtie I)) = 3$, *provided* $\mathrm{Z}(D)$ *is not an ideal*" [16, Theorem 4.12].

Theorem 3.10. *Under the above notation, assume $f^{-1}(J) \neq 0$. If $Z(A)$ is not an ideal of A, f is surjective and $f^{-1}(Z(B)) \subseteq Z(A)$, then* $\operatorname{diam}(\Gamma(R)) = 3$.

Proof. We first prove the following claims.

Claim 1. $Z(R)$ is not an ideal of R.

Indeed, let $a \neq b \in Z(A)$ such that $a - b \notin Z(A)$. Clearly $(a, f(a))$ and $(b, f(b)) \in Z(R)$. If $f(a - b) = 0$, then $a - b \in f^{-1}(Z(B)) \subseteq Z(A)$, which is absurd. Hence, $f(a - b) \neq 0$. Assume that $(a - b, f(a - b)) \in Z(R)$. Then there is $0 \neq (c, f(c) + j) \in R$ such that $(a - b, f(a - b))(c, f(c) + j) = 0$. Necessarily, $c = 0$. Hence $j \neq 0$ and $jf(a - b) = 0$; that is, $f(a - b) \in Z(B)$ and so $a - b \in f^{-1}(Z(B)) \subseteq Z(A)$, the desired contradiction. Therefore, $Z(R)$ is not an ideal of R.

Next, recall from [10, Proposition 2.6] that the prime ideals of R arise exclusively under the following two forms:

$$P^f =: P \bowtie^f J = \{(p, f(p) + j) \mid p \in P,\ j \in J\}$$

$$\overline{Q}^f = \{(a, f(a) + j) \mid a \in A,\ j \in J,\ f(a) + j \in Q\}$$

for some prime ideal P of A, and some prime ideal Q of B not containing J.

Claim 2. If P is minimal in A, then P^f is minimal in R.

Let H be a prime ideal of R such that $H \subseteq P^f$. Assume that $H := \overline{Q}^f$, for some prime ideal Q of B not containing J. Then, $f^{-1}(Q) \subseteq P$ and so $P = f^{-1}(Q)$ by minimality. Moreover, for any $a \in A$ with $f(a) \in J$, $(a, 0) \in H$ and so $a \in P$. It follows that $f^{-1}(J) \subseteq P$. But, f being surjective yields

$$J = f(f^{-1}(J)) \subseteq f(f^{-1}(Q)) = Q$$

which is absurd. So, necessarily, $H = P_1'^f$, for some prime ideal P_1 of A. Minimality forces $P_1 = P$, and then $H = P^f$, as desired.

Claim 3. If Q is minimal in B with $J \nsubseteq Q$, then $\overline{Q}^f$ is minimal in R.

Let H be a prime ideal of R such that $H \subseteq \overline{Q}^f$. Observe that, for any $j \in J \setminus Q$, $(0, j) \in P^f$ for any prime ideal P of A whilst $(0, j) \notin \overline{Q}^f$. So, necessarily, $H = \overline{Q_1}^f$ for some prime ideal Q_1 of B with $J \nsubseteq Q_1$. Let $x \in Q_1$. Then, $x = f(a)$ for some $a \in A$. Hence, $(a, x) \in H$ and so $x \in Q$. That is, $Q_1 \subseteq Q$. By minimality, $Q_1 = Q$ and therefore $H = \overline{Q}^f$, as desired.

