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t-REDUCTIONS AND t-INTEGRAL CLOSURE OF IDEALS $^{(\star)}$

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ABSTRACT. Let R be an integral domain and I a nonzero ideal of R. An ideal $J \subseteq I$ is a t-reduction of I if $(JI^n)_t = (I^{n+1})_t$ for some integer $n \geq 0$. An element $x \in R$ is t-integral over I if there is an equation $x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0$ with $a_i \in (I^i)_t$ for $i = 1, ..., n$. The set of all elements that are t-integral over I is called the t-integral closure of I. This paper investigates the t-reductions and t-integral closure of ideals. Our objective is to establish satisfactory t-analogues of well-known results, in the literature, on the integral closure of ideals and its correlation with reductions. Namely, Section 2 identifies basic properties of t-reductions of ideals and features explicit examples discriminating between the notions of reduction and t-reduction. Section 3 investigates the concept of t-integral closure of ideals, including its correlation with t-reductions. Section 4 studies the persistence and contraction of t-integral closure of ideals under ring homomorphisms. All along the paper, the main results are illustrated with original examples.

1. INTRODUCTION

Throughout, all rings considered are commutative with identity. Let R be a ring and I an ideal of R. An ideal $J \subseteq I$ is a reduction of I if $JI^n = I^{n+1}$ for some positive integer n. An ideal which has no reduction other than itself is called a basic ideal [12, 13, 23]. The notion of reduction was introduced by Northcott and Rees and its usefulness resides mainly in two facts: "First, it defines a relationship between two ideals which is preserved under homomorphisms and ring extensions; secondly, what we may term the reduction process gets rid of superfluous elements of an ideal without disturbing the algebraic multiplicities associated with it" [23]. The main purpose of their paper was to contribute to the analytic theory of ideals in Noetherian (local) rings via minimal reductions.

Reductions happened to be a very useful tool for the theory of integral dependence over ideals. Let I be an ideal in a ring R. An element $x \in R$ is integral over I if there is an equation $x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0$ with $a_i \in I^i$ for $i = 1, ..., n$. The set of all elements that are integral over I is called the integral closure of I, and is denoted by $\bar{I}$. If $I = \bar{I}$, then I is called integrally closed. It turned out that an element $x \in R$ is integral over I if and only if I is a reduction of $I + Rx$; and if I is finitely generated, then $I \subseteq \bar{J}$ if and only if J is a reduction of I [17, Corollary 1.2.5]. This correlation allowed to prove a number of crucial results in the theory including the fact that the integral closure of an ideal is an ideal [17, Corollary 1.3.1]. For a full treatment of this topic, we refer the reader to Huneke and Swanson's book "Integral closure of ideals, rings, and modules" [17].

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Let R be a domain with quotient field K, I a nonzero fractional ideal of R, and let $I^{-1} := (R : I) = \{x \in K \mid xI \subseteq R\}$. The v- and t-closures of I are defined, respectively, by $I_v := (I^{-1})^{-1}$ and $I_t := \cup J_v$, where J ranges over the set of finitely generated subideals of I. The ideal I is a v-ideal (or divisorial) if $I_v = I$ and a t-ideal if $I_t = I$. Under the ideal t-multiplication $(I, J) \mapsto (IJ)_t$ the set $F_t(R)$ of fractional t -ideals of R is a semigroup with unit R. Recall that factorial domains, Krull domains, GCDs, and PvMDs can be regarded as t-analogues of the principal domains, Dedekind domains, Bézout domains, and Prüfer domains, respectively. For instance, a domain is Prüfer (resp., a PvMD) if every nonzero finitely generated ideal is invertible (resp., t-invertible). For some relevant works on v- and t-operations, we refer the reader to [10, 16, 19, 20, 21, 24, 25, 26, 27].

This paper investigates the t-reductions and t-integral closure of ideals. Our objective is to establish satisfactory t-analogues of well-known results, in the literature, on the integral closure of ideals and its correlation with reductions. Namely, Section 2 identifies basic properties of t-reductions of ideals and features explicit examples discriminating between the notions of reduction and t-reduction. Section 3 investigates the concept of t-integral closure of ideals, including its correlation with t-reductions. Section 4 studies the persistence and contraction of t-integral closure of ideals under ring homomorphisms. All along the paper, the main results are illustrated with original examples.

2. t-Reductions of ideals

This section identifies basic ideal-theoretic properties of the notion of t-reduction including its behavior under localizations. As a prelude to this, we provide explicit examples discriminating between the notions of reduction and t-reduction.

Recall that, in a ring R, a subideal J of an ideal I is called a reduction of I if $JI^n = I^{n+1}$ for some positive integer n [23]. An ideal which has no reduction other than itself is called a basic ideal [12, 13].

Definition 2.1 (cf. [15, Definition 1.1])**.** Let R be a domain and I a nonzero ideal of R. An ideal $J \subseteq I$ is a *t-reduction* of I if $(JI^n)_t = (I^{n+1})_t$ for some integer $n \geq 0$ (and, a fortiori, the relation holds for $n \gg 0$). The ideal J is a *trivial t-reduction* of I if $J_t = I_t$. The ideal I is *t-basic* if it has no t-reduction other than the trivial t-reductions.

At this point, recall a basic property of the t-operation (which, in fact, holds for any star operation) that will be used throughout the paper. For any two nonzero ideals I and J of a domain, we have $(IJ)_t = (I_tJ)_t = (IJ_t)_t = (I_tJ_t)_t$. So, obviously, for nonzero ideals $J \subseteq I$, we always have:

$$J \text{ is a } t\text{-reduction of } I \Leftrightarrow J \text{ is a } t\text{-reduction of } I_t \Leftrightarrow J_t \text{ is a } t\text{-reduction of } I_t.$$

Notice also that a reduction is necessarily a t-reduction; and the converse is not true, in general, as shown by the next example which exhibits a domain R with two t-ideals $J \subsetneq I$ such that J is a t-reduction but not a reduction of I.

