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MINIMAL REDUCTIONS AND CORE OF IDEALS IN PULLBACKS

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Abstract. This paper deals with minimal reductions and core of ideals in various settings of pullback constructions with the aim of building original examples, where we explicitly compute the core. To this purpose, we use techniques and objects from multiplicative ideal theory to investigate the existence of minimal reductions in Section 2 and then develop explicit formulas for the core in Section 3. The last section features illustrative examples and counterexamples.

1. Introduction

Throughout this paper, all rings are commutative with identity. Let R be a ring and I an ideal of R. An ideal $J \subseteq I$ is a reduction of I if $JI^n = I^{n+1}$ for some positive integer n. An ideal which has no reduction other than itself is called basic. The notion of reduction was introduced by Northcott and Rees with the purpose to contribute to the analytic theory of ideals in Noetherian local rings through minimal reductions. If R is a Noetherian local ring, then every non-basic ideal I admits a minimal reduction; and, in the case of infinite residue field, J is a minimal reduction of I if and only if the minimal number of generators of J coincides with the analytic spread of I [22, 32]. For ample details on this topic, we refer to Huneke and Swanson's book "Integral closure of ideals, rings, and modules" [22].

In [16, 17], Hays extended the study of reductions of ideals to more general contexts; particularly, he showed that most results on reductions of ideals do not extend beyond the class of Noetherian rings, including the existence of minimal reductions. In [37], Song and Kim extended some of Northcott-Rees' results, on minimal reductions and analytic spread in Noetherian local rings, to Noetherian semi-local rings. In [20], Heinzer, Ratliff, and Rush proved that minimally generated reductions exist by extension to a finite free local unramified extension ring. In [29], we investigated the existence of minimal reductions beyond the setting of Noetherian local rings. Precisely, we proved that there are no minimal reductions in Prüfer domains; and characterized the existence of minimal reductions in pseudo-valuation domains. Recently, Fouli and Olberding [13] investigated the existence of proper reductions and the number of generators needed for a reduction in the case of finite residue field.

The notion of core of an ideal, denoted core(I), and defined as the intersection of all reductions of I, was introduced by Judith Sally in the late 1980s and was alluded to in Rees and Sally's paper "General elements and joint reductions" [34]. The core

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of an ideal naturally appears also in the context of Briancon-Skoda's Theorem; a simple version of which states that if R is a d-dimensional regular ring and I is any ideal of R, then the integral closure of I^d is contained in core(I).

In 1995, Huneke and Swanson [21] determined the core of integrally closed ideals in two-dimensional regular local rings and established a correlation to Lipman's adjoint ideal. In a series of papers [6, 7, 33], Corso, Polini and Ulrich gave explicit descriptions for the core of certain ideals in Cohen-Macaulay local rings, extending the results of [21]. In 2003, Hyry and Smith [24] generalized the results of [21] to arbitrary dimensions and more general contexts of commutative rings. In 2005, Huneke and Trung [23] answered several open questions raised by Corso, Polini and Ulrich. In 2008 & 2009, Fouli [11] and Fouli-Polini-Ulrich [14] investigated the core of ideals in arbitrary characteristic. In 2012, Fouli and Morey [12] investigated the special case of edge ideals.

As the intersection of an a priori infinite number of ideals, the core seems difficult to compute and most of the works on this topic were done in the Noetherian case; precisely, Cohen-Macaulay rings. In [29], we undertook the first study of the notion of core beyond Noetherian settings. Our main results featured explicit formulas for the core in valuation domains and pseudo-valuation domains. Recently, in [30], we investigated the core of ideals in one-dimensional Noetherian domains.