Now, by Claim 1, $Z(R)$ is not an ideal of R. So, if R is non-reduced, then $\mathrm{diam}(\Gamma(R)) = 3$ by [15, Corollary 2.5]. Next, assume that R is reduced. Then, A is reduced and $\mathrm{Nil}(B) \cap J = (0)$ by [9, Proposition 5.4]. Suppose that $\mathrm{diam}(\Gamma(R)) \leq 2$. By [15, Theorem 2.2], R has exactly two minimal prime ideals. If $\mathrm{diam}(\Gamma(A)) = 3$, by [15, Theorem 2.6(4)], A has at least three distinct minimal prime ideals, which lift in R to three distinct minimal prime ideals by Claim 2, absurd. So, necessarily, $\mathrm{diam}(\Gamma(A)) \leq 2$. Since A is reduced and $Z(A)$ is not an ideal, by [15, Theorem 2.2], A has exactly two minimal prime ideals; say, P_1 and P_2. By Claim 2, $(P_1)'^f$, and $(P_2)'^f$ are two distinct minimal prime ideals of R. Further, since $\mathrm{Nil}(B) \cap J = (0)$, there is a minimal prime ideal Q of B such that $J \nsubseteq Q$ (otherwise, $J \subseteq \mathrm{Nil}(B)$ forces J to be null). Hence, by Claim 3, $\overline{Q}^f$ is a minimal prime ideal of R and therefore R has more than two minimal prime ideals, absurd. It follows that $\mathrm{diam}(\Gamma(R)) = 3$, completing the proof of the theorem.

As a first consequence, we recover the result on duplications.

Corollary 3.11 ([16, Theorem 4.12]). *Let D be a ring and I a nonzero proper ideal of D. If $Z(D)$ is not an ideal of D, then diam*$(\Gamma(D \bowtie I)) = 3$. □

Next, we show how one may use Theorem 3.10 to construct original examples of amalgamations R with $\mathrm{diam}(\Gamma(R)) = 3$.

Example 3.12. In this example A is non-reduced. Let $g : \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2$ be the ring homomorphism defined by $g(0) = g(2) = 0$ and $g(1) = g(3) = 1$. Let $A := \mathbb{Z}_2 \times \mathbb{Z}_4$, $B := \mathbb{Z}_2 \times \mathbb{Z}_2$, $J := 0 \times \mathbb{Z}_2$, and $f : A \longrightarrow B$ be the ring homomorphism defined by $f(a,b) = (a, g(b))$. Let $R := A \bowtie^f J$. Clearly, $f^{-1}(J) \neq 0$, f is surjective, and it is easy to check that $f^{-1}(\mathrm{Z}(B) = \mathrm{Z}(A)$ and $\mathrm{Z}(A)$ is not an ideal. By Theorem 3.10, $\mathrm{diam}(\Gamma(R)) = 3$. Moreover, one can check that $|\,\mathrm{Z}(R)^{\star}| = 11$, and $\mathrm{gr}(\Gamma(R)) = 3$ since $((0,0),(0,1)) - ((0,2),(0,0)) - ((1,0),(1,0)) - ((0,0),(0,1))$ is a cycle in $\Gamma(R)$.

Example 3.13. In this example A is reduced. Let $g : \mathbb{Z}_6 \longrightarrow \mathbb{Z}_3$ be the ring homomorphism defined by $g(0) = g(3) = 0$, $g(1) = g(4) = 1$ and $g(2) = g(5) = 2$. Let $A := \mathbb{Z}_6 \times \mathbb{Z}_3$, $B := \mathbb{Z}_3 \times \mathbb{Z}_3$, $J := 0 \times \mathbb{Z}_3$, and $f : A \longrightarrow B$ be the ring homomorphism defined by $f(a,b) = (g(a), b)$. Let $R := A \bowtie^f J$. Clearly, $f^{-1}(J) \neq 0$, f is surjective, and it is easy to check that $f^{-1}(\mathrm{Z}(B)) \subseteq \mathrm{Z}(A)$ and $\mathrm{Z}(A)$ is not an ideal. By Theorem 3.10, $\mathrm{diam}(\Gamma(R)) = 3$.

Another result on duplications asserts that "*if D is a non-reduced ring with $\mathrm{Z}(D)$ not an ideal of D and $I \subseteq \mathrm{Z}(D)$, then* $\mathrm{diam}(\Gamma(D \bowtie I)) = 2$ *provided* $\mathrm{diam}(\Gamma(D) = 2)$" [16, Corollary 4.14]. This result does not carry up to amalgamations, in general, and here are two illustrative examples with A and $f(A) + J$ being non-reduced, respectively.