Example 2.2. We use a construction from [18]. Let x be an indeterminate over $\mathbb{Z}$ and let $R := \mathbb{Z}[3x, x^2, x^3]$, $I := (3x, x^2, x^3)$, and $J := (3x, 3x^2, x^3, x^4)$. Then $J \subsetneq I$ are two finitely generated t-ideals of R such that:

$$JI^n \subsetneq I^{n+1} \forall\ n \in \mathbb{N} \ \text{ and } \ (JI)_t = (I^2)_t.$$

Proof. I, being a height-one prime ideal [18], is a t-ideal of R. Next, we prove that J is a t-ideal. We first claim that $J^{-1} = \frac{1}{x}\mathbb{Z}[x]$. Indeed, notice that $\mathbb{Q}(x)$ is the quotient field of R and since $3x \subseteq J$, then $J^{-1} \subseteq \frac{1}{3x}R$. So, let $f := \frac{g}{3x} \in J^{-1}$ where $g = \sum_{i=0}^{m} a_i x^i \in \mathbb{Z}[x]$ with $a_1 \in 3\mathbb{Z}$. Then the fact that $x^3 f \in R$ implies that $a_i \in 3\mathbb{Z}$ for $i = 0, 2, ..., m$; i.e., $g \in 3\mathbb{Z}[x]$. Hence $f \in \frac{1}{x}\mathbb{Z}[x]$, whence $J^{-1} \subseteq \frac{1}{x}\mathbb{Z}[x]$. The reverse inclusion holds since $\frac{1}{x}J\mathbb{Z}[x] = (3, 3x, x^2, x^3)\mathbb{Z}[x] \subseteq R$, proving the claim. Next, let $h \in (R : \mathbb{Z}[x]) \subseteq R$. Then $xh \in R$ forcing $h(0) \in 3\mathbb{Z}$ and thus $h \in (3, 3x, x^2, x^3)$. So, $(R : \mathbb{Z}[x]) \subseteq (3, 3x, x^2, x^3)$, hence $(R : \mathbb{Z}[x]) = \frac{1}{x}J$. It follows that $J_t = J_v = \left(R : \frac{1}{x}\mathbb{Z}[x]\right) = x(R : \mathbb{Z}[x]) = J$, as desired.

Next, let $n \in \mathbb{N}$. It is to see that $x^3 x^{2n} = x^{2n+3}$ is the monic monomial with the smallest degree in JI^n. Therefore $x^{2(n+1)} = x^{2n+2} \in I^{n+1} \setminus JI^n$. That is, J is not a reduction of I. It remains to prove $(JI)_t = (I^2)_t$. We first claim that $(JI)^{-1} = \frac{1}{x^2}\mathbb{Z}[x]$. Indeed, $(JI)^{-1} \subseteq (J^{-1})^2 = \frac{1}{x^2}\mathbb{Z}[x]$ and the reverse inclusion holds since

$$\frac{1}{x^2}JI\mathbb{Z}[x] = (3, 3x, x^2, x^3)(3, x, x^2)\mathbb{Z}[x] \subseteq R$$

proving the claim. Now, observe that $I^2 = (9x^2, 3x^3, x^4, x^5)$. It follows that $(IJ)_t = (IJ)_v = \left(R : \frac{1}{x^2}\mathbb{Z}[x]\right) = x^2(R : \mathbb{Z}[x]) = xJ \supseteq I^2$. Thus $(IJ)_t \supseteq (I^2)_t$, as desired. □

Observe that the domain R in the above example is not integrally closed. Next, we provide a class of integrally closed domains where the notions of reduction and t-reduction are always distinct.

Example 2.3. Let R be any integrally closed Mori domain that is not completely integrally closed (i.e., not Krull). Then there always exist nonzero ideals $J \subsetneq I$ in R such that J is a t-reduction but not a reduction of I.

Proof. These domains do exist; for instance, let $k \subsetneq K$ be a field extension with k algebraically closed and let x be an indeterminate over K. Then, $R := k + xK[x]$ is an integrally closed Mori domain [9, Theorem 4.18] that is not completely integrally closed [11, Lemma 26.5] (see [8, p. 161]).

Now, by [15, Proposition 1.5(1)], there exists a t-ideal A in R that is not t-basic; say, $B \subseteq A$ is a t-reduction of A with $B_t \subsetneq A_t$. By [4, Theorem 2.1], there exist finitely generated ideals $F \subseteq A$ and $J \subseteq B$ such that $A^{-1} = F^{-1}$ and $B^{-1} = J^{-1}$; yielding $A_t = F_t$ and $B_t = J_t$. Let $I := F + J$. Then, one can easily see, that J is a non-trivial t-reduction of I. Finally, we claim that J is not a reduction of I. Deny. Since I is finitely generated, $I \subseteq \overline{J}$ by [17, Corollary 1.2.5]. But, $\overline{J} \subseteq J_t$ by [22, Proposition 2.2]. It follows that $J_t = I_t$, the desired contradiction. □

Another crucial fact concerns reductions of t-ideals. Indeed, if J is a reduction of a t-ideal, then so is J_t; and the converse is not true, in general, as shown by the following example which features a domain R with a t-ideal I and an ideal $J \subseteq I$ such that J_t is a reduction but J is not a reduction of I.

Example 2.4. Let k be a field and let x, y, z be indeterminates over k. Let $R := k[x] + M$, where $M := (y, z)k(x)[[y, z]]$ and let $J := M^2$. Note that R is a classical pullback issued from the local Noetherian and integrally closed domain $T := k(x)[[y, z]]$. Then M is a divisorial ideal of R by [14, Corollary 5] and clearly, $\forall\, n \in \mathbb{N}$, $M^{n+2} \subsetneq M^{n+1}$; that is, J is not a reduction of M in R. On the other hand, notice that $(M : M) = T$ (since

T is integrally closed) and M is not principal in T. Therefore, by [14, Theorem 13], we have

$$(R:(R:M^2)) = (R:(M^{-1}:M)) = (R:((M:M):M)) =$$
$$(R:(T:M)) = (R:M^{-1}) = M.$$

So that $J_t = J_v = M$. Hence, J_t is trivially a reduction of M in R.

In the sequel, R will denote a domain. For convenience, recall that, for any nonzero ideals I, J, H of R, the equality $(IJ+H)_t = (I_tJ+H)_t$ always holds since $I_tJ \subseteq (I_tJ)_t = (IJ)_t \subseteq (IJ+H)_t$. This property will be used in the proof of the next basic result which examines the t-reduction of the sum and product of ideals.

Lemma 2.5. *Let $J \subseteq I$ and $J' \subseteq I'$ be nonzero ideals of R. If J and J' are t-reductions of I and I', respectively, then $J+J'$ is a t-reduction of $I+I'$ and JJ' is a t-reduction of II'.*

Proof. Let n be a positive integer. Then the following implication always holds

$$(JI^n)_t = (I^{n+1})_t \;\Rightarrow\; (JI^m)_t = (I^{m+1})_t \;\forall m \geq n. \tag{1}$$

Indeed, multiply the first equation through by I^{m-n} and apply the t-closure to both sides. By (1), let m be a positive integer such that

$$(JI^m)_t = (I^{m+1})_t \;\text{ and }\; (J'I'^m)_t = (I'^{m+1})_t. \tag{2}$$

By (2), we get

$$\begin{array}{rcl}
\Big((I+I')^{2m+1}\Big)_t & \subseteq & \Big(I^{m+1}(I+I')^m + I'^{m+1}(I+I')^m\Big)_t \\
& = & \Big((I^{m+1})_t(I+I')^m + (I'^{m+1})_t(I+I')^m\Big)_t \\
& = & \Big((JI^m)_t(I+I')^m + (J'I'^m)_t(I+I')^m\Big)_t \\
& = & \Big(JI^m(I+I')^m + J'I'^m(I+I')^m\Big)_t \\
& \subseteq & \Big((J+J')(I+I')^{2m}\Big)_t \\
& \subseteq & \Big((I+I')^{2m+1}\Big)_t
\end{array}$$

and then equality holds throughout, proving the first statement. The proof of the second statement is straightforward via (2). ∎

The next basic result examines the transitivity for t-reduction.

