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CORE OF IDEALS IN ONE-DIMENSIONAL NOETHERIAN DOMAINS $^{(\star)}$

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ABSTRACT. This paper contributes to the study of the core of ideals in Noetherian rings. We use techniques and objects from multiplicative ideal theory to develop explicit formulas for the core in various classes of one-dimensional Noetherian domains. All results are illustrated with original examples, where we explicitly compute the core.

1. INTRODUCTION

Throughout, all rings considered are commutative with identity. Let R be a ring and I an ideal of R. An ideal $J \subseteq I$ is a reduction of I if $JI^n = I^{n+1}$ for some positive integer n. The notion of reduction was introduced by Northcott and Rees [31] to contribute to the analytic theory of ideals in Noetherian local rings with infinite residue field, and later was studied by Hays [15, 16] in arbitrary Noetherian rings and Prüfer domains. The core of I, denoted core(I), is the intersection of all reductions of I. The core was initially introduced by Sally [37] and appeared, among others, in the context of Briancon-Skoda's Theorem, which asserts that if R is regular with dimension d, then core(I) contains the integral closure of I^d [21, Chapter 13].

Most of the works on the core dealt with special settings of Noetherian rings. Huneke and Swanson [20] investigated the core of integrally closed ideals in two-dimensional regular local rings. Corso, Polini and Ulrich [6, 7, 35] provided explicit formulas for the core and extended the results in [20] to Cohen-Macaulay local rings. Hyry and K. E. Smith [23] generalized the results in [20] to arbitrary dimensions and more general rings. Huneke and Trung [22] answered several open questions raised in the literature. Polini, Ulrich, and Vitulli [36] examined the core of zero-dimensional monomial ideals. In 2008, Fouli, Polini and Ulrich [11, 12] studied the core in arbitrary characteristic and, among others, characterized Cayley-Bacharach sets of points in terms of the structure of the core of the maximal ideal of their homogeneous coordinate ring. B. Smith [39] established a formula for the core of some special ideals in a polynomial ring over a field.

In [26], we established explicit formulas for the core of ideals in valuation domains and pseudo-valuation domains as well as in various classes of Prüfer domains. In particular, under some ideal-theoretic conditions, we investigated settings where the equality core(I) = I^2I^{-1} holds for every non-zero ideal I. This paper contributes to the study of the core of ideals in Noetherian rings. Similarly to [26], we use techniques and objects from multiplicative ideal theory to develop explicit formulas for the core in various classes of one-dimensional Noetherian

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domains. All results are illustrated with original examples, where we explicitly compute the core.

Throughout this paper, given a domain R, we shall denote by $\mathrm{qf}(R)$ its quotient field, by $\overline{R}$ its integral closure, by $U(R)$ the group of its units, and by $\mathrm{Max}(R)$ the set of its maximal ideals. By a *non-trivial* domain, we mean a domain which is not a field. If R/M is infinite for each $M \in \mathrm{Max}(R)$, R is said to have infinite residue fields. For any given nonzero ideal I of R, let

$$(I : I) := \left\{x \in \mathrm{qf}(R) \mid xI \subseteq I\right\} ; \; I^{-1} := \left\{x \in \mathrm{qf}(R) \mid xI \subseteq R\right\}.$$

2. Results

Let R be a Noetherian ring and I an ideal of R. A reduction J of I is called *minimal* if no ideal strictly contained in J is a reduction of I. In general, there may not be a minimal reduction of I. However, if R is local with maximal ideal M, I always admits a minimal reduction [31, Section 2, Theorem 1] or [21, Theorem 8.3.5], which is not unique in general. In fact, if R/M is infinite, then any l non-special elements of I generate a minimal reduction of I, where l denotes the analytic spread of I [31, Section 5, Theorem 1]. Also, recall that I is called *basic* if it has no proper reduction (i.e., $\mathrm{core}(I) = I$). Invertible ideals and idempotent ideals are basic [26, Lemma 2.1].

A domain R is said to have the trace property if for each nonzero ideal I of R, either I is invertible in R or $I(R : I)$ is a prime ideal of R (for brevity, we say R is a TP-domain) [8, 9, 28]. Common examples of TP-domains are valuation domains [1, Theorem 2.8], pseudo-valuation domains [18, Example 2.12], and Dedekind domains [9, Corollary 2.5].

A Noetherian TP-domain R is either Dedekind or a one-dimensional domain with a unique non-invertible maximal ideal M such that M^{-1} coincides with its integral closure $\overline{R}$ [9, Theorem 3.5]. In particular, $\dim(R) \leq 1$. In this vein, the first main result of this paper (Theorem 2.1) shows that, in Noetherian settings, the class of domains satisfying $\mathrm{core}(I) = I^2I^{-1}$ for all nonzero ideals lies strictly between the two classes of TP-domains and one-dimensional domains; and the equivalence holds in a large class of Noetherian domains. To this purpose, it is worthwhile recalling Hays' result that "*in a Noetherian domain R with infinite residue fields, each nonzero ideal of R has an invertible reduction if and only if* $\dim(R) \leq 1$" [15, Theorem 4.4]; and Huckaba-Papick's result that "*given a Noetherian local domain* (R, M), *the fractional ideal* M^{-1} *is a domain if and only if either* $\dim(R) \geq 2$ *or* $\dim(R) = 1$ *with* $R \subsetneq \overline{R}$ [19, Theorem 3.0].