Example 3.14. This is an example where A is a non-reduced ring, $\mathrm{Z}(A)$ is an ideal of A, $J \subseteq \mathrm{Z}(f(A) + J)$, $\mathrm{diam}(\Gamma(A) = \mathrm{diam}(\Gamma(f(A) + J)) = 2$, but $\mathrm{diam}(R) = 3$. Let $A := \mathbb{Z}_4[X]/(X^2)$ and let x denote the class of X mod (X^2). Then, clearly A is non-reduced and $\mathrm{Z}(A) = \{0, 2, x, 2x, 3x, 2+x, 2+2x, 2+3x\}$ is an ideal of A. Let $B := \mathbb{Z}_4 \times \mathbb{Z}_4$, $J := 0 \times \mathbb{Z}_4$ and $f : A \mapsto B$ be the ring homomorphism defined by $f(a+bx) = (a,a)$. Hence, $f^{-1}(J) \neq 0$, $f(A)+J = B$ and so $\mathrm{diam}(\Gamma(A)) = \mathrm{diam}(\Gamma(B)) = 2$. We also have $J \subseteq \mathrm{Z}(B) = \mathrm{Z}(f(A) + J)$. We claim that $\mathrm{diam}(\Gamma(R)) = 3$. Indeed, $(1,(1,2)) = (1, f(1) + (0,1)) \in R$, $(0,(0,2)) = (0, f(0) + (0,2)) \in R$ and $(1,(1,2))(0,(0,2)) = 0$. So $(1,(1,2)) \in \mathrm{Z}(R)^*$. Also $(x,(0,1)) = (x, f(x)+(0,1)) \in R$, $(3x,(0,0)) = (3x, f(3x)+(0,0)) \in R$ and $(x,(0,1))(3x,(0,0)) = 0$ and so $(x,(0,1)) \in \mathrm{Z}(R)^*$. Finally, notice that $(1,(1,2))(x,(0,1)) = (x,(0,2)) \neq 0$. Also, if $d((1,(1,2)),(x,(0,1)) = 2$, then let $(1,(1,2)) - U = (a+bx, (a,a) + (0,k)) - (x,(0,1))$ be a path in $\Gamma(R)$. Then $a + bx = 0$ forces $a = b = 0$. So $U = (0,(0,k)$. But $(0,(0,k))(x,(0,1)) = 0$ yields $k = 0$ and so $U = 0$, absurd. Hence $d((1,(1,2)),(x,(0,1)) = 3$ and therefore $\mathrm{diam}(\Gamma(R)) = 3$, as desired.

Example 3.15. This is an example where $f(A)+J$ is non-reduced, $\mathrm{Z}(f(A)+J)$ is an ideal of $f(A)+J$, $J \subseteq \mathrm{Z}(f(A)+J)$, $\mathrm{diam}(\Gamma(A) = \mathrm{diam}(\Gamma(f(A)+J)) = 2$, but $\mathrm{diam}(R) = 3$. Let $A := \mathbb{Z}_6$, $B := \mathbb{Z}_2[X]/(X^3)$ and let $J := xB$, where x denotes the class of X mod (X^3). We have $\mathrm{Z}(B) = \{0, x, x^2, x+x^2\} = (x) = J$. Let $f : A \mapsto B$ be the ring homomorphism defined by $f(0) = f(2) = f(4) = 0; f(1) = f(3) = f(5) = 1$. Then, $f^{-1}(J) \neq 0$, $f(A) + J = B$ and so $\operatorname{diam}(\Gamma(A)) = \operatorname{diam}(\Gamma(B)) = 2$. We claim that $\operatorname{diam}(\Gamma(R)) = 3$. Indeed, $(3,1) = (3, f(3)+0) \in R$, $(2,0) = (2, f(2)+0) \in R$ and $(3,1)(2,0) = 0$. Then $(3,1) \in \mathrm{Z}(R)^*$. Also $(2,x) = (2, f(2)+x \in R$, $(0,x^2) = (0, f(0)+x^2) \in R$ and $(2,x)(0,x^2) = 0$. Then $(2,x) \in \mathrm{Z}(R)^*$. Notice that $(3,1)(2,x) = (0,x) \neq (0,0)$. If $d((3,1),(2,x)) = 2$, then let $(3,1) - U = (a, f(a)+j) - (2,x)$ be a path in $\Gamma(R)$. Then $3a = 0 = 2a$ yields $a = 0$. So $U = (0,j)$. But $(3,1)(0,j) = 0$ forces $j = 0$ and so $U = 0$, absurd. Hence $d((3,1),(2,x)) = 3$ and therefore $\operatorname{diam}(\Gamma(R)) = 3$, as desired. Moreover, one can check that $|\,\mathrm{Z}(R)^\star| = 12$, and $\operatorname{gr}(\Gamma(R)) = 3$ since $(0,x) - (0,x^2) - (2,0) - (0,x)$ is a cycle in $\Gamma(R)$.