Lemma 2.6. *Let $K \subseteq J \subseteq I$ be nonzero ideals of R. Then:*

(a) *If K is a t-reduction of J and J is a t-reduction of I, then K is a t-reduction of I.*
(b) *If K is a t-reduction of I, then J is a t-reduction of I.*

Proof. For any positive integer m, we always have

$$(JI^m)_t = (I^{m+1})_t \;\Rightarrow\; (J^nI^m)_t = (I^{m+n})_t \;\forall n \geq 1. \tag{3}$$

Indeed, multiply the first equation through by J^{n-1}, apply the t-closure to both sides, and conclude by induction on n. Let $(KJ^n)_t = (J^{n+1})_t$ and $(JI^m)_t = (I^{m+1})_t$, for some positive integers n and m. By (3), we get

$$(I^{m+n+1})_t = (J^{n+1}I^m)_t = \Big((J^{n+1})_tI^m\Big)_t = \Big((KJ^n)_tI^m\Big)_t = (KI^{m+n})_t$$

proving (a). The proof of (b) is straightforward. ∎

The next basic result examines the t-reduction of the power of an ideal.

Lemma 2.7. *Let $J \subseteq I$ be nonzero ideals of R and let n be a positive integer. Then:*

(a) *J is a t-reduction of $I \Leftrightarrow J^n$ is a t-reduction of I^n.*
(b) *If $J = (a_1, ..., a_k)$, then: J is a t-reduction of $I \Leftrightarrow (a_1^n, ..., a_k^n)$ is a t-reduction of I^n.*

Proof. (a) The "only if" implication holds by Lemma 2.5. For the converse, suppose $(J^n I^{nm})_t = (I^{nm+n})_t$ for some positive integer m. Then

$$(I^{nm+n})_t = (JJ^{n-1}I^{nm})_t \subseteq (JI^{nm+n-1})_t \subseteq (I^{nm+n})_t$$

and so equality holds throughout, as desired.

(b) Assume that J is a t-reduction of I. From [17, (8.1.6)], we always have the following equality

$$\left(a_1^n, ..., a_k^n\right)\left(a_1, ..., a_k\right)^{(k-1)(n-1)} = \left(a_1, ..., a_k\right)^{(n-1)k+1} \tag{4}$$

and, multiplying (4) through by J^{k-1}, we get $\left(a_1^n, ..., a_k^n\right)J^{nk-n} = J^{nk}$. Therefore $(a_1^n, ..., a_k^n)$ is a t-reduction of J^n and a fortiori of I^n by (a) and Proposition 2.6. The converse holds by (a) and Proposition 2.6. □

The next basic result examines the t-reduction of localizations.

Lemma 2.8. *Let $J \subseteq I$ be nonzero ideals of R and let S be a multiplicatively closed subset of R. If J is a t-reduction of I, then $S^{-1}J$ is a t-reduction of $S^{-1}I$.*

Proof. Assume that $(JI^n)_t = (I^{n+1})_t$ for some positive integer n. Let t_1 denote the t-operation with respect to $S^{-1}R$. By [21, Lemma 3.4], we have:
$((S^{-1}I)^{n+1})_{t_1} = (S^{-1}(I^{n+1}))_{t_1} = (S^{-1}((I^{n+1})_t))_{t_1} = (S^{-1}((JI^n)_t))_{t_1} = (S^{-1}(JI^n))_{t_1}$
$= ((S^{-1}J)(S^{-1}I)^n)_{t_1}$. □

It is worthwhile noting here that, in a PvMD, J is a t-reduction of I if and only if J is t-locally a reduction of I; i.e., JR_M is a reduction of IR_M for every maximal t-ideal M of R [15, Lemma 2.2].

3. t-Integral closure of ideals

This section investigates the concept of t-integral closure of ideals and its correlation with t-reductions. Our objective is to establish satisfactory t-analogues of (and in some cases generalize) well-known results, in the literature, on the integral closure of ideals and its correlation with reductions.

Definition 3.1. Let R be a domain and I a nonzero ideal of R. An element $x \in R$ is t-integral over I if there is an equation

$$x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0 \text{ with } a_i \in (I^i)_t \ \forall i = 1, ..., n.$$

The set of all elements that are t-integral over I is called the t-integral closure of I, and is denoted by $\widetilde{I}$. If $I = \widetilde{I}$, then I is called t-integrally closed.

Notice that the t-integral closure of the ideal R is always R, whereas the t-integral closure of the ring R (also called pseudo-integral closure) may be larger than R; e.g., consider any non v-domain [3, 8]. Also, we have $J \subseteq I \Rightarrow \widetilde{J} \subseteq \widetilde{I}$. More ideal-theoretic properties are provided in Remark 3.8.

It is well-known that the integral closure of an ideal is an ideal which is integrally closed [17, Corollary 1.3.1]. Next, we establish a t-analogue for this result.

Theorem 3.2. *The t-integral closure of an ideal is an integrally closed ideal. In general, it is not t-closed and, a fortiori, not t-integrally closed.*

The proof of this theorem relies on the following lemma which sets a t-analogue for the notion of Rees algebra of an ideal [17, Chapter 5]. Recall, for convenience, that the Rees algebra of an ideal I (in a ring R) is the graded subring of $R[x]$ given by $R[Ix] := \bigoplus_{n\geq 0} I^n\ x^n$ [17, Definition 5.1.1] and whose integral closure in $R[x]$ is the graded ring $\bigoplus_{n\geq 0} \overline{I^n}\ x^n$ [17, Proposition 5.2.1].