Theorem 2.1. *Let R be a Noetherian domain with infinite residue fields. Consider the following conditions:*

(1) *R is a TP domain,*
(2) $\mathrm{core}(I) = I^2I^{-1}$, *for each nonzero ideal I of R,*
(3) $\dim(R) \leq 1$.

Then:

$$(1) \Longrightarrow (2) \Longrightarrow (3)$$

and both implications are irreversible in general. Moreover, if (R, M) *is local such that* M^{-1} *is a local domain with maximal ideal M, then the above conditions are equivalent.*

The proof of this theorem draws on the following lemma, which will also be used later in this paper.

Lemma 2.2. *Let R be a domain, I a finitely generated ideal, and $S := \bigcup_{n\geq 1}(I^n : I^n)$. Suppose I has an invertible reduction J_o and let n_o be the smallest positive integer such that $J_oI^{n_o} = I^{n_o+1}$. Then:*

1. $S = (I^n : I^n)$, *for any* $n \geq n_o$.
2. $JS = J_oS = IS$, *for any reduction* J *of* I.
3. $J_o(R : S) = I(R : S) \subseteq \text{core}(I)$.

Proof. (1) Notice first that the sequence of rings $\left\{(I^n : I^n)\right\}_{n\geq 1}$ is increasing. Now, let r be an integer ≥ 1 and let $x \in (I^{n_o+r} : I^{n_o+r})$. Then

$$xJ_oI^{n_o+r-1} = xI^{n_o+r} \subseteq I^{n_o+r} = J_oI^{n_o+r-1}.$$

Since J_o is invertible, $xI^{n_o+r-1} \subseteq I^{n_o+r-1}$; that is, $(I^{n_o+r} : I^{n_o+r}) = (I^{n_o+r-1} : I^{n_o+r-1})$. By induction on r, we obtain $(I^{n_o+r} : I^{n_o+r}) = (I^{n_o} : I^{n_o})$, for all $r \geq 1$. It follows that $S = (I^n : I^n)$, for any $n \geq n_o$, proving the first assertion of the lemma.

(2) We have $J_o^{-1}I^{n_o+1} = I^{n_o}$ and so $J_o^{-1}I \subseteq (I^{n_o} : I^{n_o}) = S$ by (1). Hence $J_o^{-1}IS$ is an ideal of S and whence

$$(J_o^{-1}IS)I^{n_o} = J_o^{-1}I^{n_o+1} = I^{n_o}.$$

By [27, Theorem 76], there is $x \in J_o^{-1}IS$ such that $(1+x)I^{n_o} = 0$, which forces $J_o^{-1}IS = S$. That is, $J_oS = IS$. Next, let J be an arbitrary reduction of I. Then $JI^n = I^{n+1}$, for some positive integer n. Without loss of generality, we may assume that $n \geq n_o$. Therefore, $J_oI^n = I^{n+1} = JI^n$ and so $J_o^{-1}JI^n = I^n$. Hence, $J_o^{-1}J \subseteq (I^n : I^n) = S$ by (1); that is, $J_o^{-1}JS$ is an ideal of S. Again by [27, Theorem 76], we obtain $J_o^{-1}JS = S$. It follows that $JS = J_oS = IS$, as desired.

(3) Straightforward by (2), completing the proof of the lemma. □

Proof of Theorem 2.1. (1) ⇒ (2) Assume R is a TP-domain. As recalled above, either R is a Dedekind domain or R is a one-dimensional domain with a unique non-invertible maximal ideal M_o such that $M_o^{-1} = \overline{R}$ [9, Theorem 3.5]. In the first case, every ideal is invertible and hence basic; that is, (2) holds trivially. Next, assume that $\dim(R) = 1$ and R has a unique non-invertible maximal ideal M_o with $M_o^{-1} = \overline{R}$. Let I be a nonzero ideal of R. Without loss of generality, we assume that I is not invertible. Since R is one-dimensional with infinite residue fields, I has an invertible reduction by [15, Theorem 4.4]. Let J_o be an invertible reduction of I and let n_o be the smallest positive integer such that $J_oI^{n_o} = I^{n_o+1}$. Observe that $J_o \subsetneqq I$. Next, let $f(R)$ denote the set of all fractional ideals of R and let

$$\begin{array}{rl} T & := (I : I) \\ S & := \bigcup_{n\geq 1}(I^n : I^n) \\ F_T & := \bigcap\left\{F \in f(R) \mid FT = T\right\} \\ F_S & := \bigcap\left\{F \in f(R) \mid FS = S\right\}. \end{array}$$

Claim 1. $J_oF_S \subseteq \text{core}(I) \subseteq J_oF_T$.