In this vein, we would like to make the following conjecture:

Conjecture 3.16. Under the above notation, assume that both A and $f(A)+J$ are non-reduced, $Z(A)$ and $Z(f(A)+J)$ are ideals respectively of A and $f(A)+J$, $J \subseteq Z(f(A)+J)$, and $\operatorname{diam}(\Gamma(A)) = \operatorname{diam}(\Gamma(f(A)+J)) = 2$. Then, $\operatorname{diam}(\Gamma(R)) = 2$.

4. On the girth of amalgamations

This section deals with the girth of the zero-divisor graph of the amalgamation $R := A \bowtie^f J$ for various settings of A and $f(A)+J$.

Consider the following conditions:

($\mathcal{C}_5$) For any nonzero distinct elements $i, j \in J$, $ij \neq 0$.
($\mathcal{C}_6$) For any nonzero distinct elements $a, b \in f^{-1}(J)$, $ab \neq 0$.
($\overline{\mathcal{C}_5}$) There are nonzero distinct elements $i, j \in J$ with $ij = 0$.
($\overline{\mathcal{C}_6}$) There are nonzero distinct elements $a, b \in f^{-1}(J)$ with $ab = 0$.

Theorem 4.1. *Under the above notation, assume $f^{-1}(J) \neq 0$.*

1. *If ($\overline{\mathcal{C}_5}$) or ($\overline{\mathcal{C}_6}$) holds, then* $\operatorname{gr}(\Gamma(R)) = 3$.
2. *If "A is a domain and $|J| = 2$" or "$f(A)+J$ is a domain and $|f^{-1}(J)| = 2$", then* $\operatorname{gr}(\Gamma(R)) = \infty$.
3. *If "A is a domain, $|J| \geq 3$, and ($\mathcal{C}_5$) holds" or "$f(A)+J$ is a domain, $|f^{-1}(J)| \geq 3$, and ($\mathcal{C}_6$) holds", then* $\operatorname{gr}(\Gamma(R)) = 4$.

Proof. (1) Let i, j be nonzero distinct elements of J with $ij = 0$ and let $0 \neq a \in f^{-1}(J)$. Clearly, $(0,i) - (a,0) - (0,j) - (0,i)$ is a cycle in $\Gamma(R)$ and so $\operatorname{gr}(\Gamma(R)) = 3$. Let a, b be nonzero distinct elements of $f^{-1}(J)$ with $ab = 0$ and let $0 \neq i \in J$. Then, $(a,0) - (0,i) - (b,0) - (a,0)$ is a cycle in $\Gamma(R)$ and therefore $\operatorname{gr}(\Gamma(R)) = 3$.

