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CORE OF IDEALS IN INTEGRAL DOMAINS

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To Evan Houston on the occasion of his sixty-fifth birthday

Abstract. This paper uses objects and techniques from multiplicative ideal theory to develop explicit formulas for the core of ideals in various classes of integral domains (not necessarily Noetherian). We also investigate the existence of minimal reductions (originally established by Rees and Sally for local Noetherian rings). All results are illustrated by original examples in Noetherian and non-Noetherian settings, where we explicitly compute the core and validate some open questions recently raised in the literature.

1. Introduction

All rings considered are integral domains (i.e., commutative with identity and without zero-divisors). Let R be a domain and I an ideal of R. An ideal J is a reduction of I if $J \subseteq I$ and $JI^n = I^{n+1}$ for some positive integer n. This notion was introduced by D. G. Northcott and D. Rees [44] and has recently played a crucial role in the study of Rees algebras of ideals. The notion of core of an ideal, denoted core(I), and defined as the intersection of all reductions of I, was introduced by Judith Sally in the late 1980s and was alluded to in Rees and Sally's paper [49]. The core of an ideal naturally appears also in the context of Briancon-Skoda's Theorem; a simple version of which states that if R is a d-dimensional regular ring and I is any ideal of R, then the integral closure of I^d is contained in core(I).

In 1995, Huneke and Swanson [31] determined the core of integrally closed ideals in two-dimensional regular local rings and established a correlation to Lipman's adjoint ideal. Recently, in a series of papers [12, 13, 47], Corso, Polini and Ulrich gave explicit descriptions for the core of certain ideals in Cohen-Macaulay local rings, extending the results of [31]. In 1997, Mohan [43] investigated the core of a module over a two-dimensional regular local ring and was inspired by the original work of Huneke and Swanson. In 2003, Corso, Polini and Ulrich [14] determined the core of projective dimension one modules and recovered, in particular, the result by Mohan. In 2003, Hyry and K. E. Smith [34] generalized the results in [31] to arbitrary dimensions and more general rings. In 2005, Huneke and Trung [33] answered several open questions raised by Corso, Polini and Ulrich. In 2007, Polini, Ulrich, and Vitulli [48] gave some remarkable results on the computation of the core of zero-dimensional monomial ideals. In 2008, Fouli, Polini and Ulrich

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[20] studied the core in arbitrary characteristic and, in 2010, the same authors [21] investigated the annihilators of graded components of the canonical module and the core of standard graded algebras. In this latter paper, for example, the authors characterized Cayley-Bacharach sets of points in terms of the structure of the core of the maximal ideal of their homogeneous coordinate ring. Finally, in 2011, B. Smith [51] established a formula for the core of certain strongly stable ideals that satisfy some local properties and used a result of Polini and Ulrich which showed that the core of such an ideal I is the largest monomial ideal contained in K, for any general minimal reduction K of I.

As the intersection of an a priori infinite number of ideals, the core seems extremely difficult to determine and there are few computed examples in the literature. Little is known about the structure and the properties of core(I) and most of the works on this topic were done in the Noetherian case; precisely, in Cohen-Macaulay rings, where minimal reductions of an ideal exist and have pleasing property of carrying most of the information about the origin ideal.

A domain R is said to have the trace property if for each nonzero ideal I of R, either I is invertible in R or $I(R : I)$ is a prime ideal of R [17, 18, 41]. Valuation domains [2, Theorem 2.8] and pseudo-valuation domains [29, Example 2.12] have the trace property. We notice that, in a domain with the trace property, every ideal I satisfies $I^2I^{-1} \subseteq \mathrm{core}(I)$. Recall that a nonzero ideal I of a domain R is stable (resp., strongly stable) if it is invertible (resp., principal) in its endomorphism ring $(I : I)$ (cf. [2, 37]). One can easily show that a stable ideal I satisfies $I^2I^{-1} \subseteq \mathrm{core}(I)$.

In this paper, we use techniques and objects from multiplicative ideal theory to develop an explicit formula for the core of ideals in various classes of integral domains (not necessarily Noetherian), including valuation and Prüfer domains. Based on the above basic observations, our main goal in the second section is to prove the formula $\mathrm{core}(I) = I^2I^{-1}$ for nonzero ideals I and examine this formula under the effect of (strong) stability. We also investigate the special case of powers of prime ideals. The third section studies the existence of minimal reductions in Noetherian and non-Noetherian settings (originally established by Rees and Sally for local Noetherian rings [49]). Throughout, we provide illustrative examples and answer some questions about the core recently raised in the literature. Any unreferenced material on "multiplicative ideal theory" is standard as in [18, 22], on "commutative ring theory" as in [39], and on "reduction theory" as in [32].

2. Core of ideals

This section establishes explicit formulas for the core of nonzero ideals (resp., prime ideals) in valuation domains (resp., Prüfer domains) and pseudo-valuation domains issued from finite field extensions (resp., arbitrary pseudo-valuation domains). We also investigate the effect of (strong) stability on the core in large classes of domains.

Let R be a domain with quotient field K. For a nonzero (fractional) ideal I of R, let $I^{-1} := (R : I) = \{x \in K \mid xI \subseteq R\}$. The ideal I is invertible in R if $II^{-1} = R$. Notice at this point that invertible ideals (and also idempotent ideals) have no proper reductions. We record this basic fact in the following lemma.

Lemma 2.1. *Let R be a domain and I a nonzero ideal of R. Assume I to be invertible or idempotent. Then* $\mathrm{core}(I) = I$.

Proof. Assume I is invertible and let J be a reduction of I. Then $JI^n = I^{n+1}$ for some positive integer n. By composing the two sides by I^{-1} and using the fact $II^{-1} = R$, we get $JI^{n-1} = I^n$ and by iterating this process n times we obtain $J = I$. The same holds for idempotent ideals; if $I^2 = I$, then $I = I^{n+1} = JI^n = JI \subseteq J \subseteq I$, hence $J = I$. □

Recall that R is a pseudo-valuation domain if R is local and shares its maximal ideal with a valuation overring V or, equivalently, if R is a pullback determined by the following diagram of canonical homomorphisms

$$\begin{array}{ccc} R = \varphi^{-1}(k) & \twoheadrightarrow & k \\ \downarrow & & \downarrow \\ V & \overset{\varphi}{\twoheadrightarrow} & K := \frac{V}{\mathfrak{m}} \end{array}$$

where $\mathfrak{m}$ is the maximal ideal of V and k a subfield of K (cf. [26, 27] and also [4, 5, 11, 15, 46]). A fortiori, $\mathfrak{m}$ is the maximal ideal of R with residual field k [3, Proposition 2.6]. Also recall a basic concept that will be used throughout this paper. A domain R is said to have the trace property if for each nonzero ideal I of R, either I is invertible in R or II^{-1} is a prime ideal of R [18]. Valuation domains [2, Theorem 2.8] and pseudo-valuation domains [29, Example 2.12] & [36, Theorem 15] have the trace property. The next basic lemma examines the effect of this notion on the core of ideals (and their powers).

Lemma 2.2. *Let R be a domain with the trace property, e.g., a (pseudo-) valuation domain. Then*

(a) $I^2I^{-1} \subseteq \operatorname{core}(I)$*, for every nonzero ideal I of R.*
(b) $I^2I^{-1} = \operatorname{core}(I)$ *for every nonzero ideal I of R if and only if* $\operatorname{core}(I^n) = I^{n+1}I^{-1}$ *for every nonzero ideal I of R and every integer $n \geq 1$.*

Proof. (a) Let J be a reduction of I, then $JI^n = I^{n+1}$ for some $n \geq 1$. By [29, Remark 2.13 (b)], the trace property implies

$$I^n(I^n)^{-1} = I^nI^{-n} = II^{-1}. \tag{1}$$

Therefore $I^2I^{-1} = III^{-1} = II^nI^{-n} = I^{n+1}I^{-n} = JI^nI^{-n} \subseteq J$ and hence $I^2I^{-1} \subseteq \operatorname{core}(I)$.

(b) We only need to check the "only if" assertion. Let n be an integer ≥ 1. We have $\operatorname{core}(I^n) = (I^n)^2(I^n)^{-1} = I^{2n}I^{-n}$. By the trace property, $I^nI^{-n} = II^{-1}$. It follows that $\operatorname{core}(I^n) = I^{2n}I^{-n} = I^nII^{-1} = I^{n+1}I^{-1}$, as desired. □

The first main result of this section gives a formula for the core of ideals in valuation domains. In view of Lemma 2.1, we restrict to non-invertible ideals. Throughout $Z(R,I)$ will denote the set of all zero-divisors of a domain R modulo an ideal I.