Indeed, let J be an arbitrary reduction of I. Then, there is a positive integer $n \geq n_o$ such that $JI^n = I^{n+1} = J_oI^n$. Hence $J_o^{-1}JI^n = I^n$, whence $J_o^{-1}J \subseteq (I^n : I^n) = S$ by Lemma 2.2. Hence $J_o^{-1}JS$ is an ideal of S with $(J_o^{-1}JS)I^n = J_o^{-1}I^{n+1} = I^n$. We appeal to [27, Theorem 76] to conclude that $J_o^{-1}JS = S$. Therefore $F_S \subseteq J_o^{-1}J$ and so $J_oF_S \subseteq J$. It follows that $J_oF_S \subseteq \text{core}(I)$. On the other hand, let $F \in f(R)$ with $FT = T$. Then

$$JFT = J_oT \subseteq IT = I.$$

Hence $J_oF \subseteq I$. Further, we have

$$JFI^{n_o} = JFSI^{n_o} = JSI^{n_o} = J_oI^{n_o} = I^{n_o+1}.$$

That is, J_oF is a reduction of I and so $\text{core}(I) \subseteq J_oF$. Hence $J_o^{-1}\text{core}(I) \subseteq F$. Therefore $J_o^{-1}\text{core}(I) \subseteq F_T$. Consequently, $J_oF_S \subseteq \text{core}(I) \subseteq J_oF_T$, proving the claim.

Claim 2. $(R : S) = M_o$.

First, observe that $(R : S) \subsetneqq R$; otherwise, $R = S$ and then $J_o = I$ by Lemma 2.2, which is absurd. Therefore the inclusions $R \subset S \subseteq \overline{R} = M_o^{-1}$ yield

$$M_o = M_{ov} \subseteq (R : S) \subsetneqq R$$

and hence $(R : S) = M_o$, proving the claim.

Now, R is a one-dimensional Noetherian TP-domain, M_o is its unique non-invertible maximal ideal, and I is not invertible, then necessarily $II^{-1} = M_o$. Therefore, by Claim 1, we obtain

$$J_oM_o \subseteq J_oF_S \subseteq \text{core}(I) \subseteq J_oF_T$$

and then

$$M_o \subseteq F_S \subseteq J_o^{-1}\text{core}(I) \subseteq F_T \subseteq R.$$

Claim 3. $J_o^{-1}\text{core}(I) \subsetneqq R$.

Deny and assume that $J_o^{-1}\text{core}(I) = R$. Then, $\text{core}(I) = J_o$. Now, let JR_{M_o} be a reduction of IR_{M_o}. By [15, Corollary 3.7], $(J \cap I) + IM_o$ is a reduction of I in R and hence

$$J_oR_{M_o} \subseteq (J \cap I)R_{M_o} + IM_oR_{M_o}.$$

However, R_{M_o} is a TP-domain, then by [26, Lemma 2.2] we have

$$IM_oR_{M_o} = (I^2I^{-1})R_{M_o} = (IR_{M_o})^2(IR_{M_o})^{-1} \subseteq \text{core}(IR_{M_o}).$$

It follows that $J_oR_{M_o} \subseteq JR_{M_o}$ and thus

$$\text{core}(IR_{M_o}) = J_oR_{M_o}.$$

This is absurd since R_{M_o} is local Noetherian with infinite residue field and IR_{M_o} is not basic [15, Theorem 3.6], and then IR_{M_o} has more than one (in fact, infinitely many) minimal reduction(s) [31, Section 5, Theorem 1].

By Claim 3, we have $M_o = J_o^{-1}\text{core}(I)$ and consequently

$$\text{core}(I) = J_oM_o = J_oSM_o = ISM_o = IM_o = I^2I^{-1}$$

as desired.

(2) $\Rightarrow$ (3) Let I be a nonzero basic ideal of R. Then, (2) yields

$$I = \text{core}(I) = I^2I^{-1}.$$

We appeal to [27, Theorem 76] to conclude that $II^{-1} = R$. Next, let $k \geq 1$ and let J be a reduction of I^k. Then, there is a positive integer n such that $JI^{nk} = I^{kn+k}$. Composing the two sides by I^{-1} and using the fact that I is invertible, we obtain

$$JI^{nk-1} = I^{nk+k-1}.$$

Iterating this process kn times, we get $J = I^k$; that is, I^k is basic. Therefore, by [15, Theorem 2.7], $\dim(R) \leq 1$.

This completes the proof of the first statement of the theorem. Further, the irreversibility of the two implications is handled by Examples 3.1 and 3.2 in the next section.

Next, assume that (R,M) is local such that M^{-1} is local with maximal ideal M. We prove (3) $\Rightarrow$ (1). Assume that $\dim(R)=1$. If $M(M^{-1}:M)\subseteq M$, then

$$(M^{-1}:M)=(M:M)=M^{-1}.$$

Since $\dim(M^{-1})=1$, we get

$$M=M_{t_1}=M_{v_1}=M^{-1}$$

where t_1- and v_1- denote the t- and v- operations with respect to M^{-1}, which is absurd. Therefore, necessarily, M is invertible in M^{-1} and hence M^{-1} is a DVR. It follows that R is a pseudo-valuation domain issued from M^{-1}; precisely, by [2, Proposition 2.6], R is the pullback issued from the following diagram:

$$\begin{array}{ccc} R:=\phi^{-1}(k) & \longrightarrow & k=R/M \\ \downarrow & & \downarrow \\ M^{-1} & \xrightarrow{\phi} & K=M^{-1}/M. \end{array}$$

Consequently, R is a TP-domain, completing the proof of the theorem. □

Observe that the "*infinite residue fields*" assumption was used only in the proof of the implication (1) $\Longrightarrow$ (2). Moreover, note that Examples 3.1 and 3.2 feature one-dimensional Noetherian local domains (R,M) such that M^{-1} is local with maximal ideal $\neq M$ and Condition (2) above holds in 3.1 but not in 3.2. This shows that the assumption "M^{-1} is local with maximal ideal M" is neither necessary nor superfluous. Also note that the result is not true, in general, if one restricts Condition (2) to the nonzero prime ideals of R, as shown by Example 3.3.