(2) Assume that A is a domain and let $J = \{0, i\}$. Suppose that $\Gamma(R)$ contains a cycle of length n, say, $x_1 - x_2 - \cdots - x_n - x_1$ with $x_k := (a_k, f(a_k) + i_k)$ for $k = 1, \ldots, n$. If $a_1 \neq 0$, then $a_2 = a_n = 0$. So $i_2 \neq 0$ and $i_n \neq 0$. Thus

$i_2 = i_n = i$ and so $x_2 = x_n$, absurd. If $a_1 = 0$, then $x_1 = (0, i)$. Since $a_2a_3 = 0$, then either $a_2 = 0$ or $a_3 = 0$, yielding $x_2 = x_1$ or $x_3 = x_1$, absurd. It follows that $\mathrm{gr}(\Gamma(R)) = \infty$.

Next, assume that $f(A)+J$ is a domain and $|f^{-1}(J)| = 2$. Suppose that $\Gamma(R)$ contains a cycle of length n, say, $x_1 - x_2 - \cdots - x_n - x_1$ with $x_k := (a_k, f(a_k)+i_k)$ for $k = 1, \ldots, n$. If $f(a_1) + i_1 \neq 0$, then $f(a_2) + i_2 = f(a_n) + i_n = 0$. So $0 \neq a_2$ and $0 \neq a_n \in f^{-1}(J)$ with $a_2 \neq a_n$, absurd. If $f(a_1) + i_1 = 0$, then $0 \neq a_1 \in f^{-1}(J)$, and since $(f(a_2) + i_2)(f(a_3) + i_3) = 0$, we obtain $0 \neq a_2 \in f^{-1}(J)$ or $0 \neq a_3 \in f^{-1}(J)$, absurd. It follows that $\mathrm{gr}(\Gamma(R)) = \infty$.

(3) Assume that A is a domain. We claim that $|A| \geq 4$. Indeed, if $A = \{0, 1\}$, then $1 \in f^{-1}(J)$ (since $f^{-1}(J) \neq 0$ by hypothesis) which forces $J = B$, absurd. Assume that $|A| = 3$ and set $A = \{0, 1, a\}$. Necessarily, $a \in f^{-1}(J)$ with $a^2 = 1$, hence $1 = (f(a))^2 \in J$, absurd, proving the claim. Next, assume that $|J| \geq 3$ and $(\mathcal{C}_5)$ holds. Let $a \in f^{-1}(J) \setminus \{0, 1\}$ and let $b \in A \setminus \{0, 1, a\}$. Then, $0 \neq ab \in f^{-1}(J)$ with $a \neq ab$. Now, let i, j be two nonzero distinct elements of J. Then $(a, 0) - (0, i) - (ab, 0) - (0, j) - (a, 0)$ is a cycle in $\Gamma(R)$ and so $\mathrm{gr}(\Gamma(R)) \leq 4$. Suppose that $(x, f(x) + i) - (y, f(y) + j) - (z, f(z) + r) - (x, f(x) + i)$ is a cycle in $\Gamma(R)$. If $x \neq 0$, necessarily, $y = z = 0$, whence $j \neq 0$ and $r \neq 0$ with $rj = 0$, absurd. If $x = 0$, then $i \neq 0$. Moreover, if $y = 0$, then $j \neq 0$ with $ij = 0$, absurd; and if $y \neq 0$, then $z = 0$ and so $r \neq 0$, yielding $ir = 0$, absurd. Consequently, $\mathrm{gr}(\Gamma(R)) = 4$.

Next, assume that $f(A)+J$ is a domain, $|f^{-1}(J)| \geq 3$, and $(\mathcal{C}_6)$ holds. Let a, b be two nonzero distinct elements of $f^{-1}(J)$ and let $0 \neq i \in J$. Clearly, $(a, 0) - (0, i) - (b, 0) - (0, i^2) - (a, 0)$ is a cycle in $\Gamma(R)$ and so $\mathrm{gr}(\Gamma(R)) \leq 4$. Suppose that $\mathrm{gr}(\Gamma(R)) = 3$ and let $(a, f(a) + i) - (b, f(b) + j) - (c, f(c) + r) - (a, f(a) + i)$ be a cycle in $\Gamma(R)$. If $f(a) + i \neq 0$, necessarily, $f(b) + j = f(c) + r = 0$, hence b, c are two nonzero distinct elements of $f^{-1}(J)$ with $bc = 0$, absurd. If $f(a) + i = 0$, then $0 \neq a \in f^{-1}(J)$, and $(f(b) + j)(f(c) + r) = 0$ forces $f(b) + j = 0$ or $f(c) + r = 0$, hence $0 \neq b \in f^{-1}(J)$ with $ab = 0$ or $0 \neq c \in f^{-1}(J)$ with $ac = 0$, absurd. Consequently, $\mathrm{gr}(\Gamma(R)) = 4$, completing the proof of the theorem.