Theorem 2.3. *Let V be a valuation domain and I a non-invertible ideal of V. Then*

$$\operatorname{core}(I) = I^2I^{-1} = IZ(V,I).$$

Proof. Since $II^{-1} \subsetneqq V$, the trace property ensures that $Q := II^{-1}$ is a prime ideal of V and by [2, Theorem 2.8]

$$(I : I) = V_Q. \tag{2}$$

By Lemma 2.2, $IQ \subseteq \operatorname{core}(I)$. Suppose by way of contradiction that $IQ \subsetneqq \operatorname{core}(I)$ and let $x \in \operatorname{core}(I) \setminus IQ$. Since V is a valuation domain, $IQ \subset xV$. So $x^{-1}IQ \subset V$ and by [30, Corollary 3.6]

$$x^{-1}I \subseteq (V : Q) = Q^{-1} = (Q : Q) = V_Q.$$

By (2), I is an ideal of V_Q and so is $x^{-1}I$. Therefore either $x^{-1}I = V_Q$ or $x^{-1}I \subseteq Q$, the maximal ideal of V_Q. The latter case is ruled out since $x \in \mathrm{core}(I) \subseteq I$. So $x^{-1}I = V_Q$, that is, $I = xV_Q$. Let $\mathfrak{m}$ denote the maximal ideal of V. Necessarily, $Q \subsetneqq \mathfrak{m}$, since $Q = \mathfrak{m}$ would yield $I = xV$, absurd (recall $II^{-1} \subsetneqq V$). Let $m \in \mathfrak{m} \setminus Q$ and set $J := mxV$. Then

$$JI = JIV_Q = I(xV_Q) = I^2$$

so that J is a reduction of I. Hence $x \in J$ and therefore $1 \in mV \subseteq \mathfrak{m}$, the desired contradiction. Consequently,

$$\mathrm{core}(I) = IQ = I^2I^{-1}.$$

It remains to show that $I^2I^{-1} = IZ(V,I)$. By [29, Lemma 2.3], $P := Z(V,I)$ is a prime ideal of V and $(I : I) = V_P$. By (2), $II^{-1} = Q = P$. Therefore $\mathrm{core}(I) = I^2I^{-1} = IQ = IP$, as desired. □

Two of the open questions addressed by Huneke and Swanson in [31] were: "*How does* $\mathrm{core}(I^n)$ *compare to* $\mathrm{core}(I)$*?*" and "*If I and J are two integrally closed ideals and $I \subseteq J$, is* $\mathrm{core}(I) \subseteq \mathrm{core}(J)$*?*" They proved, in the context of two-dimensional regular local rings with infinite residue field, that the latter statement always holds [31, Proposition 3.15] and $\mathrm{core}(I^n) = I^{2n-2}\,\mathrm{core}(I)$ for any integrally closed ideal I [31, Proposition 4.4]. For an ideal I of a domain R, the integral closure of I [32] is the ideal $\bar{I}$ of all elements x of R that satisfies an equation of the form $x^n + a_1x^{n-1} + \cdots + a_n = 0$ where $a_i \in I^i$ for $i = 1, \cdots, n$; and I is integrally closed if $I = \bar{I}$. It is well-known that every ideal in a valuation domain is integrally closed [32, Proposition 6.8.1].

In this vein, Theorem 2.3 offers complete answers to the aforementioned questions in the context of valuation domains. Indeed, we have $\mathrm{core}(I^n) = I^{2n}I^{-n} = I^nI^nI^{-n} = I^nII^{-1} = I^{n-1}I^2I^{-1} = I^{n-1}\,\mathrm{core}(I)$, for every nonzero ideal I and each $n \geq 1$. Moreover, notice that if I is an invertible (a fortiori, principal) ideal of $(V, \mathfrak{m})$, one may easily check that $Z(V,I) = \mathfrak{m}$. Hence $IZ(V,I) = I\mathfrak{m} \subsetneqq I = \mathrm{core}(I)$. However, if P is a non-invertible prime ideal of V, then $\mathrm{core}(P) = PZ(V,P) = P^2$. The next example uses this fact to show that the notion of core is not stable under inclusion in valuation domains.

Example 2.4. Let k be a field and let X, Y be two indeterminates over k. Consider the domain $V := k[[X]] + P$, where $P := Yk((X))[[Y]]$. Recall for convenience that V is a valuation domain since it arises as a pullback issued from the valuation domain k((X))[[Y]] (cf. [6, Theorem 2.1(h)]). Clearly, we have $YV \subsetneqq P$. Further, by [30, Corollary 3.6 and Theorem 3.8], $PP^{-1} = P$, so that P is a non-invertible prime ideal of V. By Theorem 2.3, $\mathrm{core}(P) = P^2 = Y^2k((X))[[Y]] \subsetneqq YV = \mathrm{core}(YV)$.

Let R be a Prüfer domain and I a nonzero ideal of R. For any reduction J of I, $JI = I^2$ [25, Proposition 1]. So $I^2I^{-1} = JII^{-1} \subseteq J$. Hence $I^2I^{-1} \subseteq \mathrm{core}(I)$. This inequality can be strict as shown by the next example (i.e., Theorem 2.3 doesn't extend to Prüfer domains). For this purpose, recall Hays' result that, for a domain R, every ideal has no proper reduction if and only if R is a one-dimensional Prüfer domain [24, Theorem 6.1] and [25, Theorem 10].

Example 2.5. Let R be an almost Dedekind domain which is not Dedekind [22, Example 42.6]. By [22, Theorem 36.5], R is a one-dimensional Prüfer domain with no idempotent maximal ideals. Clearly, R contains a non-invertible maximal ideal

$\mathfrak{m}$. Hence $\mathfrak{m}^{-1} = (\mathfrak{m} : \mathfrak{m}) = R$ since R is completely integrally closed. By Hays' aforementioned result, we get

$$\mathfrak{m}^2\mathfrak{m}^{-1} = \mathfrak{m}^2 \subsetneq \mathfrak{m} = \mathrm{core}(\mathfrak{m})$$

as desired.

Next, we establish an explicit formula for the core of powers of nonzero prime ideals of a Prüfer domain.

Theorem 2.6. *Let R be a Prüfer domain, P a nonzero prime ideal of R, and n a positive integer. Then*

$$\mathrm{core}(P^n) = \begin{cases} P^n, & \text{if } P \text{ is maximal} \\ P^{n+1}, & \text{if } P \text{ is not maximal} \end{cases}$$

Proof. We first prove the result in the local case, i.e., R is assumed to be a valuation domain. If P is maximal, then either P is idempotent or principal, and so is P^n. By Lemma 2.1, $\mathrm{core}(P^n) = P^n$. If P is not maximal, $P^{-1} = (P : P)$ by [30, Corollary 3.6 and Theorem 3.8]. Hence $PP^{-1} = P$. By (1), we get $P^nP^{-n} = PP^{-1} = P$. By Theorem 2.3, $\mathrm{core}(P^n) = (P^n)^2P^{-n} = P^nP = P^{n+1}$, as desired.

Assume R is Prüfer. Let J be a reduction of P^n. By [25, Proposition 1], we have

$$JP^n = P^{2n} \text{ and then } P^{2n} \subseteq J \subseteq P^n. \tag{3}$$

Suppose that P is maximal in R. By (3), $\sqrt{J} = P$ and thus J is P-primary. Moreover, JR_P is a reduction of P^nR_P in R_P and so $JR_P = P^nR_P$ by the first step. Let $x \in P^n$. Then $xu \in J$ for some $u \in R \setminus P$. Therefore $x \in J$ and hence $P^n \subseteq J$, whence $\mathrm{core}(P^n) = P^n$. Suppose that P is not maximal in R. We claim that $P^{n+1} \subseteq J$. Indeed it suffices to check it locally. Let $\mathfrak{m}$ be a maximal ideal of R. If $P \nsubseteq \mathfrak{m}$, then $J \nsubseteq \mathfrak{m}$ by (3). Therefore $P^{n+1}R_\mathfrak{m} = R_\mathfrak{m} = JR_\mathfrak{m}$. If $P \subseteq \mathfrak{m}$, then $JR_\mathfrak{m}$ is a reduction of $P^nR_\mathfrak{m}$ in $R_\mathfrak{m}$. Further, $PR_\mathfrak{m}$ is a non-maximal prime ideal of $R_\mathfrak{m}$. By the first step, $P^{n+1}R_\mathfrak{m} \subseteq JR_\mathfrak{m}$, as claimed. Consequently, we get

$$P^{n+1} \subseteq \mathrm{core}(P^n).$$

Conversely, let $\mathfrak{m}$ be a maximal ideal of R containing P. Then $PR_P = PR_\mathfrak{m}$ and hence $P^nR_P = P^nR_\mathfrak{m}$. Let J be a reduction of $P^nR_\mathfrak{m}$. By (3), we get

$$JP^nR_\mathfrak{m} = P^{2n}R_\mathfrak{m} \text{ and then } P^{2n} \subseteq J \cap R \subseteq P.$$