Lemma 3.3. *Let R be a domain, I a t-ideal of R, and x an indeterminate over R. Let $R_t[Ix] := \bigoplus_{n\geq 0}(I^n)_t x^n$. Then $R_t[Ix]$ is a graded subring of $R[x]$ and its integral closure in $R[x]$ is the graded ring $\bigoplus_{n\geq 0}\widetilde{I^n}x^n$.*

Proof. That $R_t[Ix]$ is $\mathbb{N}$-graded follows from the fact that $(I^i)_t\cdot(I^j)_t \subseteq (I^{i+j})_t, \forall i,j\in\mathbb{N}$. Let $\overline{R_t[Ix]}$ denote its integral closure in $R[x]$. By [17, Theorem 2.3.2], $\overline{R_t[Ix]}$ is an $\mathbb{N}$-graded ring. Let $k\in\mathbb{N}$ and let S_k denote the homogeneous component of $\overline{R_t[Ix]}$ of degree k. We shall prove that $S_k = \widetilde{I^k}x^k$. Let $s := s_k x^k \in S_k$, for some $s_k\in R$. Then, $s^n + a_1 s^{n-1} + \cdots + a_n = 0$ for some positive integer n and $a_i\in R_t[Ix]$, $i = 1,\ldots,n$. Expanding each $a_i = \sum_{j=0}^{k_i} a_{i,j}x^j$ with $a_{i,j}\in(I^j)_t$, the coefficient of the monomial of degree kn in the above equation is $s_k^n + \sum_{i=1}^n a_{i,ik}s_k^{n-i} = 0$, with $a_{i,ik}\in(I^{ik})_t$. It follows that $s_k\in\widetilde{I^k}$ and thus $S_k\subseteq\widetilde{I^k}x^k$. For the reverse inclusion, let $z_k := y_k x^k\in\widetilde{I^k}x^k$, for some $y_k\in\widetilde{I^k}$. Then, $y_k^n + a_1 y_k^{n-1} + \cdots + a_n = 0$ for some positive integer n and $a_j\in(I^{kj})_t$, $j = 1,\ldots,n$. Multiplying through by x^{kn} yields the equation $z_k^n + a_1x^k z_k^{n-1} + \cdots + a_n x^{kn} = 0$, with $a_j x^{kj}\in(I^{kj})_t x^{kj}\subseteq R_t[Ix]$, $j = 1,\ldots,n$. That is, $z_k\in\overline{R_t[Ix]}$. But z_k is homogeneous of degree k in $\overline{R_t[Ix]}$. Therefore $z_k\in S_k$ and hence $\widetilde{I^k}x^k\subseteq S_k$, completing the proof of the lemma. □

Definition 3.4. The t-Rees algebra of an ideal I (in a domain R) is the graded subring of $R[x]$ given by $R_t[Ix] := \bigoplus_{n\geq 0}(I^n)_t x^n$.

Proof of Theorem 3.2. Let R be a domain and I a nonzero ideal of R. Since $\widetilde{I} = \widetilde{I_t}$, we may assume I to be a t-ideal. We first prove that $\widetilde{I}$ is an ideal. Clearly, $\widetilde{I}$ is closed under multiplication. Next, we show that $\widetilde{I}$ is closed under addition. Let $a,b\in\widetilde{I}$. Then, by Lemma 3.3, ax and $bx\in\overline{R_t[Ix]}$. Hence, $ax+bx = (a+b)x\in\overline{R_t[Ix]}$. Again, by Lemma 3.3, $a+b\in\widetilde{I}$, as desired. Next, we prove that $\widetilde{I}$ is integrally closed. For this purpose, observe that, $\forall n\in\mathbb{N}$, $(S_1)^n\subseteq S_n$, forcing

$$\left(\widetilde{I}\right)^n \subseteq \widetilde{I^n}\ \forall n\in\mathbb{N}. \tag{5}$$

Consider the Rees algebra of the ideal $\widetilde{I}$, $R[\widetilde{I}x] = \bigoplus_{n\geq 0}\left(\widetilde{I}\right)^n x^n$. Therefore $R[\widetilde{I}x]\subseteq \overline{R_t[Ix]}$ and hence $\overline{R[\widetilde{I}x]}\subseteq\overline{R_t[Ix]}$. Now, a combination of Lemma 3.3 and [17, Proposition 5.2.1] yields $\bigoplus_{n\geq 0}\overline{\left(\widetilde{I}\right)^n}x^n\subseteq\bigoplus_{n\geq 0}\widetilde{I^n}x^n$. In particular, $\overline{\widetilde{I}}\subseteq\widetilde{I}$; that is, $\widetilde{I}$ is integrally closed. The proof of the last statement of the theorem is handled by Example 3.10(b), where we provide a domain with an ideal I such that $\widetilde{I}\subsetneqq(\ \widetilde{I}\ )_t$. That is, $\widetilde{I}$ is not a t-ideal and, hence, not t-integrally closed since $(\ \widetilde{I}\ )_t\subseteq\widetilde{\widetilde{I}}$ always holds. □

The next result shows that the t-integral closure collapses to the t-closure in the class of integrally closed domains. It also completes two existing results in the literature on the integral closure of ideals (Gilmer [11] and Mimouni [22]).

Theorem 3.5. *Let R be a domain. The following assertions are equivalent:*

(a) *R is integrally closed;*
(b) *Every principal ideal of R is integrally closed;*
(c) *Every t-ideal of R is integrally closed;*
(d) *$\overline{I} \subseteq I_t$ for each nonzero ideal I of R;*
(e) *Every principal ideal of R is t-integrally closed;*
(f) *Every t-ideal of R is t-integrally closed;*
(g) *$\widetilde{I} = I_t$ for each nonzero ideal I of R.*

Proof. (a) $\Leftrightarrow$ (b) and (a) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) are handled by [11, Lemma 24.6] and [22, Proposition 2.2], respectively. Also, (g) $\Leftrightarrow$ (f) $\Rightarrow$ (e) $\Rightarrow$ (b) are straightforward. So, it remains to prove (a) $\Rightarrow$ (g). Assume R is integrally closed and let I be a nonzero ideal of R. The inclusion $I_t \subseteq \widetilde{I}$ holds in any domain. Next, let $\alpha \in \widetilde{I}$.

Claim 1. *There exists a finitely generated ideal $J \subseteq I$ such that $\alpha \in \widetilde{J}$.*

Indeed, α satisfies an equation of the form $\alpha^n + a_1\alpha^{n-1} + ... + a_n = 0$ with $a_i \in (I^i)_t$ $\forall i = 1,...,n$. Now, let $i \in \{1,...,n\}$. Hence, there exists a finitely generated ideal $F_i \subseteq I^i$ such that $a_i \in F_{iv}$. Further, each generator of F_i is a finite combination of elements of the form $\prod_{1\leq j\leq i} c_j \in I^i$. Let J denote the subideal of I generated by all c_j's emanating from all F_i's. Clearly, $a_i \in (J^i)_t$ $\forall i = 1,...,n$. That is, $\alpha \in \widetilde{J}$, proving the claim.

Claim 2. $\widetilde{J} \subseteq J_t$.