The special case where both A and $f(A) + J$ are domains is given below.

Corollary 4.2. *Assume that both A and $f(A)+J$ are domains with $f^{-1}(J) \neq 0$. Then,* $\mathrm{gr}(\Gamma(R)) = 4$.

Proof. The proof is straightforward via Theorem 4.1 since $(\mathcal{C}_5)$ and $(\mathcal{C}_6)$ always hold in the domain setting, and $|J| = 2$ or $|f^{-1}(J)| = 2$ would yield $J = B$ (whereas J is by hypothesis proper).

We recover a well known result on the girth of duplications, as shown below.

Corollary 4.3 ([16, Proposition 3.2]). *Let D be a domain and let I be a nonzero proper ideal of D. Then,* $\mathrm{gr}(\Gamma(D \bowtie I)) = 4$. $\square$

The special case, where neither A nor $f(A) + J$ is a domain, is given below.

Proposition 4.4. *Assume that* $\mathrm{Z}(A) \neq 0$ *and f is injective with* $f^{-1}(J) \neq 0$. *Then,* $\mathrm{gr}(\Gamma(R)) = 3$.

Proof. Let a and b be nonzero elements of A such that $ab = 0$ (possibly, $a = b$) and let $0 \neq c \in f^{-1}(J)$. If $ca = 0$ or $cb = 0$, then $c \in f^{-1}(J) \cap \mathrm{Z}(A)$. If $ca \neq 0$ and $cb \neq 0$, then $ca \in f^{-1}(J) \cap \mathrm{Z}(A)$. Thus, without loss of generality, we assume that $f^{-1}(J) \cap \mathrm{Z}(A) \neq \{0\}$. Next, let $0 \neq x \in f^{-1}(J) \cap \mathrm{Z}(A)$ and let $0 \neq y \in A$ such that $xy = 0$. Since f is injective, $f(x) \neq 0$ and $f(y) \neq 0$. It follows that $(y, f(y)) - (x, 0) - (0, f(x)) - (y, f(y))$ is a cycle in $\Gamma(R)$ and therefore $\mathrm{gr}(\Gamma(R)) = 3$.

We recover a well known result on the girth of duplications, as shown below.

Corollary 4.5 ([16, Proposition 3.1]). *Let D be a ring with* $\mathrm{Z}(D) \neq 0$ *and let I be a nonzero proper ideal of D. Then,* $\mathrm{gr}(\Gamma(D \bowtie I)) = 3$. $\square$

We close this section with some illustrative examples. First, we show how one may use Theorem 4.1 to construct original examples of amalgamations R with $\mathrm{gr}(\Gamma(R)) = 3$, 4, or ∞.

Example 4.6. Let $A := \mathbb{Z}_6$, $B := \mathbb{Z}_6 \times \mathbb{Z}_6$, and $J := 2\mathbb{Z}_6 \times \mathbb{Z}_6$. Consider the ring homomorphism $f : A \to B$, defined by $f(a) = (a, a)$; and let $R := A \bowtie^f J$. Clearly, $f^{-1}(J) = 2A$ and $(\overline{\mathcal{C}_5})$ holds. By Theorem 4.1(1), $\mathrm{gr}(\Gamma(R)) = 3$.

Next, we revisit Example 3.5 to compute the girth using Theorem 4.1.