At this point, recall that a domain is divisorial if all its nonzero (fractional) ideals are divisorial. Divisorial domains have been studied by Bass [4] and Matlis [29] for the Noetherian case, Heinzer [17] for the integrally closed case, Bastida-Gilmer [3] for the $D+M$ rings, and Bazzoni [5] for more general settings. In this vein, recall Matlis' result that *a non-trivial Noetherian local domain is divisorial if and only if* $\dim(R)=1$ *and* M^{-1}/R *is a simple R-module* [5, Theorem A]. Finally, recall that two of Hays' results assert that, *in a one-dimensional local Noetherian domain with infinite residue field, every nonzero ideal has a principal reduction*; and *a nonzero ideal is basic if and only if it is principal* [15, Theorem 4.4 & Corollary 4.5]. A recent result, due to Fouli and Olberding, ensures that *if* (R,M) *is a one-dimensional local Noetherian domain, then* $\left|\mathrm{Max}(\overline{R})\right|\leq\left|R/M\right|$ *if and only if every ideal of R has a principal reduction* [10, Corollary 3.3].

Throughout, for an ideal I, we denote by $\mathrm{pcore}(I)$ the *principal core* of I; i.e., the intersection of all principal reductions of I. The second main result of this paper (Theorem 2.4) establishes correlation between the core and principal core in a special class of Noetherian local divisorial domains, and will be used to compute the core explicitly for various examples in the next section.

Notice that, in general, $\mathrm{core}(I)\subsetneq\mathrm{pcore}(I)$, as shown by Example 3.7. The following lemma sheds more light on the structure of principal reductions, and will be used in the proof of Theorem 2.4 and, later, in the construction of examples.

Lemma 2.3. *Let R be a domain and I a finitely generated ideal. Suppose I has a principal reduction a_oR and let*

$$\begin{array}{lll} S & := & \bigcup_{n\geq 1}(I^n : I^n) \\ U_o & := & U(S)\cap(I : a_oR) = \left\{u\in U(S) \mid a_ou\in I\right\}\supseteq U(R) \\ P_o & := & \bigcap_{u\in U_o} uR. \end{array}$$

Then, J is a principal reduction of I if and only if $J = a_ouR$ for some $u \in U_o$. Moreover, $\mathrm{pcore}(I) = a_oP_o$.

Proof. Let $J := bR$ be a principal reduction of I. By Lemma 2.2, $a_oS = bS$; that is, $ba_o^{-1}S = S$. Hence $J = a_o(ba_o^{-1})R$ with $ba_o^{-1}\in U_o$. Conversely, let $u \in U_o$. By Lemma 2.2, $S = (I^n : I^n)$, where n is the smallest positive integer such that $a_oI^n = I^{n+1}$. It follows that

$$a_ouI^n = a_ouSI^n = a_oSI^n = a_oI^n = I^{n+1}$$

and thus a_ouR is a principal reduction of I.

By the first statement, we have

$$\begin{array}{lll} \mathrm{pcore}(I) & = & \bigcap_{u\in U_o} a_ouR \\ & = & a_o\bigcap_{u\in U_o} uR \\ & = & a_oP_o \end{array}$$

completing the proof of the lemma. □

Theorem 2.4. *Let (R,M) be a one-dimensional Gorenstein (i.e., non-trivial Noetherian divisorial) local domain such that $\left|\mathrm{Max}(\overline{R})\right| \leq \left|R/M\right|$ and M^{-1} is a TP-domain. Let I be a non-basic ideal of R and let aR be an arbitrary principal reduction of I.*

(1) *If M^{-1} is local, then* $\mathrm{core}(I) = \mathrm{pcore}(I)$.
(2) *If M^{-1} is not local, then* $\mathrm{core}(I) = aM \cap \mathrm{pcore}(I)$.

Proof. First, recall that the assumption "$\left|\mathrm{Max}(\overline{R})\right| \leq \left|R/M\right|$" ensures the existence of principal reductions by [10, Corollary 3.3]. Then, notice that M is not principal (since M^{-1} is a ring by assumption) and throughout this proof, let

$$\begin{array}{ll} T & := M^{-1} = (M : M) \\ S & := \bigcup_{n\geq 1}(I^n : I^n) \\ Q & := (R : S). \end{array}$$

Moreover, since I is not principal (i.e., not invertible), $II^{-1} \subseteq M$ and then

$$I \subset R \subseteq T \subseteq (II^{-1})^{-1} = (I_v : I_v) = (I : I) \subseteq S$$

forcing I to be an ideal of T. So, for any non-principal reduction J of I in R, as seen above, J is an ideal of T and hence a reduction of I in T. Whence, $\mathrm{core}_T(I) \subseteq J$. It follows that

$$\mathrm{core}(I) = \mathrm{pcore}(I) \cap \mathrm{core}_T(I)$$

where $\mathrm{core}_T(I)$ denotes the core of I as an ideal of T.