We claim that $I := J \cap R$ is a reduction of P^n. Indeed, let N be a maximal ideal of R. If $P \nsubseteq N$, then $I \nsubseteq N$ and so

$$IR_N = R_N = P^nR_N \text{ and } IP^nR_N = R_N = P^{2n}R_N.$$

If $P \subseteq N$, then

$$IR_N \subseteq (P^nR_\mathfrak{m} \cap R)R_N = (P^nR_p \cap R)R_N = P^nR_N \text{ and}$$

$$IP^nR_N = IP^nR_P = IP^nR_\mathfrak{m} = JP^nR_\mathfrak{m} = P^{2n}R_\mathfrak{m} = P^{2n}R_P = P^{2n}R_N.$$

It follows that $I \subseteq P^n$ and $IP^n = P^{2n}$, as claimed. Therefore $\mathrm{core}(P^n) \subseteq J \cap R$. Set $\Delta := \{\text{all reductions of } P^nR_\mathfrak{m}\}$. By the first step, we obtain

$$\mathrm{core}(P^n) \subseteq \bigcap_{J\in\Delta}(J \cap R) = (\bigcap_{J\in\Delta} J) \cap R = P^{n+1}R_\mathfrak{m} \cap R.$$

Thus, $\mathrm{core}(P^n) \subseteq P^{n+1}R_{\mathfrak{m}}$, for each maximal ideal $\mathfrak{m}$ of R with $P \subseteq \mathfrak{m}$. Moreover, for each maximal ideal $\mathfrak{m}$ of R with $P \nsubseteq \mathfrak{m}$, we obviously have

$$\mathrm{core}(P^n) \subseteq P^n \subseteq R_{\mathfrak{m}} = P^{n+1}R_{\mathfrak{m}}.$$

Consequently, $\mathrm{core}(P^n) \subseteq P^{n+1}$ and therefore $\mathrm{core}(P^n) = P^{n+1}$, completing the proof of the theorem. □

Now, we are ready to provide an example of a (Prüfer) domain R with an ideal I such that $I^2I^{-1} \subsetneq \mathrm{core}(I) \subsetneq I$. This example is based on [19, Example 8.4.1].

Example 2.7. Let E be the (Bézout) ring of entire functions, $\mathfrak{m}$ a maximal ideal of E of infinite height, V a non-trivial valuation domain on K, and R the pullback determined by the following diagram

$$\begin{array}{ccc} R := \varphi^{-1}(V) & \twoheadrightarrow & V \\ \downarrow & & \downarrow \\ E & \stackrel{\varphi}{\twoheadrightarrow} & K := \frac{E}{\mathfrak{m}} \end{array}$$

where φ is the canonical homomorphism and the vertical arrows are inclusion maps. Then $I := \mathfrak{m}^2$ is an ideal of R satisfying

$$I^2I^{-1} \subsetneq \mathrm{core}(I) \subsetneq I.$$

Proof. By [19, Example 8.4.1], R is a Prüfer domain, $\mathfrak{m}$ is a non-maximal prime ideal of R with $\mathfrak{m}^2 \subsetneq \mathfrak{m}$ and $(R : \mathfrak{m}) = (E : \mathfrak{m}) = (\mathfrak{m} : \mathfrak{m}) = E$, and $P := \bigcap_{n\geq 1} \mathfrak{m}^n$ is a nonzero prime ideal of R properly contained in $\mathfrak{m}$. Now, by Theorem 2.6, $\mathrm{core}(I) = \mathrm{core}(\mathfrak{m}^2) = \mathfrak{m}^3$ and $I^2I^{-1} = \mathfrak{m}^4(R : \mathfrak{m}^2) = \mathfrak{m}^4((R : \mathfrak{m}) : \mathfrak{m}) = \mathfrak{m}^4(E : \mathfrak{m}) = \mathfrak{m}^4E = \mathfrak{m}^4$. Moreover, we claim that $\mathfrak{m}^{n+1} \subsetneq \mathfrak{m}^n$, for every positive integer n. Deny and assume $\mathfrak{m}^{n+1} = \mathfrak{m}^n$ for some n. Then, by induction on k, we have $\mathfrak{m}^n = \mathfrak{m}^k$ for all $k \geq n$. So $P = \mathfrak{m}^n$ which yields $P = \mathfrak{m}$, absurd. It follows that $\mathfrak{m}^4 \subsetneq \mathfrak{m}^3 \subsetneq \mathfrak{m}^2$, as desired. □

The formula in Theorem 2.3 holds for the class of pseudo-valuation domains issued from algebraic field extensions, as shown by the next result.

Theorem 2.8. *Let R be a pseudo-valuation domain, V its associated valuation overring, and $\mathfrak{m}$ its maximal ideal. Then the following statements are equivalent:*

(a) $\dfrac{V}{\mathfrak{m}}$ *is an algebraic extension of* $\dfrac{R}{\mathfrak{m}}$;
(b) $\mathrm{core}(I) = I^2I^{-1}$, *for every nonzero ideal I of R.*

Proof. (a) $\Longrightarrow$ (b) In view of Theorem 2.3, we may assume $R \subsetneq V$. Let's envisage two cases. *Case* 1*: I is an ideal of V.* Let $\mathrm{core}_V(I)$ denote the core of I in V. Since R has the trace property, by Lemma 2.2(a) and Theorem 2.3, we get

$$I^2I^{-1} \subseteq \mathrm{core}(I) \subseteq \mathrm{core}_V(I) = I^2(V : I). \tag{4}$$

If I is not invertible in V, then $I(V : I) \subseteq \mathfrak{m}$. So

$$(V : I) \subseteq (\mathfrak{m} : I) \subseteq I^{-1} \subseteq (V : I); \text{ that is, } I^{-1} = (V : I). \tag{5}$$

Whence $I^2(V : I) = I^2I^{-1}$. By (4), $\mathrm{core}(I) = I^2I^{-1}$. Next, assume that I is invertible in V, i.e., $I = aV$ for some nonzero $a \in I$. So

$$I^2I^{-1} = I^2(R : I) = I^2(R : aV) = I^2a^{-1}(R : V) = I^2a^{-1}\mathfrak{m} = a\mathfrak{m}. \tag{6}$$

Set $J := aR$. Since $JV = I$, then J is a reduction of I. Hence $\mathrm{core}(I) \subseteq J$. Let $z \in \mathrm{core}(I)$. Then $z = ay$ for some $y \in R$. We claim that $y \in \mathfrak{m}$. Deny and assume $y \notin \mathfrak{m}$. Since $(R, \mathfrak{m})$ is local, then $y^{-1} \in R$. Let $u \in V \setminus R$ and set

$$B := uaR \subset aV = I.$$

Necessarily, $u \notin \mathfrak{m}$ and hence $u^{-1} \in V$. So $BV = I$ and then B is a reduction of I. Then $ya = z \in uaR$. Therefore $yu^{-1} \in R$ and then $u^{-1} \in R$. So $u^{-1} \in \mathfrak{m}$, the desired contradiction. It follows that $y \in \mathfrak{m}$ and hence $z = ay \in a\,\mathfrak{m}$. By (6),

$$\mathrm{core}(I) \subseteq a\,\mathfrak{m} = I^2I^{-1} \subseteq \mathrm{core}(I)$$

and thus $\mathrm{core}(I) = I^2I^{-1}$.

Case 2*: I is not an ideal of V*. Let $k := \frac{R}{\mathfrak{m}}$ and $K := \frac{V}{\mathfrak{m}}$. Then

$$I = a\varphi^{-1}(W)$$

for some nonzero $a \in I$ and k-vector space W such that $k \subseteq W \subsetneqq K$ (cf. [6, Theorem 2.1(n)]), where φ denotes the canonical homomorphism from V onto K. If $k = W$, then $I = a\varphi^{-1}(k) = aR$, hence $\mathrm{core}(I) = I = I^2I^{-1}$. Next, suppose that $k \subsetneqq W$. Then $(k : W) = 0$. Hence

$$I^2I^{-1} = I^2a^{-1}\varphi^{-1}(k : W) = I^2a^{-1}\varphi^{-1}(0) = I^2a^{-1}\,\mathfrak{m} = a\,\mathfrak{m}. \tag{7}$$

Let $\{1, \omega_\alpha\}_{\alpha\in\Omega}$ be a basis of W as a k-vector space. For each $\alpha \in \Omega$, let

$$r_\alpha := [k(\omega_\alpha) : k] \text{ and } F_\alpha := k + \sum_{\alpha\neq\beta\in\Omega} k\omega_\beta.$$

Clearly $W = F_\alpha + k\omega_\alpha$ and, since ω_α is algebraic over k, we have

$$k\omega_\alpha^{r_\alpha} \subseteq \sum_{i=0}^{r_\alpha-1} k\omega_\alpha^i \subseteq \sum_{i=0}^{r_\alpha-1} F_\alpha^{r_\alpha-i}\omega_\alpha^i = F_\alpha W^{r_\alpha-1}.$$