Let T be a domain, M a maximal ideal of T, K its residue field, $\varphi : T \longrightarrow K$ the canonical surjection, D a proper subring of K, and $k := \mathrm{qf}(D)$. Let R be the pullback issued from the following diagram of canonical homomorphisms:

$$(\square) \quad \begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & K = T/M. \end{array}$$

So, $R := \varphi^{-1}(D) \subsetneq T$. All along this paper, we shall refer to the diagram ($\square$) as *generic*, and we say R is a generic pullback issued from (T,M,D). If $T = V$ is a valuation domain, we say R is a classical pullback issued from (V,M,D) and in the special case when $D = k$, we say R is a pseudo-valuation domain (PVD, for short) issued from (V,M,k). Also, we will assume familiarity with the prime ideal structure along with basic ideal-theoretic properties of generic and classical pullbacks as in [1, 3, 4, 5, 8, 9, 15, 18, 25, 26].

In [31], we investigated the reductions of ideals in various contexts of pullbacks of type ($\square$). The present paper deals with minimal reductions and core of ideals in the same settings with the aim of building original examples, where we explicitly compute the core. To this purpose, we use techniques and objects from multiplicative ideal theory to investigate the existence of minimal reductions in Section 2 and then develop explicit formulas for the core in Section 3. The last section features illustrative examples and counterexamples.

2. Minimal reductions

A reduction J of I is called *minimal* if no ideal strictly contained in J is a reduction of I. An ideal that has no reduction other than itself is called *basic* [22, Definition 8.3.1]. This section investigates the existence of minimal reductions of some classes of ideals in pullback constructions. In this vein, the case when $M \subsetneq I$ is quite simple on account of our previous work in [31], as shown below.

Proposition 2.1. *Let R be a generic pullback issued from (T,M,D) and I an ideal of R. Suppose $M \subsetneq I$; i.e., $I = \varphi^{-1}(I_o)$, for some nonzero ideal I_o of D. Then, the minimal reductions of I have the form $J := \varphi^{-1}(J_o)$, where J_o is a minimal reduction of I_o.*

Proof. By [31, Theorem 2.1], the reductions of I have the form $J := \varphi^{-1}(J_o)$, where J_o is a reduction of I_o in D. Further, one can easily check that J is minimal if and only if J_o is minimal. □

It is worthwhile noting that if I is an ideal of R with $IT = T$, then $M \subsetneq I$. So, throughout, we restrict our study to classes of ideals of R which survive in T (i.e., $IT \subsetneq T$). We start with two useful technical lemmas which correlate the existence of minimal reductions to the maximality of the conductor ideal in the general setting of extensions of domains (which, obviously, covers pullback constructions).

Lemma 2.2. *Let $A \subseteq B$ an extension of domains such that the conductor ideal $(A : B)$ is maximal in B. Let I be a nonzero ideal of both A and B that is finitely generated in $(I : I)$. Then, I has a minimal reduction in A only if $(A : B)$ is maximal in A.*

Proof. Set $M := (A : B)$ and $S := (I : I)$, and assume that I has a minimal reduction J in A. We claim that J is finitely generated. Indeed, let n be a positive integer such that $JI^n = I^{n+1}$. As I^n and I^{n+1} are finitely generated in S, write

$$I^n = \sum_{i=1}^{r} b_i S \text{ and } I^{n+1} = \sum_{j=1}^{s} c_j S.$$

Then, for each $j \in \{1, \ldots, s\}$, we have

$$c_j = \sum_{i=1}^{r} x_{ij} b_i$$

where $x_{ij} \in JS$. Next, for each (i, j), write

$$x_{ij} = \sum_{k=1}^{n_{ij}} \lambda_{ijk} \mu_{ijk}$$

where $\lambda_{ijk} \in J$ and $\mu_{ijk} \in S$. Now, let J_o be the ideal of A (finitely) generated by all λ_{ijk}. Then $J_o \subseteq J$ and $x_{ij} \in J_o S$, for all i, j. Hence $c_j \in J_o S I^n = J_o I^n$, for each j, and so

$$I^{n+1} \subseteq J_o I^n \subseteq JI^n = I^{n+1}.$$

Whence, $J_o I^n = I^{n+1}$; i.e., J_o is a reduction of I in A. By minimality, $J = J_o$, proving the claim. Next, let $J =: (a_1, ..., a_m)A$, for some nonzero $a_1, ..., a_m \in J$, and suppose, by way of contradiction, that M is not maximal in A. Then $M \subsetneq N$, for some maximal ideal N of A. Necessarily, $NB = B$ and hence