(1) Assume T is local with maximal ideal N. Since, $R \subset T$ is an integral extension, then $N\cap R = M$ and hence $U(R) \subsetneqq U(T)$. Let $u \in U(T)\setminus R$. Then, $u^{-1} \in U(S)$ and $au^{-1} \in IT = I$. By Lemma 2.3, $au^{-1}R$ is a principal reduction of I. Hence $\mathrm{pcore}(I) \subseteq au^{-1}R$. By consequence, $\mathrm{pcore}(I) \subsetneqq aR$ and so $a^{-1}\mathrm{pcore}(I) \subsetneqq R$. Thus, $a^{-1}\mathrm{pcore}(I) \subseteq M$ and so

$$\mathrm{pcore}(I) \subseteq aM.$$

We claim that $M^2 \subseteq Q$. By hypothesis, T is a TP-domain. By [9, Theorem 3.5], two cases are possible. Either T is Dedekind and so

$$T \subseteq S \subseteq \overline{S} = \overline{R} = \overline{T} = T$$

yielding

$$Q = (R : S) = (R : T) = M \supseteq M^2.$$

Or T is not Dedekind and has a unique non-invertible maximal ideal N with $(T : N) = \overline{T}$. Again, we obtain

$$T \subseteq S \subseteq \overline{S} = \overline{T} = (T : N) = (R : MN).$$

Hence

$$M^2 \subseteq MN \subseteq (MN)_v \subseteq (R : S) = Q$$

proving the claim.

By Lemma 2.2, $a(T : S) \subseteq \text{core}_T(I)$. Moreover, we have

$$(T : S) = (M^{-1} : S) = (R : MS) = ((R : S) : M) = (Q : M).$$

Since $M^2 \subseteq Q$, then $M \subseteq (Q : M) = (T : S)$ and so

$$\text{pcore}(I) \subseteq aM \subseteq a(T : S) \subseteq \text{core}_T(I).$$

It follows that $\text{core}(I) = \text{pcore}(I)$, completing the proof of (1).

(2) Assume T is not local and let N_1, N_2 be two maximal ideals of T. We claim that T has exactly two maximal ideals. Indeed, if N is a maximal ideal of T, then

$$N_1 \cap R = N_2 \cap R = N \cap R = M$$

as the extension $R \subset T$ is integral. Further, the fact that it is a minimal extension forces $N_1 \cap N_2 = M \subseteq N$ by [14, Lemma 2.1]. Hence $N = N_1$ or $N = N_2$, as claimed. Further, in view of [9, Theorem 3.5], we envisage two cases. Assume T is Dedekind. Then, $I = aT$ and so $T = (I : I)$; that is, I is strongly stable in R. Since I is by hypothesis non-principal, $R \subsetneq T$ and hence $(R : T) = M$. By [26, Theorem 2.12], we have

$$\begin{aligned} \text{core}(I) &= I^2 I^{-1} \\ &= I^2 a^{-1} M \\ &= aM \\ &= aM \cap \text{pcore}(I). \end{aligned}$$

Next, assume T is not Dedekind. In this case, say, N_1 is invertible but N_2 is not, and hence the trace of every non-invertible (nonzero) ideal of T equals N_1. If I is invertible in T, then $I = aT$ and, as seen above, we are done. So, assume I is not invertible in T and let n be a positive integer such that $aI^n = I^{n+1}$. By [18, Remark 2.13 (b)], the trace property implies

$$I^n(T : I^n) = I(T : I) = N_1.$$

So Theorem 2.1 yields

$$\begin{aligned} aN_1 &= aI^n(T : I^n) \\ &= I^{n+1}(T : I^n) \\ &= IN_1 \\ &= \text{core}_T(I). \end{aligned}$$

Hence, we get

$$\begin{array}{rcl} \mathrm{core}(I) & = & \mathrm{core}_T(I)\cap\Big(aR\cap\mathrm{pcore}(I)\Big) \\ & = & \Big(aN_1\cap aR\Big)\cap\mathrm{pcore}(I) \\ & = & aM\cap\mathrm{pcore}(I) \\ & \subseteq & aN_1\cap\mathrm{pcore}(I) \\ & = & \mathrm{core}_T(I)\cap\mathrm{pcore}(I) \\ & = & \mathrm{core}(I). \end{array}$$

Consequently, $\mathrm{core}(I) = aM\cap\mathrm{pcore}(I)$, completing the proof of the theorem. □

Illustrative examples are provided in the next section.

3. Examples

In this section, all results obtained in the previous section are illustrated with original examples, where we explicitly compute the core. The first example features a Noetherian local domain R in which $\mathrm{core}(I) = I^2I^{-1}$ for each nonzero ideal I and R is not a TP-domain. This shows that the implication (1) ⇒ (2) of Theorem 2.1 is not reversible in general. For this purpose, recall that a nonzero ideal I of a domain R is stable (resp., strongly stable) if it is invertible (resp., principal) in its endomorphism ring $(I : I)$ (cf. [1, 24, 25, 32, 33, 34, 38, 39]).