Indeed, we first prove that $J^{-1} = (\widetilde{J})^{-1}$. Clearly, $(\widetilde{J})^{-1} \subseteq J^{-1}$. For the reverse inclusion, let $x \in J^{-1}$ and $y \in \widetilde{J}$. Then y satisfies an equation of the form $y^n + a_1y^{n-1} + ... + a_n = 0$ with $a_i \in (J^i)_t$ $\forall i = 1,...,n$. It follows that $(yx)^n + a_1x(yx)^{n-1} + \cdots + a_nx^n = 0$ with $a_ix^i \in (J^i)_t(J^{-1})^i \subseteq (J^i)_t(J^i)^{-1} = (J^i)_t((J^i)_t)^{-1} \subseteq R$. Hence $yx \in R$. Thus, $x \in (\widetilde{J})^{-1}$, as desired. Therefore, $\widetilde{J} \subseteq (\widetilde{J})_v = J_v = J_t$, proving the claim.

Now, by the above claims, we have $\alpha \in \widetilde{J} \subseteq J_t \subseteq I_t$. Consequently, $\widetilde{I} = I_t$, completing the proof of the theorem. ◻

In case all ideals of a domain are t-integrally closed, then it must be Prüfer. This is a well-known result in the literature:

Corollary 3.6 ([11, Theorem 24.7])**.** *A domain R is Prüfer if and only if every ideal of R is (t-)integrally closed.*

Now, we examine the correlation between the t-integral closure and t-reductions of ideals. In this vein, recall that, for the trivial operation, two crucial results assert that $x \in \overline{I} \Leftrightarrow I$ *is a reduction of* $I + Rx$ [17, Corollary 1.2.2] and *if I is finitely generated and $J \subseteq I$, then: $I \subseteq \overline{J} \Leftrightarrow J$ is a reduction of I* [17, Corollary 1.2.5]. Next, we establish t-analogues of these two results.

Proposition 3.7. *Let R be a domain and let $J \subseteq I$ be nonzero ideals of R.*

(a) *$x \in \widetilde{I} \Rightarrow I$ is a t-reduction of $I + Rx$.*

(b) *Assume I is finitely generated. Then: $I \subseteq \widetilde{J} \Rightarrow J$ is a t-reduction of I.*
Moreover, both implications are irreversible in general.

Proof. (a) Let $x \in \widetilde{I}$. Then, $x^n + a_1x^{n-1} + \cdots + a_n = 0$ for some $a_i \in (I^i)_t$ for each $i \in \{1,\dots,n\}$. Hence

$$x^n \in I_t x^{n-1} + \cdots + (I^n)_t \subseteq \left(I_t x^{n-1} + \cdots + (I^n)_t\right)_t \subseteq \left(I(I+Rx)^{n-1}\right)_t.$$

It follows that $(I+Rx)^n \subseteq \left(I(I+Rx)^{n-1}\right)_t$. Hence, $((I+Rx)^n)_t = \left(I(I+Rx)^{n-1}\right)_t$. Thus, I is a t-reduction of $I+Rx$.

(b) Assume $I = (a_1,\dots,a_n)$, for some integer $n \geq 1$ and $a_i \in R$ $\forall i = 1,\dots,n$. Suppose that $I \subseteq \widetilde{J}$. By (a), J is a t-reduction of $J + Ra_i$, for each $i \in \{1,\dots,n\}$. By Lemma 2.5, J is a t-reduction of $J + (a_1,\dots,a_n) = I$, as desired.

The converse of (a) is not true, in general, as shown by Example 3.10(a). Also, (b) can be irreversible even with I and J both being finitely generated. For instance, consider the integrally domain R of Example 2.3 with two ideals $J \subsetneqq I$, where J is a non-trivial t-reduction of I (i.e., $J_t \subsetneqq I_t$). By Theorem 3.5, $\widetilde{J} = J_t \not\supseteq I$. □

Next, we collect some ideal-theoretic properties of the integral closure of ideals.

Remark 3.8. Let R be a domain and let I, J be nonzero ideals of R. Then:

(1) $I \subseteq \overline{I} \subseteq \widetilde{I} \subseteq \sqrt{I_t}$. Example 3.9(a) features a t-ideal for which these three containments are strict. However, note that radical (and, a fortiori, prime) t-deals are necessarily t-integrally closed.

(2) $\widetilde{I \cap J} \subseteq \widetilde{I} \cap \widetilde{J}$. The inclusion can be strict, for instance, in any integrally closed domain that is not a PvMD by [1, Theorem 6] and Theorem 3.5. Another example is provided in the non-integrally closed case by Example 3.9(c).

(3) $\widetilde{I} + \widetilde{J} \subseteq \widetilde{I+J}$. The inclusion can be strict. For instance, in $\mathbb{Z}[x]$, we have $\widetilde{(2)} + \widetilde{(x)} = (2,x)$ and $(2,x)^{-1} = \mathbb{Z}[x]$ so that $\widetilde{(2,x)} = (2,x)_t = \mathbb{Z}[x]$ (via Theorem 3.5).

(4) By (5), $\forall\, n \geq 1$, $(\widetilde{I})^n \subseteq \widetilde{I^n}$. The inclusion can be strict, as shown by Example 3.9(b).

(5) $\forall\, x \in R$, $x\,\widetilde{I} \subseteq \widetilde{xI}$. Indeed, let $y \in x\,\widetilde{I}$. Then, there is an equation of the form $y^n + (xa_1)y^{n-1} + \cdots + x^n a_n = 0$ with $x^i a_i \in x^i (I^i)_t = ((xI)^i)_t$, $i = 1,\dots,n$. Hence, $y \in \widetilde{xI}$. Note that $x\,\widetilde{I} = \widetilde{xI}$, $\forall\, x \in R$ and $\forall\, I$ ideal $\Leftrightarrow R$ is integrally closed (Theorem 3.5).

We close this section by the two announced examples.

Example 3.9. Let $R := \mathbb{Z}[\sqrt{-3}][2x, x^2, x^3]$, $I := (2x^2, 2x^3, x^4, x^5)$, and $J := (x^3)$, where x is an indeterminate over $\mathbb{Z}$. Then I is a t-ideal of R such that

(a) $I \subsetneqq \overline{I} \subsetneqq \widetilde{I} \subsetneqq \sqrt{I}$.
(b) $(\widetilde{I})^2 \subsetneqq \widetilde{I^2}$.
(c) $\widetilde{J \cap I} \subsetneqq \widetilde{J} \cap \widetilde{I}$.