It follows that $W^{r_\alpha} \subseteq F_\alpha W^{r_\alpha-1}$ and hence $F_\alpha W^{r_\alpha-1} = W^{r_\alpha}$. Therefore $J_\alpha := a\varphi^{-1}(F_\alpha)$ is a reduction of I since $JI^{r_\alpha-1} = I^{r_\alpha}$. By (7), we get

$$a\,\mathfrak{m} \subseteq \mathrm{core}(I) \subseteq \bigcap_{\alpha\in\Omega} J_\alpha = a\varphi^{-1}(\bigcap_{\alpha\in\Omega} F_\alpha) = a\varphi^{-1}(k) = aR \tag{8}$$

Suppose by way of contradiction that $a\,\mathfrak{m} \subsetneqq \mathrm{core}(I)$ and let $x \in \mathrm{core}(I) \setminus a\,\mathfrak{m}$. By (8), $x = ay$ for some $y \in R \setminus \mathfrak{m}$. Since R is local with maximal ideal $\mathfrak{m}$, $a = xy^{-1} \in \mathrm{core}(I)$, whence $\mathrm{core}(I) = aR$. Now let

$$F := \sum_{\alpha\in\Omega} k\omega_\alpha.$$

We claim that $FW^{r_\alpha-1} = W^{r_\alpha}$ for each $\alpha \in \Omega$. Indeed, let $\alpha \in \Omega$. Then

$$W^{r_\alpha} = (k + F)^{r_\alpha} = k + \sum_{i=1}^{r_\alpha} F^i.$$

On the other hand, we have

$$FW^{r_\alpha-1} = F(k + \sum_{i=1}^{r_\alpha-1} F^i) = \sum_{i=1}^{r_\alpha} F^i.$$

Since $[k(\omega_\alpha):k]=r_\alpha$, 1 can be written as a combination of $(\omega_\alpha^i)_{1\leq i\leq r_\alpha}$; that is

$$k\subseteq\sum_{i=1}^{r_\alpha}F^i.$$

So $FW^{r_\alpha-1}=W^{r_\alpha}$, as claimed. It follows that $J:=a\varphi^{-1}(F)$ is a reduction of I and hence $a\varphi^{-1}(k)=aR=\mathrm{core}(I)\subseteq J=a\varphi^{-1}(F)$. Whence $k\subseteq F$, the desired contradiction. Therefore, $\mathrm{core}(I)=a\,\mathfrak{m}=I^2I^{-1}$.

(b) $\Longrightarrow$ (a) Suppose that K is transcendental over k and let x be a transcendental element of K over k. Let $W:=k+kx$ and consider the ideal of R given by $I:=a\varphi^{-1}(W)$, where $0\neq a\in\mathfrak{m}$ and φ denotes the canonical homomorphism from V onto K. Then

$$I^2I^{-1}\subsetneqq\mathrm{core}(I)$$

Indeed, let J be a reduction of I. Then for some positive integer n we have

$$JI^n=I^{n+1}\tag{9}$$

Hence

$$a^nJV=JI^nV=I^{n+1}V=a^{n+1}V.$$

Since $W\subsetneqq K$ we obtain

$$JV=IV=aV\supsetneqq a\varphi^{-1}(W)=I\supseteq J.\tag{10}$$

Thus J is not an ideal of V. By [6, Theorem 2.1(n)]

$$J=b\varphi^{-1}(F)\tag{11}$$

for some $0\neq b\in J$ and k-vector space F with $k\subseteq F\subsetneqq K$. Combine (10) with (11) to get $aV=bV$, that is, $a^{-1}b$ is a unit in V. Then $\mu:=\varphi(a^{-1}b)$ is a nonzero element of K. Using the fact $a^{-1}b\,\mathfrak{m}=\mathfrak{m}$, one can easily check that $a^{-1}b\varphi^{-1}(F)=\varphi^{-1}(\mu F)$. So that

$$J=a\varphi^{-1}(\mu F).\tag{12}$$

We claim that $k\subsetneqq F$. Deny. By (9) and (12), we get

$$\mu W^n=\mu kW^n=W^{n+1}\subset k[x].\tag{13}$$

Therefore

$$\mu\in(W^{n+1}:W^n)=W\subset k[x].\tag{14}$$

By (13) and (14), μ is invertible in $k[x]$ (since $1\in W^{n+1}$); that is, $\mu\in k$. It follows that $W^{n+1}=W^n$, the desired contradiction. Then $k\subsetneqq F\subseteq\mu^{-1}W$ and hence $\mu F=W$ since $\dim_k(\mu^{-1}W)=\dim_k(W)=2$. So $J=I$ and therefore I has no proper reduction. Finally, (7) leads to $I^2I^{-1}=a\,\mathfrak{m}\subsetneqq a\varphi^{-1}(W)=I=\mathrm{core}(I)$, as desired. This completes the proof of the theorem. ◻

Remark 2.9. In the first case envisaged in the proof of (a) ⇒ (b), we do not make use of the algebraicity of K over k. Consequently, "*if R is a pseudo-valuation domain issued from V, then* $\mathrm{core}(I)=I^2I^{-1}$ *for every common nonzero ideal I of R and V.*"

Next, we establish an explicit formula for the core of powers of nonzero prime ideals of a (non-trivial) arbitrary pseudo-valuation domain.

Proposition 2.10. *Let R be a non-trivial pseudo-valuation domain, P a nonzero prime ideal of R, and n a positive integer. Then*

$$\mathrm{core}(P^n)=P^{n+1}.$$

Proof. Let V be the associated valuation overring of R and $\mathfrak{m}$ its maximal ideal. In view of Lemma 2.1, we may assume that P is not idempotent. It follows that $P = PV_P$ is principal in V_P; that is, $P = aV_P$ for some nonzero $a \in P$. Let $J := a^nR$. Since $JV_P = P^n$, then J is a reduction of P^n. So $\mathrm{core}(P^n) \subseteq J$. Now, let $z \in \mathrm{core}(P^n)$. Then

$$z = a^n y$$

for some $y \in R$. We claim that $y \in P$. Deny. Then $y^2 \notin P$. Let $A := y^2a^nR$. Since $AV_P = P^n$, A is a reduction of P^n. Hence $\mathrm{core}(P^n) \subseteq A$, whence $z \in A$. Therefore $y^{-1} \in R$. Since R is not a valuation domain, there exists $u \in V \setminus R$. Let $F := ua^nR$. Clearly, $FV_P = P^n$. So F is a reduction of P^n. Then $z \in F$ and hence $y \in uR$, i.e., $yu^{-1} \in R$. It follows that $u^{-1} \in R$ and necessarily $u^{-1} \in \mathfrak{m} \subseteq V$, the desired contradiction. Consequently, $y \in P$ and thus

$$\mathrm{core}(P^n) \subseteq P^{n+1}.$$

Conversely, let J be a reduction of P^n. By Theorem 2.6, $JV_P = P^n$. Hence $P^{n+1} = JP \subseteq J$. So $P^{n+1} \subseteq \mathrm{core}(P^n)$ and therefore $\mathrm{core}(P^n) = P^{n+1}$. □

Recall that a nonzero ideal I of a domain R is stable (resp., strongly stable) if it is invertible (resp., principal) in its endomorphism ring $T := (I : I)$ (cf. [2, 37]). Sally and Vasconcelos [50] used stability to settle Bass' conjecture on one-dimensional Noetherian rings with finite integral closure. Recent developments on this notion, due to Olberding [45], prepared the ground to address the correlation between stability and several concepts in multiplicative ideal theory [10, 23, 38]. Next, we examine the effect of (strong) stability on the core of ideals.

Lemma 2.11. *Let R be a domain and I a stable ideal of R. Then:*

(a) *J is a reduction of I if and only if $JI = I^2$ if and only if $JT = I$.*
(b) $I^2I^{-1} \subseteq \mathrm{core}(I)$.

Proof. (a) Assume that J is a reduction of I. Then $JI^n = I^{n+1}$ for some positive integer $n \geq 1$. By stability, we obtain

$$I^n = I^{n+1}(T : I) = JI^n(T : I) = JI^{n-1}.$$

By iterating this process, we get $I^2 = JI$. The latter equality yields $JT = JI(T : I) = I^2(T : I) = I$. Finally, $JT = I$ obviously forces J to be a reduction of I, as desired.

(b) By (a), $I^2I^{-1} = JII^{-1} \subseteq J$ for any reduction J of I. So $I^2I^{-1} \subseteq \mathrm{core}(I)$. □

Theorem 2.12. *Let R be a domain and I a strongly stable ideal of R, i.e., $I := a(I : I)$ for some nonzero element a of I. Set $T := (I : I)$, $Q := (R : T)$, and $P := \bigcap_{F \in \Sigma} F$ where $\Sigma :=$ {all finitely generated R-modules F with $FT = T$}. Then*

$$aQ = I^2I^{-1} \subseteq \mathrm{core}(I) = aP.$$

Moreover, if any one of the following conditions holds

- *Q is maximal in R*
- *R is Prüfer*
- *T is local*

then $\mathrm{core}(I) = I^2I^{-1}$.