Example 3.1. Let k be a field and X an indeterminate over k. Let $R := k[[X^2,X^5]]$. Then, R is a one-dimensional Noetherian local domain with maximal ideal $M = (X^2,X^5)$. We claim that R is a strongly stable divisorial domain. Indeed, it is easy to check that $M^{-1} = k[[X^2,X^3]]$. Further, $k[[X^2,X^5]] \subsetneq k[[X^2,X^3]]$ is a minimal extension; that is, M^{-1}/R is a simple R-module. Hence R is a divisorial domain by [5, Theorem A]. Next, let I be a non-invertible ideal of R and set $T := (I : I)$. Then $II^{-1} \subseteq M$ and so

$$k[[X^2,X^3]] = M^{-1} \subseteq (II^{-1})^{-1} = (I_v : I_v) = (I : I) = T \subseteq k[[X]].$$

Since the extension $k[[X^2,X^3]] \subsetneq k[[X]]$ is minimal, either $T = k[[X]]$ or $T = k[[X^2,X^3]]$. In the first case, $k[[X]]$ is a DVR and so I would be invertible in T. Next, assume $T = k[[X^2,X^3]]$ and $I(T : I) \subseteq (X^2,X^3)$. Then, we get

$$k[[X]] = (X^2,X^3)^{-1} \subseteq (T : (I(T : I))) = (I_{v_1} : I_{v_1})$$

where v_1- denotes the v-operation with respect to T. Similar arguments as above ensure (via [5, Theorem A]) that T is a divisorial domain. So $(I_{v_1} : I_{v_1}) = (I : I) = T$ forcing $k[[X]] = T$, which is absurd. Necessarily, I is invertible in T. Consequently, in both cases, I is strongly stable. Since T is local, by [26, Theorem 2.12], $\mathrm{core}(I) = I^2I^{-1}$, as desired. But, R is not a TP-domain as $M^{-1} \subsetneq \overline{R} = k[[X]]$. □

The next example features a one-dimensional Noetherian local domain with maximal ideal M such that $\mathrm{core}(M) = M^3 \subsetneq M^2M^{-1}$. This shows that the implication (2) ⇒ (3) of Theorem 2.1 is not reversible in general.

Example 3.2. Let k be a field and X an indeterminate over k. Let $R := k[[X^3,X^4]]$. Then, R is a one-dimensional Noetherian local domain with maximal ideal $M := (X^3,X^4)$ and hence

$$T := (M : M) = M^{-1} = k[[X^3,X^4,X^5]].$$

Since $R \subsetneq T$ is a minimal extension, R is a divisorial domain [5, Theorem A]. Also, we have
$$S := R^M = (M^2 : M^2) = k[[X]]$$
and so
$$(R : S) = M^2 = X^6 k[[X]].$$
Now, let $f \in \text{core}(M)$, say $f = X^3\left(a_o + a_3X^3 + a_4X^4 + a_6X^6 + \dots\right)$ and let $J_1 := X^3(X+1)R$ and $J_{-1} := X^3(X-1)R$. Both J_1 and J_{-1} are subideals of M with
$$J_1M^2 = J_{-1}M^2 = M^3$$
that is, both are (principal) reductions of M in R. Hence $f \in J_1 \cap J_{-1}$ and a routine verification shows that $a_o = a_3 = a_4 = 0$. So $f \in X^9k[[X]] = M^3$; that is, $\text{core}(M) \subseteq M^3$. On the other hand, let J be any arbitrary reduction of M in R. By Lemma 2.2, $JS = MS$ and so $JM^2 = M^3 \subseteq J$. It follows that $M^3 \subseteq \text{core}(M)$ and thus $\text{core}(M) = M^3 \subsetneq M^2 = M^2T = M^2M^{-1}$. □

The next example features a two-dimensional Noetherian local domain with maximal ideal M such that M^{-1} is local with maximal ideal M, and $\text{core}(P) = P^2P^{-1}$ for every nonzero prime ideal P. This shows that the implication (2) $\Longrightarrow$ (3) in Theorem 2.1 is not true, in general, if the formula "$\text{core}(I) = I^2I^{-1}$" holds only for the nonzero prime ideals.