Example 4.7. Let $A := \mathbb{Z}_4$, $B := \mathbb{Z}_2[X]$, $J := XB$, and $f : A \longrightarrow B$ be the ring homomorphism defined by $f(0) = f(2) = 0$ and $f(1) = f(3) = 1$. Let $R := A \bowtie^f J$. Clearly, $f^{-1}(J) = \{0, 2\}$ and, by Theorem 4.1(2), $\mathrm{gr}(\Gamma(R)) = \infty$.

Example 4.8. Let $A =: \mathbb{Z}$, $B =: \mathbb{Z}_6$, $J = 2B$, and $f : A \longrightarrow B$ be the canonical ring homomorphism. Let $R := A \bowtie^f J$. Clearly, $f^{-1}(J) = 2A$, $J = \{0, 2, 4\}$ and $(\mathcal{C}_5)$ holds. By Theorem 4.1(3), $\mathrm{gr}(\Gamma(R)) = 4$. Moreover, $\mathrm{Z}^\star(J) = \{3\}$. Hence $\mathrm{Z}(R)^\star = \mathbb{E}_1 \cup \mathbb{E}_2 \cup \mathbb{E}_3 = \{(0, 2), (0, 4)\} \cup \{(6n + k, 0) \mid n \in \mathbb{Z}^\star,\ k = 0, 2, 4\} \cup \{(6n + k, 3) \mid n \in \mathbb{Z}^\star,\ k = 1, 3, 5\}$. By Theorem 3.1, $\mathrm{diam}(\Gamma(R)) = 2$.

The next two examples show that the domain assumption in Assertions (2) and (3) of Theorem 4.1 is not superfluous.

Example 4.9. For Assertion (2) of Theorem 4.1, let $A := \mathbb{Z}_4$ and let $B := \mathbb{Z}_2[X]/(X^2)$. Let $J := (x) = \{0, x\}$ (i.e., $|J| = 2$), where x denotes the class of X mod (X^2), and $f : A \longrightarrow B$ be the ring homomorphism defined by $f(0) = f(2) = 0$ and $f(1) = f(3) = 1$. Let $R := A \bowtie^f J$. Clearly, $f^{-1}(J) = \{0, 2\}$ (i.e., $|f^{-1}(J)| = 2$), $f(A) + J = B$, and it is easy to check that $\mathrm{Z}(R)^* = \{(0, x), (2, 0), (2, x)\}$ and $(0, x) - (2, 0) - (2, x) - (0, x)$ is a cycle in $\Gamma(R)$; that is, $\mathrm{gr}(\Gamma(R)) = 3 < \infty$. Notice that $\mathrm{diam}(\Gamma(R)) = 1$.

Example 4.10. For Assertion (3) of Theorem 4.1, consider the amalgamation $R := A \bowtie^f J$ of Example 4.7. Clearly, A is not a domain, $|J| > 3$ (in fact, infinite), $(\mathcal{C}_5)$ holds (since B is a domain) whilst $\mathrm{gr}(\Gamma(R)) = \infty$.

The next example shows that the injectivity assumption in Proposition 4.4 is not superfluous.

Example 4.11. Let $A := \mathbb{Z}_4$, $B := \mathbb{Z}_2 \times \mathbb{Z}_2$, $J := 0 \times \mathbb{Z}_2$, and $f : A \longrightarrow B$ be the ring homomorphism defined by $f(0) = f(2) = 0$ and $f(1) = f(3) = 1$. Let $R := A \bowtie^f J$. Clearly $f^{-1}(J) = \{0, 2\}$ and $f(A) + J = B$. Also, we have

$$\mathrm{Z}(R)^\star = \{x := (0, (0, 1)), y := (1, (1, 0)), z := (2, (0, 0)),$$

$$u := (2, (0, 1)), v := (3, (1, 0))\}.$$

Since $xy = xz = xv = 0$ and $zu = 0$ are the only connected vertices, there is no cycle in $\Gamma(R)$. Consequently, both A and $f(A) + J$ are non-domains with $\mathrm{gr}(\Gamma(R)) = \infty$. Notice that $\mathrm{diam}(\Gamma(R)) = 3$.

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