Proof. The first equality is straightforward and the inclusion is ensured by Lemma 2.11. Next, we prove $core(I) = aP$. For any $F \in \Sigma$, $aF \subseteq I$ and $aFT = I$ so that aF is a reduction of I. Now, let J be a reduction of I. By Lemma 2.11, $JT = I$. Hence $a^{-1}JT = T$, whence $1 = \Sigma_{i=1}^{n} e_i u_i$ for some positive integer n and $e_i \in a^{-1}J$ and $u_i \in T$ for each i. Let $F := \Sigma_{i=1}^{n} Re_i$. Clearly, $F \in \Sigma$ and $aF \subseteq J$. Consequently, each reduction of I contains a reduction of the form aF where F ranges over Σ. It follows that

$$\mathrm{core}(I) = \bigcap_{F \in \Sigma} aF = a\Big(\bigcap_{F \in \Sigma} F\Big) = aP.$$

Notice at this point that P is an ideal of R with $Q \subseteq P$. Next, assume that Q is maximal in R. Necessarily, $R \subsetneqq T$. Suppose by way of contradiction that $P = R$ and consider the following pullback diagram of canonical homomorphisms

$$\begin{array}{ccc} R = \varphi^{-1}(k) & \twoheadrightarrow & k := \frac{R}{Q} \\ \downarrow & & \downarrow \\ T & \overset{\varphi}{\twoheadrightarrow} & \frac{T}{Q} \end{array}$$

We claim that for every $u \in T \setminus R$, $\varphi(u)$ is not algebraic over k. Deny. Let $u \in T \setminus R$ such that $\omega := \varphi(u)$ is algebraic over k and set

$$r := [k(\omega) : k] \text{ and } W := \sum_{i=1}^{r-1} k\omega^i.$$

One can check that $W\frac{T}{Q} = \frac{T}{Q}$ and hence $\varphi^{-1}(W)T = T$. Thus $\varphi^{-1}(W) \in \Sigma$ and then $R = \varphi^{-1}(k) \subseteq \varphi^{-1}(W)$. Hence $k \subseteq W$, absurd. Now, let $u \in T \setminus R$ and set

$$F := (1 - u, u^2)R.$$

Clearly, $FT = T$. Therefore $F \in \Sigma$ and then $R \subseteq F$. Hence $\alpha u^2 - \beta u + \beta - 1 = 0$ for some $\alpha, \beta \in R$. Obviously, α or $\beta \notin Q$. It follows that $\varphi(u)$ is algebraic over k, the desired contradiction. It follows that $Q = P$, as desired

Let $x \in P$ and $0 \neq u \in T$. Assume R is a Prüfer domain. Then $(u^{-1}R \cap R)T = u^{-1}T \cap T$. Therefore $F := u^{-1}R \cap R \in \Sigma$ and hence $x \in F$. Whence $xu \in R$. Assume T is a local domain. If $u^{-1} \in T$, then $u^{-1}R \in \Sigma$ and hence $xu \in R$. If $u^{-1} \notin T$, then $(1 + u)^{-1} \in T$. So a similar argument yields $x(1 + u) \in R$, hence $xu \in R$. In both cases, we get $P \subseteq Q$, leading to the conclusion. □

In the next section, we will provide two illustrative examples for Theorem 2.12; namely, Example 3.1 (Noetherian context) and Example 3.8.

3. Minimal reductions

Recall from [24, 44] that an ideal which has no reduction other than itself is called a basic ideal. In [49], Rees and Sally considered the intersection of minimal reductions (i.e., the core) in order to counteract the lack of uniqueness of minimal reductions.

Let $(R, \mathfrak{m})$ be a Noetherian local ring (i.e., not necessarily a domain) and I a non-basic ideal of R, Northcott and Rees proved that I admits at least one minimal reduction (with respect to inclusion) [44, Section 2, Theorem 1] or [32, Theorem 8.3.6]. This reduction is not unique in general. Indeed, if the residue field is infinite, they proved that any l non-special elements of I generate a minimal reduction of I, where l denotes the analytic spread of I [44, Section 5, Theorem 1]. In other words, in a Noetherian local ring with infinite residue field, we have:

$$\text{"core}(I) \textit{ is a reduction of } I \textit{ if and only if } I \textit{ is basic."} \tag{15}$$

This section studies the existence of minimal reductions as well as the validity of the above fact beyond the context of Noetherian local rings.

As a contrast to (15) , we start by providing an example of a Cohen-Macaulay domain where we use Theorem 2.12 to compute explicitly the core as well as all minimal reductions for one of its (non-basic) maximal ideals. Moreover, this example validates a conjecture by Corso, Polini, and Ulrich [13, Conjecture 5.1] which was mentioned (and studied) later in [33, 47, 52]. The conjecture sustains that if R is a Cohen-Macaulay ring, I is an ideal of R of analytic spread ≥ 1 (subject to some additional assumptions), and J is a minimal reduction of I with reduction number r, then

$$\mathrm{core}(I) = (J^{r+1} : I^r). \tag{16}$$

For this purpose, let $\mathrm{Red}(I)$ (resp., $\mathrm{MinRed}(I)$) denote the set of all reductions (resp., minimal reductions) of an ideal I, and $|A|$ denote the cardinality of a set A.

Example 3.1. Let k be an arbitrary field and X an indeterminate over k. Let $R := k[X^2, X^3]$ and $I := (X^2, X^3)$. Then:

(a) $\mathrm{core}(I) = X^2 I = X^4 k[X] = I^2$.
(b) Every proper (minimal) reduction of I has reduction number 1 and has the form
$$J_x := X^2(1 + xX, X^2),\ x \in k.$$
(c) $\big|\mathrm{Red}(I)\big| = \big|\mathrm{MinRed}(I)\big| + 1 = \big|k\big| + 1$.
(d) $\forall x \in k$, $\mathrm{core}(I) = (J_x^2 : I)$.

The proof requires the following result, known in the literature, but with no clear reference. So we offer here a proof. Recall that a domain D is said to be divisorial if each of its nonzero ideals is divisorial. These domains have been studied by Bass [7], Matlis [42], Heinzer [28], Bazzoni and Salce [8], and Bazzoni [9].

Lemma 3.2. *Let k be a field and X an indeterminate over k. Then $R := k[X^2, X^3]$ is a divisorial domain.*

Proof. Note that R is a (one-dimensional) Noetherian domain. So R is divisorial if and only if $R_\mathfrak{m}$ is divisorial, for every $\mathfrak{m} \in \mathrm{Max}(R)$. Let $\mathfrak{m} \in \mathrm{Max}(R)$ with $\mathfrak{m} \neq (X^2, X^3)$. Then, $R_\mathfrak{m}$ is a rank-one DVR since $R_\mathfrak{m} = (S^{-1}R)_{S^{-1}\mathfrak{m}} = (S^{-1}k[X])_{S^{-1}\mathfrak{m}}$, where S is a multiplicatively closed subset of R (and $k[X]$) given by $S := \{X^n \mid n \in \mathbb{N} \text{ and } n \neq 1\}$. Next, let $\mathfrak{m} := (X^2, X^3)$. Then $\mathfrak{m}^{-1} = k[X] = R + XR$. Hence $(\mathfrak{m}R_\mathfrak{m})^{-1} = R_\mathfrak{m} + XR_\mathfrak{m}$, whence $(\mathfrak{m}R_\mathfrak{m})^{-1}$ is a 2-generated $R_\mathfrak{m}$-module. Therefore $R_\mathfrak{m}$ is divisorial since a local Noetherian domain $(D, \mathfrak{m})$ is divisorial if and only if $\dim(D) = 1$ and $\mathfrak{m}^{-1}$ is a 2-generated D-module [7, Theorems 6.2 and 6.3] and [42, Theorem 3.8]. □

Proof. of Example 3.1 (a) Notice that I is a maximal ideal in R. Set $T := (I : I) = I^{-1}$. Then

$$T = k[X]$$

which is the (complete) integral closure of R. Hence $I = X^2 T$, whence I is strongly stable. By Theorem 2.12, $\mathrm{core}(I) = X^2(R : T) = X^2 I = X^4 k[X]$, as desired.