Example 3.3. Let $\mathbb{Q}$ be the field of rational numbers and X, Y two indeterminates over $\mathbb{Q}$. Let $M := (X,Y)\mathbb{Q}(\sqrt{2})[[X,Y]]$, $T := \mathbb{Q}(\sqrt{2}) + M$, and $R := \mathbb{Q} + M$. Observe that R and T are two-dimensional Noetherian local domains with the same spectrum [2, 13] and $M^{-1} = (M : M) = T$. Let P be a height-one prime ideal of R. Then P is a height-one prime ideal of T and hence principal (since T is a UFD). Therefore, $T = (P : P)$ and thus P is a strongly stable ideal of R. By [26, Theorem 2.12], $\text{core}(P) = P^2P^{-1}$. It remains to prove that
$$\text{core}(M) = M^2M^{-1} = M^2T = M^2.$$
Indeed, since M is a 2-generated height-two ideal of T (i.e., it is of the principal class), then M is a basic ideal of T by [15, Theorem 2.3]. Next, let J be a reduction of M in R. Then, JT is a reduction of M in T and so $JT = M$, hence $JM = M^2$, whence $M^2 \subseteq J$. It follows that $M^2 \subseteq \text{core}(M)$. Conversely, let $f \in \text{core}(M)$ and let $J_o := (X,Y)R$. Then, $\sqrt{2}J_oM = J_oM = M^2$ and so $\text{core}(M) \subseteq J_o \cap \sqrt{2}J_o$. So, write
$$f = Xg + Yh = Xu\sqrt{2} + Yv\sqrt{2}$$
for some $g, h, u, v \in R$, and write
$$g = a_1 + m_1,\ h = a_2 + m_2,\ u = b_1 + m'_1,\ v = b_2 + m'_2$$
where $a_1, a_2, b_1, b_2 \in \mathbb{Q}$ and $m_1, m_2, m'_1, m'_2 \in M$. Then, we have
$$(a_1 - b_1\sqrt{2})X + (a_2 - b_2\sqrt{2})Y = (m'_1\sqrt{2} - m_1)X + (m'_2\sqrt{2} - m_2)Y \in M^2.$$
Necessarily, $a_1 = b_1 = a_2 = b_2 = 0$ and hence $f = m_1X + m_2Y \in M^2$. Consequently, $\text{core}(M) = M^2 = M^2M^{-1}$, as desired. □

The following three examples illustrate Theorem 2.4. The first two examples provide local Noetherian divisorial domains (R, M) such that M^{-1} is a *local* TP-domain, where we compute the core for a non-basic ideal I. While R is a strongly stable domain in the first example, it is not in the second example.

Example 3.4. Let k be an infinite field and X an indeterminate over k, and let $R := k[[X^2, X^5]]$. In Example 3.1, we saw that R is a local Noetherian strongly stable divisorial domain with maximal ideal $M = (X^2, X^5)$ and $M^{-1} = k[[X^2, X^3]]$. Since $\left(M^{-1} : (X^2, X^3)\right) = \overline{M^{-1}} = k[[X]]$, M^{-1} is a *local* TP-domain [9, Theorem 3.5]. Next, let $I := (X^4, X^5)R = X^4k[[X]]$. Now, consider S, U_o, and P_o from Lemma 2.3. Clearly, $X^4I = I^2$ and then Lemma 2.2 yields $S = (I : I) = k[[X]]$. Moreover, $U_o = U(S)$ and $P_o \subseteq M$. Let $f \in P_o$. Then, $f \in (1 + X)R \cap M$ and, through polynomial identification, one can check that $f \in X^4k[[X]]$; that is, $P_o \subseteq X^4k[[X]]$. The reverse inclusion trivially holds since I is an ideal of S. It follows that $\text{pcore}(I) = X^4P_o = X^8k[[X]]$. By Theorem 2.4, $\text{core}(I) = \text{pcore}(I) = X^8k[[X]]$. □

Example 3.5. Let k be an infinite field, X an indeterminate over k, and let $R := k[[X^3, X^4]]$ and $I := (X^7, X^8) \subseteq M := (X^3, X^4)$. Then, (R, M) is divisorial [30, Theorem 2.2] and $T := M^{-1} = k[[X^3, X^4, X^5]]$ is *local* with maximal ideal $N := (X^3, X^4, X^5)$. Since $(T : N) = \overline{T} = k[[X]]$, T is a TP-domain [9, Theorem 3.5]. Notice that I (and a fortiori R) is not stable since $(I : I) = T$ and so $I(T : I) = I\left(X^{-4}k[[X]]\right) = N$. Next, consider S, U_o, and P_o from Lemma 2.3. Clearly $I^2 = (X^{14}, X^{15}, X^{16})$ and $I^3 = (X^{21}, X^{22}, X^{23})$, yielding $X^7I = (X^{14}, X^{15}) \subsetneq I^2$ and $X^7I^2 = I^3$; that is, X^7R is a principal reduction of I and so $S = (I^2 : I^2) = k[[X]]$ by Lemma 2.2. Moreover, one can easily check that $U_o = \left\{a + bX + X^3g \mid a, b \in k \text{ and } g \in k[[X]]\right\}$. Let $f \in P_o$. Then, $f \in M \cap \left(\bigcap_p (1 + pX)R\right)$, where p ranges over the positive prime integers. Through polynomial identification, we get $f \in X^6k[[X]]$; that is, $P_o \subseteq X^6k[[X]]$. On the other hand, for every $u \in U_o$, $u^{-1}X^6k[[X]] = X^6k[[X]] \subseteq R$ and so $X^6k[[X]] \subseteq uR$. Hence $X^6k[[X]] \subseteq P_o$ and therefore $P_o = X^6k[[X]]$. It follows that $\text{pcore}(I) = X^7P_o = X^{13}k[[X]]$. By Theorem 2.4, $\text{core}(I) = \text{pcore}(I) = X^{13}k[[X]]$. □

The third example provides a local Noetherian divisorial domain (R, M) such that M^{-1} is a *non-local* TP-domain, where we compute the core for a non-basic ideal.