Proof. We first show that I is a t-ideal. Clearly, $\frac{1}{x^2}\mathbb{Z}[\sqrt{-3}][x] \subseteq I^{-1}$. For the reverse inclusion, let $f \in I^{-1} \subseteq x^{-4}R$. Then $f = \frac{1}{x^4}(a_0 + a_1x + \cdots + a_nx^n)$ for some $n \in \mathbb{N}$, $a_0 \in \mathbb{Z}[\sqrt{-3}]$, $a_1 \in 2\mathbb{Z}[\sqrt{-3}]$, and $a_i \in \mathbb{Z}[\sqrt{-3}]$ for $i \geq 2$. Since $2x^2 f \in R$, then $a_0 = a_1 = 0$. It follows that $f \in \frac{1}{x^2}\mathbb{Z}[\sqrt{-3}][x]$. Therefore $I^{-1} = \frac{1}{x^2}\mathbb{Z}[\sqrt{-3}][x]$. Next, let $g \in (R : \mathbb{Z}[\sqrt{-3}][x]) \subseteq R$. Then $xg \in R$, forcing $g(0) \in 2\mathbb{Z}[\sqrt{-3}]$ and hence $g \in$

$(2,2x,x^2,x^3)$. So $(R:\mathbb{Z}[\sqrt{-3}][x]) \subseteq (2,2x,x^2,x^3)$. The reverse inclusion is obvious. Thus, $(R:\mathbb{Z}[\sqrt{-3}][x]) = (2,2x,x^2,x^3)$. Consequently, we obtain

$$I_t = I_v = (R:\frac{1}{x^2}\mathbb{Z}[\sqrt{-3}][x]) = x^2(R:\mathbb{Z}[\sqrt{-3}][x]) = I.$$

(a) Next, we prove the strict inclusions $I \subsetneq \overline{I} \subsetneq \widetilde{I} \subsetneq \sqrt{I}$. For $I \subsetneq \overline{I}$, notice that $(1+\sqrt{-3})x^2 \in \overline{I}\setminus I$ as $\left((1+\sqrt{-3})x^2\right)^3 = -8x^6 \in I^3$ and $1+\sqrt{-3} \notin 2\mathbb{Z}[\sqrt{-3}]$.

For $\overline{I} \subsetneq \widetilde{I}$, we claim that $x^3 \in \widetilde{I}\setminus\overline{I}$. Indeed, let $f \in (I^2)^{-1} \subseteq x^{-8}R$. Then there are $n \in \mathbb{N}$, $a_i \in \mathbb{Z}[\sqrt{-3}]$ for $i \in \{0,2,\dots,n\}$, and $a_1 \in 2\mathbb{Z}[\sqrt{-3}]$ such that $f = \frac{1}{x^8}(a_0 + a_1x + \cdots + a_nx^n)$. Since $4x^4f \in R$, then $a_0 = a_1 = a_2 = a_3 = 0$. Therefore, $(I^2)^{-1} \subseteq \frac{1}{x^4}\mathbb{Z}[\sqrt{-3}][x]$. The reverse inclusion is obvious. Hence, $(I^2)^{-1} = \frac{1}{x^4}\mathbb{Z}[\sqrt{-3}][x]$. It follows that

$$(I^2)_t = (I^2)_v = (R:\frac{1}{x^4}\mathbb{Z}[\sqrt{-3}][x]) = x^4(R:\mathbb{Z}[\sqrt{-3}][x]) = x^2I.$$

Hence $x^6 \in (I^2)_t$ and thus $x^3 \in \widetilde{I}$. It remains to show that $x^3 \notin \overline{I}$. By [17, Corollary 1.2.2], it suffices to show that I is not a reduction of $I+(x^3)$. Let $n \in \mathbb{N}$. It is easy to see that x^4x^{3n} is the monic monomial with the smallest degree in $I\left(I+(x^3)\right)^n$. Therefore, $x^{3(n+1)} = x^{3n+3} \in \left(I+(x^3)\right)^{n+1} \setminus I\left(I+(x^3)\right)^n$. Hence, I is not a reduction of $I+(x^3)$, as desired.

For $\widetilde{I} \subsetneq \sqrt{I}$, we claim that $x^2 \in \sqrt{I}\setminus\widetilde{I}$. Obviously, $x^2 \in \sqrt{I}$. In order to prove that $x^2 \notin \widetilde{I}$, it suffices by Proposition 3.7 to show that I is not a t-reduction of $I+(x^2)$. To this purpose, notice that $I+(x^2) = (x^2)$. Suppose by way of contradiction that $(I(I+(x^2))^n)_t = ((I+(x^2))^{n+1})_t$ for some $n \in \mathbb{N}$. Then $(x^2)^{n+1} = x^{2n+2} \in (I(I+(x^2))^n)_t = x^{2n}I$. Consequently, $x^2 \in I$, absurd.

(b) We first prove that $\widetilde{I} = (2x^2,(1+\sqrt{-3})x^2,x^3,x^4)$. In view of (a) and its proof, we have $(2x^2,(1+\sqrt{-3})x^2,x^3,x^4) \subseteq \widetilde{I}$. Next, let $\alpha := (a+b\sqrt{-3})x^2 \in \widetilde{I}$ where $a,b \in \mathbb{Z}$. If $b=0$, then $a \neq 1$ as $x^2 \notin \widetilde{I}$. Moreover, since $2x^2 \in \widetilde{I}$, a must be even; that is, $\alpha \in (2x^2)$. Now assume $b \neq 0$. If $a = 0$, then $b \neq 1$ as $\sqrt{-3}x^2 \notin \widetilde{I}$. Moreover, since $2\sqrt{-3}x^2 \in \widetilde{I}$, b must be even; that is, $\alpha \in (2x^2)$. So suppose $a \neq 0$. Then similar arguments force a and b to be of the same parity. Further, if a and b are even, then $\alpha \in (2x^2)$; and if a and b are odd, then $\alpha \in (2x^2,(1+\sqrt{-3})x^2)$. Finally, we claim that $\widetilde{I}$ contains no monomials of degree 1. Deny and let $ax \in \widetilde{I}$, for some nonzero $a \in 2\mathbb{Z}[\sqrt{-3}]$. Then, by [17, Remark 1.1.3(7)], $ax \in \widetilde{I} \subseteq \widetilde{(x^2)} = \overline{(x^2)} \subseteq \overline{x^2\mathbb{Z}[\sqrt{-3}][x]}$. By [17, Corollary 1.2.2], (x^2) is a reduction of (ax,x^2) in $\mathbb{Z}[\sqrt{-3}][x]$, absurd. Consequently, $\widetilde{I} = (2x^2,(1+\sqrt{-3})x^2,x^3,x^4)$. Now, we are ready to check that $(\widetilde{I})^2 \subsetneq \widetilde{I^2}$. For this purpose, recall that $(I^2)_t = x^2I$. So, $2x^4 \in \widetilde{I^2}$. We claim that $2x^4 \notin (\widetilde{I})^2$. Deny. Then, $2x^4 \in (4x^4, 2(1+\sqrt{-3})x^4)$, which yields $x^2 \in (2x^2,(1+\sqrt{-3})x^2) \subseteq \widetilde{I}$, absurd.