$$JNI^n = JI^n = I^{n+1}.$$

That is, JN is a reduction of I in A. By minimality, $JN = J$. So, for each $i \in \{1, \ldots, m\}$, we have

$$a_i = \sum_{j=1}^{m} \gamma_{ij} a_j$$

for some elements γ_{ij} of N, which yields a system of the form $CX = 0$, where C denotes the matrix

$$\begin{pmatrix} \gamma_{11}-1 & \gamma_{12} & \dots & \gamma_{1m} \\ \gamma_{21} & \gamma_{22}-1 & \dots & \gamma_{2m} \\ \dots & \dots & \dots & \dots \\ \gamma_{m1} & \gamma_{m2} & \dots & \gamma_{mm}-1 \end{pmatrix}$$

and X denotes the nonzero vector

$$\begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_m \end{pmatrix}.$$

Therefore, det(C) = 0 and hence $\gamma - 1 = 0$, for some $\gamma \in N$, the desired contradiction. Consequently, M is a maximal ideal of A, completing the proof of the lemma. □

Note that Example 4.2 shows that Lemma 2.2 is not true, in general, if I is not supposed to be an ideal of (both A and) B.

The second lemma establishes the converse, of the above result, for a special class of ideals. Recall that a nonzero ideal I of a domain is *stable* (resp., *strongly stable*) if it is invertible (resp., principal) in its endomorphism ring $(I : I)$ [2, 27, 28, 35, 36].

Lemma 2.3. *Let $A \subseteq B$ an extension of domains such that the conductor ideal $(A : B)$ is maximal in B. Let I be an ideal of A such that $I = aB$, for some $0 \neq a \in (A : B)$. Then, $(A : B)$ is maximal in A only if aA is a minimal reduction of I in A.*

Proof. Assume that $M := (A : B)$ is maximal in A. First, notice that $(I : I) = B$ and hence I is a strongly stable ideal of A. By [29, Lemma 2.11], we have

$$I^2 I^{-1} \subseteq \text{core}(I)$$

and, moreover, $J \subseteq I$ is a reduction of I if and only if $JB = I$. Whence, aA is a reduction of I. Moreover, let J be a reduction of I such that $J \subseteq aA$. Then, $a^{-1}J$ is an ideal of A. If $a^{-1}J \subseteq M$, then $J \subseteq aM$ and so

$$aB = I = JB \subseteq aM$$

which is absurd. Therefore, necessarily, we have

$$a^{-1}J + M = A.$$

It follows that

$$\begin{array}{rcl} aA & = & J + aM \\ & = & J + a^2(A : aB) \\ & = & J + I^2 I^{-1} \\ & \subseteq & J + \text{core}(I) \\ & \subseteq & J \end{array}$$

and, thus, aA is a minimal reduction of I, completing the proof of the lemma. □

As a application of Lemma 2.2 and Lemma 2.3, we obtain the next result on the existence of minimal reductions for a sub-class of (strongly) stable ideals in generic pullbacks.

Proposition 2.4. *Let R be a generic pullback issued from (T, M, D) and let $I \subseteq M$ be a strongly stable (resp., stable) ideal of R with $(I : I) = T$. Then, $I = aT$ (resp., $IR_M = aT_M$), for some $0 \neq a \in I$ and the following assertions are equivalent:*

(1) *aR (resp., $aR+IM$) is a proper minimal reduction of I in R;*
(2) *I has a proper minimal reduction in R;*
(3) *$D=k$.*