Example 3.6. Let k be an infinite field, X an indeterminate over k, and let $R := k + X(X-1)k[X]_{(X)\cup(X+1)}$. Then R is one-dimensional local Noetherian domain with maximal ideal $M := X(X-1)k[X]_{(X)\cup(X+1)}$. Since $M^{-1} = k[X]_{(X)\cup(X+1)}$ is a semilocal Dedekind domain with exactly two maximal ideals N_1 and N_2 with $M = N_1N_2$, M^{-1}/M is a 2-dimensional R/M-vector space. Hence R is a divisorial domain [5, Theorem A]. Also, since $M^{-1} = \bar{R}$ is Dedekind, R is a TP-domain [9, Theorem 3.5]. Next, let I be a non-basic ideal of R with a principal reduction a_oR and consider S, U_o, and P_o from Lemma 2.3. Then, $II^{-1} = M$ and so $M^{-1} = (II^{-1})^{-1} = (II^{-1} : II^{-1}) = (I_v : I_v) = (I : I)$. Hence, $M^{-1} \subseteq S \subseteq \overline{R} = M^{-1}$; that is, $S = (I : I) = M^{-1}$. It follows that $U_o = U(S)$ and so, for every $u \in U(S)$ and $m \in M$, $mu^{-1} \in M \subseteq R$. Hence $m \in uR$, whence $M \subseteq P_o$. Thus $P_o = M$. Consequently, $\text{pcore}(I) = a_oP = a_oM$ and therefore $\text{core}(I) = a_oM \cap \text{pcore}(I) = a_oM$ by Theorem 2.4. □

The last example provides a *non-local* (i.e., beyond the scope of Theorem 2.4) Noetherian divisorial domain R with a maximal ideal M such that M^{-1} is a *non-local* TP-domain, and with a non-basic ideal $I \subseteq M$ that has a principal reduction aR such that $\text{core}(I) = aM \cap \text{pcore}(I) \subsetneq \text{pcore}(I)$.

Example 3.7. Let k be an infinite field, X an indeterminate over k, $R := k[X^2, X^5]$ and $I := (X^4, X^5)R \subseteq M := (X^2, X^5)R$. Then, $M^{-1} = k[X^2, X^3]$ and, by [30, Theorem 2.2], R

is divisorial. Moreover, $\left(M^{-1} : (X^2, X^3)\right) = \overline{M^{-1}} = k[X]$ and, for any $N \in \mathrm{Max}(M^{-1})$, $N(M^{-1} : N) = N$ implies $(M^{-1} : N) = (N : N) \subseteq k[X]$ and so $N = (X^2, X^3)$. Hence, by [9, Theorem 3.5], M^{-1} is a TP-domain. Further, $X^4 I = I^2$; that is, $X^4 R$ is a principal reduction of I. By Lemma 2.2, $S := \bigcup_{n\geq 1}(I^n : I^n) = (I : I) = k[X]$ and $X^4 k[X] = I$. Since $U(R) = U(S)$, by Lemma 2.3, $X^4 R$ is the unique principal reduction of I and so $\mathrm{pcore}(I) = X^4 R$. Moreover, it is easy to check that $I^{-1} = k[X]$ and hence $Q := (R : S) = (R : I^{-1}) = I_v = I$. By Lemma 2.2, $X^8 k[X] = X^4 Q \subseteq \mathrm{core}(I)$. On the other hand, let $f := X^4 g \in \mathrm{core}(I) \subseteq \mathrm{pcore}(I)$ with $g = a_o + a_2 X^2 + X^4 g'$, for some $a_o, a_2 \in k$ and $g' \in k[X]$, and let $J := X^4(1 + X, X^4)R \subset I$. The basic fact that $1 - X^4 \in (1 + X)S$ yields $(1 + X, X^4)S = S$ and so $JS = X^4 S$. Hence $JI = I^2$; that is, J is a reduction of I. Therefore $f \in J$. It follows that $g = (1 + X)h + X^4 h'$, for some $h, h' \in R$, which forces $a_o = a_2 = 0$ and so $f \in X^8 k[X]$. Consequently, we have $\mathrm{core}(I) = X^8 k[X] = X^4 M \cap \mathrm{pcore}(I) \subsetneq X^4 R = \mathrm{pcore}(I)$. □

In view of the above example , where R is *not local*, we close this paper with the following open question:

Question 3.8. *Can Theorem 2.4 be extended to non-local settings? Namely, a possible extension may exhibit as follows: Let R be a one-dimensional Gorenstein (i.e., Noetherian divisorial) domain with infinite residue fields such that M^{-1} is a TP-domain for each $M \in \mathrm{Max}(R)$. Let I be a non-basic ideal of R and J_o an arbitrary invertible reduction of I.*

(1) *If M^{-1} is local, then* $\mathrm{core}(I) = \mathrm{icore}(I)$,
(2) *If M^{-1} is not local, then* $\mathrm{core}(I) = J_o M \cap \mathrm{icore}(I)$,

where $\mathrm{icore}(I)$ *denotes the* invertible core *of I; i.e., the intersection of all invertible reductions of I.*

Notice that the "infinite residue fields" assumption ensures the existence of invertible reductions in one-dimensional Noetherian domains [15, Theorem 4.4].

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