(c) We claim that $x^3 \in \widetilde{I}\cap\widetilde{J}\setminus\widetilde{I\cap J}$. We proved in (a) that $x^3 \in \widetilde{I}$. So, $x^3 \in \widetilde{I}\cap\widetilde{J}$. Now, observe that $I \cap J = xI$ and assume, by way of contradiction, that $x^3 \in \widetilde{I\cap J} = \widetilde{xI}$. Then x^3 satisfies an equation of the form $(x^3)^n + a_1(x^3)^{n-1} + \cdots + a_n = 0$ with $a_i \in ((xI)^i)_t = x^i(I^i)_t$, $i = 1,\dots,n$. For each i, let $a_i = x^ib_i$, for some $b_i \in (I^i)_t$. Therefore $(x^2)^n + b_1(x^2)^{n-1} + \cdots + b_n = 0$. It follows that $x^2 \in \widetilde{I}$, the desired contradiction. $\square$

Example 3.10. Let $R := \mathbb{Z} + x\mathbb{Q}(\sqrt{2})[x]$, $I := (\frac{x}{\sqrt{2}})$, and $a := \frac{x}{2}$, where x is an indeterminate over $\mathbb{Q}$. Then:

(a) I is a t-reduction of $I + aR$ and $a \notin \widetilde{I}$.

(b) $\widetilde{I} \subsetneqq (\widetilde{I})_t$ and hence $\widetilde{I} \subsetneqq \widetilde{\widetilde{I}}$.

Proof. (a) First, we prove that $(I(I+aR))_t = ((I+aR)^2)_t$. It suffices to show that $a^2 \in (I(I+aR))_t$. For this purpose, let $f \in (I(I+aR))^{-1} = (\frac{x^2}{2}, \frac{x^2}{2\sqrt{2}})^{-1} \subseteq (\frac{x^2}{2})^{-1} = \frac{2}{x^2}R$. Then, $f = \frac{2}{x^2}(a_0 + a_1x + \ldots + a_nx^n)$, for some $n \geq 0$, $a_0 \in \mathbb{Z}$, and $a_i \in \mathbb{Q}(\sqrt{2})$ for $i \geq 1$. Since $\frac{x^2}{2\sqrt{2}}f \in R$, $a_0 = 0$. It follows that $(I(I+aR))^{-1} \subseteq \frac{1}{x}\mathbb{Q}(\sqrt{2})[x]$. On the other hand, $(I(I+aR))(\frac{1}{x}\mathbb{Q}(\sqrt{2})[x]) \subseteq R$. So, we have

$$\left(I(I+aR)\right)^{-1} = \left(\frac{x^2}{2}, \frac{x^2}{2\sqrt{2}}\right)^{-1} = \frac{1}{x}\mathbb{Q}(\sqrt{2})[x] \tag{6}$$

Now, clearly, $a^2(I(I+aR))^{-1} \subseteq R$. Therefore, $a^2 \in (I(I+aR))_v = (I(I+aR))_t$, as desired.

Next, we prove that $a \notin \widetilde{I} = \bar{I}$. By [17, Corollary 1.2.2], it suffices to show that I is not a reduction of $I + aR$. Deny and suppose that $I(I+aR)^n = (I+aR)^{n+1}$, for some positive integer n. Then $a^{n+1} = (\frac{x}{2})^{n+1} \in I(I+aR)^n = \frac{x}{\sqrt{2}}(\frac{x}{\sqrt{2}}, \frac{x}{2})^n$. One can check that this yields $1 \in \sqrt{2}(\sqrt{2}, 1)^n \subseteq (\sqrt{2})$ in $\mathbb{Z}[\sqrt{2}]$, the desired contradiction.

(b) We claim that $a \in (\widetilde{I})_t$. Notice first that $x \in \widetilde{I}$ as $x^2 \in I^2 = (I^2)_t$. Therefore, $A := (x, \frac{x}{\sqrt{2}}) \subseteq \widetilde{I}$. Clearly, $A = \frac{2}{x}(\frac{x^2}{2}, \frac{x^2}{2\sqrt{2}})$. Hence, by (6), $A^{-1} = \mathbb{Q}(\sqrt{2})[x]$. However, $aA^{-1} \subseteq R$. Whence, $a \in A_v = A_t \subseteq (\widetilde{I})_t$. Consequently, $a \in (\widetilde{I})_t \setminus \widetilde{I}$. □

4. Persistence and contraction of t-integral closure

Recall that the persistence and contraction of integral closure describe, respectively, the facts that for any ring homomorphism $\varphi : R \to T$, $\varphi(\overline{I}) \subseteq \overline{\varphi(I)T}$ for every ideal I of R, and $\overline{\varphi^{-1}(J)} = \varphi^{-1}(J)$ for every integrally closed ideal J of T.

This section studies the persistence and contraction of t-integral closure. To this purpose, we first introduce the concept of t-compatible homomorphism which extends the well-known notion of t-compatible extension [2]. Throughout, we denote by t (resp. t_1) and v (resp. v_1) the t- and v- closures in R (resp., T).

Lemma 4.1. *Let $\varphi : R \longrightarrow T$ be a homomorphism of domains. Then, the following statements are equivalent:*

(a) $\varphi(I_v)T \subseteq \left(\varphi(I)T\right)_{v_1}$*, for each nonzero finitely generated ideal I of R;*

(b) $\varphi(I_t)T \subseteq \left(\varphi(I)T\right)_{t_1}$*, for each nonzero ideal I of R;*

(c) $\varphi^{-1}(J)$ *is a t-ideal of R for each t_1-ideal J of T such that* $\varphi^{-1}(J) \neq 0$.

Proof. (a) $\Rightarrow$ (c) Let J be a t_1-ideal of T and let A be any finitely generated ideal of R contained in $\varphi^{-1}(J)$. Then, $\varphi(A)T \subseteq J = J_{t_1}$. Further, $\varphi(A)T$ is finitely generated. Hence, $\left(\varphi(A)T\right)_{v_1} \subseteq J$. It follows, via (a), that $\varphi(A_v)T \subseteq \left(\varphi(A)T\right)_{v_1} \subseteq J$. Therefore, $A_v \subseteq \varphi^{-1}(J)$ and thus $\varphi^{-1}(J)$ is a t-ideal.