(b) Let $x \in k$. Clearly, $(1+xX, X^2)T = T$ so $J_xT = X^2T = I$, that is, $J_xI = I^2$. Hence J_x is a reduction of I. Conversely, let J be a proper reduction of I. By Lemma 2.11, $JT = I$, hence $T = (R : I) = (R : JT) = ((R : T) : J) = (I : J)$ which yields

$$I = JT = J(I : J) \subseteq J(R : J) = JJ^{-1} \subseteq R.$$

If $JJ^{-1} \subsetneqq R$, then $I = JJ^{-1}$. Therefore $J^{-1} = (I : J) = T$. Hence, as R is divisorial (Lemma 3.2), $J = J_v = (R : J^{-1}) = (R : T) = I$, absurd. It follows that J is an invertible ideal of R. By [40, pages 27-42], J necessarily has the form

$$J = \frac{f}{g}\left(1 + xX, X^2\right)$$

for some $f, g \in R$ and $x \in k$. We obtain

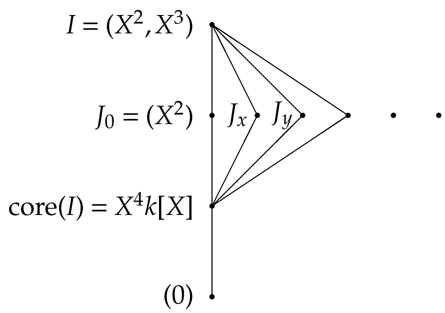
$$X^2T = I = JT = \frac{f}{g}\left(1 + xX, X^2\right)T = \frac{f}{g}T$$

which yields $f/g = cX^2$ for some nonzero $c \in k$. So $J = cJ_x = J_x$ since c is a unit in R. Finally, one can easily see that

$$J_x \subsetneqq I \ \forall\ x \ \text{ and } \ J_x \nsubseteq J_y \ \forall\ x \neq y \tag{17}$$

completing the proof of (b).

(c) Obvious by (b) and (17). We have the following diagram of inclusion relations:



(d) Let $x \in k$ and let $J := J_x = X^2(1 + xX, X^2)$. Notice that the analytic spread of I is 1 since $J_0 = (X^2)$.

Claim 1. $I^2 \subseteq (J^2 : I) \subseteq I = X^2T$.

Indeed, we have

$$J^2 = (X^4 + 2xX^5 + x^2X^6, X^6 + xX^7, X^8) \ \text{ and } \ I^3 = X^6T.$$

Obviously, $I^3 = X^4I \subseteq X^4R = J_0^2$; i.e., $I^2 \subseteq (J_0^2 : I)$. So, we may assume $x \neq 0$. Since $X^2(X^4 + 2xX^5 + x^2X^6) \in J^2$, then $X^6 + 2xX^7 \in J^2$, hence $xX^7 \in J^2$. It follows that $X^7 \in J^2$ and hence $X^6 \in J^2$. Therefore $I^3 \subseteq J^2$; i.e., $I^2 \subseteq (J^2 : I)$, as desired. Next, for $x \in k$, let $f \in (J^2 : I)$. Then $fI = fIT \subseteq J^2T = I^2$. So $f \in (I^2 : I) = (X^4T : X^2T) = X^2T = I$. Consequently, $(J^2 : I) \subseteq I$.

Claim 2. $X^5 \notin J^2$.

Deny and let $X^5 = (X^4 + 2xX^5 + x^2X^6)f_1 + (X^6 + xX^7)f_2 + X^8f_3$, for some $f_1, f_2, f_3 \in R$. This yields $f_1(0) = 0$ and $2xf_1(0) = 1$, the desired contradiction.

Claim 3. $X^{-2}(J^2 : I)$ is a proper ideal of T.

Indeed, $(J^2 : I)$ is an ideal of T and so is $X^{-2}(J^2 : I)$ by Claim 1. Assume by way of contradiction that $X^{-2}(J^2 : I) = T$; i.e., $(J^2 : I) = I$. Then $I^2 = I(J^2 : I) \subseteq J^2$, absurd by Claim 2 since $X^5 \in I^2 = X^4T$.

By Claim 3, there is $\mathfrak{m} \in \mathrm{Max}(T)$ such that $X^{-2}(J^2 : I) \subseteq \mathfrak{m}$. However, by Claim 1, $X^2T = X^{-2}I^2 \subseteq X^{-2}(J^2 : I)$. So that $\mathfrak{m} = XT$ and thus $(J^2 : I) \subseteq X^3T$. By Claim 2, $(J^2 : I) \subsetneq X^3T$. It follows that $\mathfrak{m} = X^{-3}I^2 \subseteq X^{-3}(J^2 : I) \subsetneq T$. By maximality, $\mathfrak{m} = X^{-3}(J^2 : I)$. Consequently, $(J^2 : I) = X^3\,\mathfrak{m} = X^4T = \mathrm{core}(I)$, as desired. □

The next result investigates the context of Prüfer domains. In particular, it shows that a non-basic ideal in a Prüfer domain has no minimal reduction.

Theorem 3.3. *Let R be a Prüfer domain and I a nonzero ideal of R. Then the following statements are equivalent:*

(a) *I has a minimal reduction;*
(b) $\mathrm{core}(I)$ *is a reduction of I;*
(c) *I is basic.*

Proof. We only need to establish (a) ⇒ (b) ⇒ (c).

(a) ⇒ (b) Suppose that I has a minimal reduction J_0. Let J be any reduction of I. By [25, Proposition 1], $J_0I = JI = I^2$. But since R is Prüfer, $(J_0 \cap J)I = J_0I \cap JI = I^2$ and thus $J_0 \cap J$ is a reduction of I. By minimality, $J_0 \cap J = J_0$ and so $J_0 \subseteq J$. It follows that $\mathrm{core}(I) = J_0$ is a reduction of I.

(b) ⇒ (c) Set $A := \mathrm{core}(I)$ and assume that A is a reduction of I. We first prove the result in the local case, i.e., R is assumed to be a valuation domain. Set $Q := II^{-1}$. Then $AI = I^2$ [25, Proposition 1] and $A = IQ$ by Theorem 2.3. Therefore

$$I^2 = I^2Q = I^2Q^2 = A^2$$

which yields $I = A$; this is a consequence of the fact that in a valuation domain every ideal is integrally closed [32, Proposition 6.8.1] or one can also refer to [22, Exercise 1, page 284].

Next, assume R is a Prüfer domain. In view of [24, Lemma 2.2], it suffices to show that I is locally basic. Indeed, let $\mathfrak{m}$ be a maximal ideal of R containing I. Without loss of generality we may assume that $IR_{\mathfrak{m}}(IR_{\mathfrak{m}})^{-1} \subsetneq R_{\mathfrak{m}}$. Since $R_{\mathfrak{m}}$ has the trace property, by [2, Theorem 2.8], there is some prime ideal $Q \subseteq \mathfrak{m}$ in R such that

$$IR_{\mathfrak{m}}(IR_{\mathfrak{m}})^{-1} = QR_{\mathfrak{m}} \text{ and } (IR_{\mathfrak{m}} : IR_{\mathfrak{m}}) = R_Q.$$

Since A is a reduction of I, so is $AR_{\mathfrak{m}}$ and then, by Theorem 2.3, we obtain

$$IQR_{\mathfrak{m}} = \mathrm{core}(IR_{\mathfrak{m}}) \subseteq AR_{\mathfrak{m}}.$$

We claim that $IQR_{\mathfrak{m}} = AR_{\mathfrak{m}}$. Deny. Let $a \in AR_{\mathfrak{m}}$ such that $a \notin IQR_{\mathfrak{m}}$. Necessarily, $IQR_{\mathfrak{m}} \subset aR_{\mathfrak{m}}$. Hence $a^{-1}IQR_{\mathfrak{m}} \subset R_{\mathfrak{m}}$. Since $QR_{\mathfrak{m}}$ is a trace ideal of $R_{\mathfrak{m}}$, we get

$$a^{-1}IR_{\mathfrak{m}} \subseteq (R_{\mathfrak{m}} : QR_{\mathfrak{m}}) = (QR_{\mathfrak{m}} : QR_{\mathfrak{m}}) = R_Q.$$

Moreover, $IR_{\mathfrak{m}}$ is an ideal of R_Q containing a. It follows that

$$IR_{\mathfrak{m}} = aR_Q. \tag{18}$$

Since $IR_{\mathfrak{m}}$ is not invertible in $R_{\mathfrak{m}}$, $Q \subsetneq \mathfrak{m}$. Therefore $I\mathfrak{m}R_{\mathfrak{m}} = a\mathfrak{m}R_Q = aR_Q = IR_{\mathfrak{m}}$. Further, for each maximal ideal $N \neq \mathfrak{m}$ of R, $I\mathfrak{m}R_N = IR_N$. This yields $I\mathfrak{m} = I$ and $A\mathfrak{m}I = AI = I^2$. So $A\mathfrak{m}$ is a reduction of I. Consequently,

$$A = A\mathfrak{m}. \tag{19}$$

Finally, let $B := aR_{\mathfrak{m}} \cap I$. Since R is Prüfer, we have by (18)

$$BI = (aR_{\mathfrak{m}} \cap I)I = aIR_{\mathfrak{m}} \cap I^2 = a^2R_Q \cap I^2 = I^2R_{\mathfrak{m}} \cap I^2 = I^2.$$

Therefore B is a reduction of I and hence $A \subseteq B$. Then

$$aR_{\mathfrak{m}} \subseteq AR_{\mathfrak{m}} \subseteq BR_{\mathfrak{m}} \subseteq aR_{\mathfrak{m}}.$$

So $aR_{\mathfrak{m}} = AR_{\mathfrak{m}}$. By (19), $aR_{\mathfrak{m}} = AR_{\mathfrak{m}} = A\mathfrak{m}R_{\mathfrak{m}} = a\mathfrak{m}R_{\mathfrak{m}}$, the desired contradiction. It follows that $IQR_{\mathfrak{m}} = \text{core}(IR_{\mathfrak{m}}) = AR_{\mathfrak{m}}$. Hence $\text{core}(IR_{\mathfrak{m}})$ is a reduction of $IR_{\mathfrak{m}}$. By the first step, $IR_{\mathfrak{m}}$ is basic, completing the proof of the theorem. □

In Example 2.7, we showed how to build Prüfer domains with nonzero ideals I such that $I^2I^{-1} \subsetneq \text{core}(I) \subsetneq I$. Next, we provide examples of Prüfer domains with non-trivial basic ideals.

Example 3.4. Let R be a Prüfer domain with two maximal ideals M and N such that $M^{-1} = R$ and N is invertible. Then the ideal $I := MN$ is basic.

Proof. For such an example, one can take R to be the ring of entire functions which posses infinite-height maximal ideals and (height-one) invertible maximal ideals (cf. [19, Corollary 3.1.3, Proposition 8.1.1 (5) and Example 8.4.1]). Now notice that

$$I^{-1} = (R : MN) = ((R : M) : N) = (R : N) = N^{-1}.$$

So that $II^{-1} = M$. Suppose by way of contradiction that I is not basic. Let J be a proper reduction of I, that is,

$$JI = I^2. \tag{20}$$

Then $JM = I^2I^{-1} = IM$. Since $JR_M \subsetneq IR_M$, let $a \in IR_M \setminus JR_M$. Necessarily, we have $JR_M \subsetneq aR_M$. Therefore $a^{-1}JR_M \subseteq MR_M$. We get

$$JR_M \subseteq aMR_M \subseteq IMR_M = JMR_M \subseteq JR_M$$

and then

$$JR_M = aMR_M = IMR_M. \tag{21}$$

A combination of (20) and (21) yields

$$I^2R_M = I^2MR_M = a^2MR_M.$$

Consequently, $a^2 \in a^2MR_M$ and thus $1 \in MR_M$, the desired contradiction. □

The next two results deal with the context of pseudo-valuation domains. The first one investigates the problem of when core is a reduction. Then we use this result to establish our main result on the existence of minimal reductions in pseudo-valuation domains which (unlike Prüfer domains and, a fortiori, valuation domains) possess ideals with proper minimal reductions.

Recall at this point that if R is a pseudo-valuation domain issued from V, then R and V share the same prime ideals but not, in general, the non-prime ideals. Also if P is a non-maximal prime ideal of R, then $R_P = V_P$. For more details about the spectrum of a pseudo-valuation domain or a pullback in general, we refer the reader to [1, 3, 16, 35] and most references in these papers.

In the sequel, $\mathfrak{m}$ will denote the maximal ideal of R (and V), $k := R/\mathfrak{m}$, $K := V/\mathfrak{m}$, and φ will denote the canonical homomorphism from V onto K.

Proposition 3.5. *Let R be a pseudo-valuation domain, V its associated valuation overring, and I a nonzero ideal of R.*

(a) *Assume I is an ideal of V. Then* core(I) *is a reduction of I if and only if I is basic.*
(b) *Assume I is not an ideal of V. Then* core(I) *is a reduction of I only if either I is invertible or $I^2I^{-1} \subsetneq$* core(I).

Proof. (a) We only need to prove the necessity. Suppose that core(I) is a reduction of I. Without loss of generality, we may assume that I is not invertible. By the trace property, $P := II^{-1}$ is a prime ideal of R. By Theorem 2.8 and Remark 2.9, we have

$$\mathrm{core}(I) = I^2I^{-1} = IP.$$

So there exists an integer $n \geq 1$ such that

$$PI^{n+1} = IPI^n = \mathrm{core}(I)I^n = I^{n+1}. \tag{22}$$

Now assume that I is invertible in V, i.e., $I = aV$ for some nonzero $a \in I$. Then

$$P = a^{-1}I(R : V) = a^{-1}I\mathfrak{m} = \mathfrak{m}.$$

By (22), $\mathfrak{m} = V$, absurd. Therefore I is not invertible in V. By (5), $I^{-1} = (V : I)$ and hence $(I : I) = V_P$ by [2, Theorem 2.8]. If P is not idempotent, then $P = PV_P = aV_P$ for some nonzero $a \in P$. By (22), we obtain

$$I^{n+1} = aV_PI^{n+1} = a(I : I)I^{n+1} = aI^{n+1}$$

and hence a is an invertible element in $(I^{n+1} : I^{n+1}) = (I : I) = V_P$, which is absurd. Therefore $P^2 = P$ and so $P = P^{n+1}$. Once again we appeal to (22) to get

$$(IP)^{n+1} = P^{n+1}I^{n+1} = PI^{n+1} = I^{n+1}.$$

The above equality viewed in the valuation domain V yields $I = IP$, as desired.

(b) Now I is not an ideal of V. Assume that I is not invertible. Then

$$I = a\varphi^{-1}(W)$$

for some nonzero $a \in I$ and k-vector space W with $k \subseteq W \subsetneq K$ (cf. [6, Theorem 2.1(n)]). Since I cannot be principal, then $k \subsetneq W$. Similar arguments used in (7) -in the proof of Theorem 2.8- lead to

$$P = Ia^{-1}\varphi^{-1}(0) = Ia^{-1}\mathfrak{m} = \mathfrak{m}.$$

Now, assume by way of contradiction that $I^2I^{-1} = IP = \mathrm{core}(I)$, where $P := II^{-1}$. By (22), we obtain

$$a^{n+1}\mathfrak{m} = \mathfrak{m}I^{n+1} = I^{n+1} = a^{n+1}\varphi^{-1}(W^{n+1})$$

for some integer $n \geq 1$. It follows that $\mathfrak{m} = \varphi^{-1}(W^{n+1})$, the desired contradiction. Therefore $I^2I^{-1} \subsetneq \mathrm{core}(I)$, completing the proof of the theorem. □

Remark 3.6. (1) A combination of Theorem 2.8 and Proposition 3.5 yields the following result: "*Let R be a pseudo-valuation domain, V its associated valuation overring, and I a nonzero ideal of R. Assume that K is algebraic over k. Then* core(I) *is a reduction of I if and only if I is basic.*"
(2) A possible occurrence for the assertion (b) in Proposition 3.5 happens when I is a non-invertible basic ideal of R (hence core(I) is trivially a reduction of I). For example, suppose that K is not algebraic over k and let x be a transcendental element of K over k and $0 \neq a \in \mathfrak{m}$. Then the ideal $I := a\varphi^{-1}(k + kx)$ of R verifies $I^2I^{-1} \subsetneq \mathrm{core}(I) = I$ (as established in the proof of (b) ⇒ (a) of Theorem 2.8).

The next result characterizes the ideals which admit proper minimal reductions in a pseudo-valuation domain R associated with V and also describes these minimal reductions. We will break our findings into two separate and unrelated cases for a given ideal I of R; namely, when I is an ideal of V (e.g., if I is prime) and when I is not an ideal of V. In the latter case, we restrict our study to pseudo-valuation domains issued from finite extensions. Here, too, Theorem 3.7 validates Corso, Polini, and Ulrich's conjecture mentioned in (16) in the context of pseudo-valuation domains.

For this purpose, let $U(A)$ denote the set of all units of a ring A and $\mathrm{Frac}(A)$ denote the set of all fractional ideals of A.

Theorem 3.7. *Let R be a pseudo-valuation domain and $(V, \mathfrak{m})$ its associated valuation overring with $R \subsetneq V$. Let I be a nonzero ideal of R.*

(a) *Assume I is an ideal of V. Then I has a proper minimal reduction if and only if $I = aV$ for some $0 \neq a \in I$. Moreover,* $\mathrm{MinRed}(I) = \{auR \mid u \in U(V)\}$.

(b) *Assume I is not an ideal of V and $[K : k] < \infty$. Then I has a proper minimal reduction if and only if $I = a\varphi^{-1}(W)$ for some $0 \neq a \in I$ and k-vector space W such that $k \subsetneq W \subsetneq K$ and $W^n = W^{n+1}$ for some (minimal) integer $n \geq 1$. Moreover,* $\mathrm{MinRed}(I) = \{a\varphi^{-1}(kw) \mid w \in W \setminus \{0\}\}$.

Moreover, for both cases, $\mathrm{core}(I) = (J^2 : I)$*, for each minimal reduction J of I.*

Proof. (a) Assume that I is an ideal of V and let $(I : I) = V_P$ for some nonzero prime ideal P of V. To prove the necessity, assume I has a minimal reduction $J_0 \subsetneq I$.

Claim 4. I is invertible in V_P.