(c) $\Rightarrow$ (b) Let I be a nonzero ideal of R. The ideal $J := \left(\varphi(I)T\right)_{t_1}$ is clearly a t_1-ideal of T with $\varphi^{-1}(J) \neq 0$. By (c), $\varphi^{-1}(J)$ is a t-ideal of R. Consequently, we obtain

$$I_t \subseteq \left(\varphi^{-1}(\varphi(I)T)\right)_t \subseteq \left(\varphi^{-1}\left(\varphi(I)T\right)_{t_1}\right)_t = \left(\varphi^{-1}(J)\right)_t = \varphi^{-1}(J).$$

So that $\varphi(I_t)T \subseteq J = \left(\varphi(I)T\right)_{t_1}$, as desired.

(b) $\Rightarrow$ (a) Trivial. □

Definition 4.2. A homomorphism of domains $\varphi : R \longrightarrow T$ is called t-compatible if it satisfies the equivalent conditions of Lemma 4.1.

When φ denotes the natural embedding $R \subseteq T$, this definition matches the notion of t-compatible extension (i.e., $I_tT \subseteq (IT)_{t_1}$ for every ideal I of R) well studied in the literature [2, 5, 6, 7].

Next, we announce the main result of this section which establishes persistence and contraction of t-integral closure under t-compatible homomorphisms.

Proposition 4.3. *Let $\varphi : R \longrightarrow T$ be a t-compatible homomorphism of domains, I an ideal of R, and J an ideal of T. Then:*

(a) $\varphi(\widetilde{I})T \subseteq \widetilde{\varphi(I)T}$.

(b) $\widetilde{\varphi^{-1}(J)} \subseteq \varphi^{-1}(\widetilde{J})$. *Moreover, if J is t-integrally closed, then* $\widetilde{\varphi^{-1}(J)} = \varphi^{-1}(J)$.

Proof. (a) Let $x \in \widetilde{I}$, $y := \varphi(x)$, and $z \in T$. We shall prove that $yz \in \widetilde{\varphi(I)T}$. Suppose that x satisfies the equation $x^n + a_1x^{n-1} + ... + a_n = 0$ with $a_i \in (I^i)_t$ for $i = 1, ..., n$. Then, apply φ to this equation and multiply through by z^n to obtain

$$(yz)^n + b_1z(yz)^{n-1} + ... + b_{n-1}z^{n-1}(yz) + b_nz^n = 0$$

where $b_i := \varphi(a_i) \in \varphi((I^i)_t)T \subseteq (\varphi(I^i)T)_{t_1} = \left(\left(\varphi(I)T\right)^i\right)_{t_1}$ by t-compatibility. Hence $b_iz^i \in \left(\left(\varphi(I)T\right)^i\right)_{t_1}$, for $i = 1, ..., n$. Consequently, $yz \in \widetilde{\varphi(I)T}$.

(b) Let $H := \varphi(\varphi^{-1}(J))T$. Then, by (a), we have

$$\varphi\left(\widetilde{\varphi^{-1}(J)}\right)T \subseteq \widetilde{H} \subseteq \widetilde{J}.$$

It follows that $\widetilde{\varphi^{-1}(J)} \subseteq \varphi^{-1}(\widetilde{J})$, as desired. Now, if J is t-integrally closed, then $\widetilde{\varphi^{-1}(J)} \subseteq \varphi^{-1}(\widetilde{J}) = \varphi^{-1}(J) \subseteq \widetilde{\varphi^{-1}(J)}$ and hence the equality holds. □

In the special case when both R and T are integrally closed, persistence of t-integral closure coincides with t-compatibility by Theorem 3.5. This shows that the t-compatibility assumption in Proposition 4.3 is imperative.

Corollary 4.4. *Let $R \subseteq T$ be a t-compatible extension of domains and I an ideal of R. Then:*

(a) $\widetilde{I}T \subseteq \widetilde{IT}$.

(b) $\widetilde{I} \subseteq \widetilde{IT \cap R} \subseteq \widetilde{IT} \cap R$.

Moreover, the above inclusions are strict in general.

Proof. (a) and (b) are direct consequences of Proposition 4.3. The inclusion in (a) and second inclusion in (b) can be strict as shown by Example 4.6. The first inclusion in (b) can also be strict. For instance, let R be an integrally closed domain and let $P \subsetneqq Q$ be prime ideals of R with $x \in Q \setminus P$. Then $\widetilde{(x)} = (x)$ by Theorem 3.5. While $\widetilde{xR_P \cap R} = \widetilde{R_P \cap R} = R$. That is, $\widetilde{(x)} \subsetneqq \widetilde{(x)R_P \cap R}$. □

Corollary 4.5. *Let R be a domain, I an ideal of R, and S a multiplicatively closed subset of R. Then $S^{-1}\widetilde{I} \subseteq \widetilde{S^{-1}I}$.*

Proof. It is well-known that flatness implies t-compatibility [7, Proposition 0.6]. Hence, Corollary 4.4 leads to the conclusion. ◻

For the integral closure, we always have $S^{-1}\overline{I} = \overline{S^{-1}I}$ [17, Proposition 1.1.4]. But in the above corollary the inclusion can be strict, as shown by the following example.

Example 4.6. We use a construction due to Zafrullah [25]. Let E be the ring of entire functions and x an indeterminate over E. Let S denote the set generated by the principal primes of E. Then, we claim that $R := E + xS^{-1}E[x]$ contains a prime ideal P such that $S^{-1}\widetilde{P} \subsetneqq \widetilde{S^{-1}P}$. Indeed, R is a P-domain that is not a PvMD [25, Example 2.6]. By [26, Proposition 3.3], there exists a prime t-ideal P in R such that PR_P is not a t-ideal of R_P. By Theorem 3.5, we have

$$\widetilde{P}R_P = PR_P \subsetneqq R_p = (PR_P)_t = \widetilde{PR_P}$$

since R is integrally closed. Also notice that $P = P\widetilde{R_P \cap} R \subsetneqq \widetilde{PR_P} \cap R = R$.

Corollary 4.7. *Let R be a domain and I a t-ideal that is t-locally t-integrally closed (i.e., I_M is t-integrally closed in R_M for every maximal t-ideal M of R). Then I is t-integrally closed.*

Proof. Let $\mathrm{Max}_t(R)$ denote the set of maximal t-ideals of R. By Corollary 4.5, we have

$$\begin{aligned} \widetilde{I} &\subseteq \bigcap_{M_i \in \mathrm{Max}_t(R)} (\widetilde{I})_{M_i} \\ &\subseteq \bigcap_{M_i \in \mathrm{Max}_t(R)} \widetilde{I_{M_i}} \\ &= \bigcap_{M_i \in \mathrm{Max}_t(R)} I_{M_i} \\ &= I. \end{aligned}$$

Consequently, I is t-integrally closed. ◻

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