Deny and suppose $I(V_P : I) \subsetneq V_P$. Then $I(V_P : I) = QV_P = Q$ for some prime ideal $Q \subseteq P$ of V (and a fortiori of R), whence $(V_P : I) \subseteq (Q : I)$. It follows that

$$I^{-1} = (V : I) = (V_P : I) = (Q : I). \tag{23}$$

By [1, Theorem 2.8], we get

$$P = I(V : I) = II^{-1} = I(Q : I) \subseteq Q$$

and therefore $P = Q$. Now suppose that I has a reduction J which is not an ideal of V. Then $JI^n = I^{n+1}$ for some positive integer n and $J = b\varphi^{-1}(F)$ for some nonzero $b \in J$ and k-vector subspace F of K with $k \subseteq F \subsetneq K$. Since R has the trace property (cf. (1) in the proof of Lemma 2.2), we have $P = II^{-1} = I^nI^{-n}$ and then

$$bP = JP = JII^{-1} = JI^nI^{-n} = I^{n+1}I^{-n} = III^{-1} = IP.$$

The fact that P is a trace ideal of V combined with (23) yields

$$b^{-1}V_P = b^{-1}(V : P) = (V : IP) = ((V : P) : I) = (V_P : I) = (P : I).$$

Thus $V_P = b(P : I) \subseteq J(P : I) \subseteq I(P : I) \subseteq P$, which is absurd. Hence every reduction of I is an ideal of V, whence all reductions of I contains J_0 (since they are linearly ordered in V). It follows that $\mathrm{core}(I) = J_0$ is a reduction of I and, by Remark 3.6(1), I is basic, the desired contradiction, proving the claim.

By Claim 4, $I(V_P : I) = V_P$ and hence

$$I = aV_P$$

for some nonzero $a \in I$; that is, I is strongly stable in R.

Claim 5. $\mathrm{Red}(I) = \{aF \mid F \in \mathrm{Frac}(R) \text{ with } FV_P = V_P\}$.

Indeed, let J be a reduction of I. By Lemma 2.11, $JV_P = I = aV_P$. So there is $u \in V \setminus P$ such that $b := ua \in J$ and whence $JV_P = bV_P$. Therefore, $W := \{x \in V_P \mid xb \in J\}$ is an R-submodule of V_P with $R \subseteq W \subseteq V_P$ and $J = bW$. Let $F := uW$. Clearly, $F \in \mathrm{Frac}(R)$ with $J = aF$ and $FV_P = a^{-1}JV_P = V_P$, as desired. The reverse inclusion is straightforward.

Claim 6. $P = \mathfrak{m}$.

Deny and let $x \in \mathfrak{m} \setminus P$. By Claim 5, $J_0 = aF_0$ for some $F_0 \in \mathrm{Frac}(R)$ with $F_0V_P = V_P$ and so $F_0P = P$. Thus

$$P \subsetneqq F_0 \subseteq V_P = R_P.$$

Hence there exists $y \in F_0 \setminus P$ such that $y \in R$. Set

$$F_x := P + xyR \subseteq F_y := P + yR \subseteq F_0.$$

Clearly, $F_xV_P = F_yV_P = V_P$, so that aF_x and aF_y are reductions of I. By minimality, $F_x = F_y = F_0$. Therefore, $y(1 - x\beta) \in P$ for some $\beta \in R$, hence $1 - x\beta \in P \subseteq \mathfrak{m}$, which is absurd, proving the claim.

By Claim 6, we have $I = aV$, completing the proof of the necessity. Conversely, assume $I = aV$. Claim 5 yields

$$\mathrm{Red}(I) = \{aF \mid F \in \mathrm{Frac}(R) \text{ with } FV = V\}.$$

Now, for each $u \in U(V)$, set $J_u := auR$. Obviously, J_u is a reduction of I. Further, if aF is a reduction of I with $aF \subseteq J_u$, then $u^{-1}F$ is an integral ideal of R with $u^{-1}F \nsubseteq \mathfrak{m}$ since $FV = V$. Hence $F = uR$ and thus J_u is minimal. In particular, $J_1 = aR \subsetneqq I$ is a proper minimal reduction of I, which proves the sufficiency.

Moreover, if $J := aF$ is a minimal reduction of $I = aV$, then $\mathfrak{m} \subsetneqq F$. Let $u \in F \setminus \mathfrak{m}$ and set $J_u = auR$. Necessarily, $u \in V$. So $J_u \subseteq J$ is a reduction of I and hence $J = J_u$. It follows that $\mathrm{MinRed}(I) = \{auR \mid u \in U(V)\}$, completing the proof of (a).

(b) Assume that I is not an ideal of V. Then

$$I = a\varphi^{-1}(W)$$

for some $0 \neq a \in I$ and k-vector space W with $k \subseteq W \subsetneqq K$. We first prove the following claim:

Claim 7. $\mathrm{Red}(I) = \{a\varphi^{-1}(F) \mid F$ ranges over the set of k-vector subspaces of W with $FW^n = W^{n+1}$ for some integer $n \geq 1\}$.

Indeed, let J be a reduction of I. Then $JI^n = I^{n+1}$ for some positive integer $n \geq 1$. Therefore $a^nJV = a^{n+1}V$ and thus

$$JV = aV = IV \supsetneqq I \supseteq J.$$

Hence J is not an ideal of V. Further, since V is a valuation (in fact "local" is enough), one can check that

$$JV = bV$$

for some $b \in J$, that is, $b = au$ for some $u \in U(V)$ (i.e., $\varphi(u) \neq 0$). Set

$$E := \{\varphi(x) \mid x \in V \text{ such that } xb \in J\} \ \text{ and } \ F := \varphi(u)E$$

Clearly, E is a k-vector subspace of K with $k \subseteq E \subsetneqq K$ and $J = b\varphi^{-1}(E)$. It is easily seen that F is a k-vector subspace of K with

$$I \supseteq J = au\varphi^{-1}(E) = a\varphi^{-1}(F) \ \text{ and } \ W \supseteq F.$$

Moreover, the equality $JI^n = I^{n+1}$ yields $FW^n = W^{n+1}$, as desired. It is worthwhile adding that J is minimal if and only if so is F. The reverse inclusion is straight.

Next, suppose that $[K\colon k]$ is finite. Then there is a minimal positive integer n such that $W^n = W^{n+m}$, $\forall\ m \geq 1$. In particular, $W^n = W^{2n}$, that is, W^n is a ring and hence a field (due to algebraicity). So the following holds:

Claim 8. $\mathrm{MinRed}(I) = \{a\varphi^{-1}(kw) \mid w \in W\}$.

Let $0 \neq w \in W$ and set
$$J_w := a\varphi^{-1}(kw).$$
By Claim 7, J_w is a reduction of I; and if $J := a\varphi^{-1}(F)$ is a reduction of I with $J \subseteq J_w$, then $F \subseteq kw$, hence $F = kw$ since $\dim_k kw = 1$, whence J_w is minimal. Conversely, let $J_0 := a\varphi^{-1}(F_0)$ be a minimal reduction of I and let $0 \neq w_o \in F_0 \subseteq W \subseteq W^n$. Clearly, $J_{w_o} := a\varphi^{-1}(kw_o) \subseteq J_0$, that is, $J_0 = J_{w_o}$, proving the claim.

Finally, suppose that I has a proper minimal reduction. Then, $k \subsetneqq W$; otherwise, $I = aR$ and so I would be basic. Conversely, if $k \subsetneqq W$, then by Claim 8, $J_0 := a\varphi^{-1}(k)$ is a proper minimal reduction of I.

Next, we prove the last statement. Notice first that, in both cases (a) and (b), $JI = I^2$ for any minimal reduction J of I. So I has reduction number 1. Now, assume $J_u := auR$. Then $(J^2 : I) = au^2(R : V) = au^2\,\mathfrak{m} = a\,\mathfrak{m} = I^2I^{-1} = \mathrm{core}(I)$. Finally, let $J_w := a\varphi^{-1}(kw)$.Then $(J^2 : I) = a\varphi^{-1}(kw^2 : W) = a\varphi^{-1}(0) = a\,\mathfrak{m} = \mathrm{core}(I)$, completing the proof of the theorem. □

Example 3.8. Let $\mathbb{Q}$ denote the field of rational numbers and X an indeterminate over $\mathbb{Q}$. Consider the pseudo-valuation domain $R := \mathbb{Q} + \mathfrak{m}$ issued from $\mathbb{Q}(\sqrt{2}, \sqrt{3})[[X]]$, where $\mathfrak{m} := X\mathbb{Q}(\sqrt{2}, \sqrt{3})[[X]]$. Consider the ideal of R given by
$$I := X\Big(\mathbb{Q}(\sqrt{2}) + \mathfrak{m}\Big).$$
Clearly, $T := (I\colon I) = \mathbb{Q}(\sqrt{2}) + \mathfrak{m}$. By Theorem 2.12 and Theorem 3.7, we have:

(a) $\mathrm{core}(I) = \mathfrak{m}^2$.
(b) $\mathrm{MinRed}(I) = \{wX\mathbb{Q} + \mathfrak{m}^2 \mid 0 \neq w \in \mathbb{Q}(\sqrt{2})\}$.

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