Proof. In view of Lemma 2.2 and Lemma 2.3, we only need to prove (3) $\Longrightarrow$ (1) for the stable statement. Next, suppose that I is stable (i.e., I is invertible in T). Then, $IR_M = IT_M = aT_M$, for some $0 \neq a \in I$. Now, assume $D=k$ is a field (i.e., M is maximal in R). Applying the strongly stable statement to the generic pullback (T_M, MT_M, k), we obtain that aR_M is a proper minimal reduction of IR_M. By [16, Corollary 3.7], $J_0 := aR+IM$ is a reduction of I in R. We claim that J_0 is a minimal reduction of I. Indeed, let J be a reduction of I with $J \subseteq J_0$. Since I is stable, $JT = J_0T = T$ by [29, Lemma 2.11]. Let Q be a maximal ideal of R distinct from M. Then, $Q = N \cap R$, for some maximal ideal N of T, with $R_Q = T_N$ and hence $JR_Q = JT_N = J_0T_N = J_0R_Q$. Moreover, since $JR_M \subseteq J_0R_M = aR_M$, JR_M is a reduction of $IR_M = IT_M$. By minimality, $JR_M = aR_M = J_0R_M$. It follows that $J = J_0$. Finally, the fact $k \subsetneq K$ forces $R_M \subsetneq T_M$ and hence $J_0 \subsetneq I$. Consequently, $J_0 := aR+IM$ is a proper minimal reduction of I, completing the proof of the proposition. □

In particular, for the special ideal $I = M$ in classical pullbacks, we have the following result.

Corollary 2.5. *Let R be a classical pullback issued from (V,M,D) such that M is not basic. Then, M has a minimal reduction in R if and only if R is a PVD.*

Proof. Notice that in the classical setting, M is either idempotent or a principal ideal of the valuation domain V. But, by [31, Theorem 2.1], M is basic in R if and only if $M = M^2$. So, Proposition 2.4 leads to the conclusion. □

The next result investigates the existence of minimal reductions for the class of (strongly) stable ideals in classical pullbacks.

Theorem 2.6. *Let R be a classical pullback issued from (V,M,D). Then, the following assertions are equivalent:*

(1) *Every stable ideal of R has a minimal reduction;*
(2) *Every strongly stable ideal of R has a minimal reduction;*
(3) *R is a one-dimensional PVD.*

Proof. (2) $\Longleftrightarrow$ (3) Assume that every strongly stable ideal of R has a minimal reduction. Then, Proposition 2.4 applied to aV, for some $0 \neq a \in M$, forces $D=k$; that is, R is a PVD issued from (V,M,k). Next, let P be a nonzero prime ideal of R. Let $0 \neq b \in P$ and consider the ideal of R given by $I := bV_P$. Then, $(I:I) = V_P$ and $(R:(I:I)) = (R:V_P) = P$ is a maximal ideal of V_P. By Lemma 2.2, P is a maximal ideal of R. It follows that $P = M$ and therefore $\dim(V) = \dim(R) = 1$.

Conversely, assume that R is a one-dimensional PVD. Then, $\dim(V) = 1$ and so V is completely integrally closed. Next, let I be a strongly stable ideal of R; i.e., $I = aT$, for some $0 \neq a \in I$, and where $T := (I:I)$. Clearly, aR is a reduction of I in R. Without loss of generality, we may assume that $R \subsetneq T$. Therefore,

$$R \subsetneq T \subseteq (IV:IV) = V$$

and hence

$$M = (R:V) \subseteq (R:T) \subsetneq R.$$

It follows that $(R:T)=M$. Now, assume for a contradiction, that J is a reduction of I with $J \subsetneqq aR$. Then, $a^{-1}J \subsetneqq R$ and, a fortiori, $a^{-1}J \subseteq M$. That is, $J \subseteq aM$. Moreover, the fact $J = IT = aT$ yields $aT = I = JT \subsetneqq aM$, which is absurd. Hence aR is a minimal reduction of I in R, as desired.

(3) $\Longrightarrow$ (1) Assume that R is a one-dimensional PVD. Let I be a stable ideal of R and set $T := (I : I)$. Without loss of generality, we may assume that I is not strongly stable. Therefore, since V is completely integrally closed, we have

$$R \subsetneqq T \subsetneqq (IV : IV) = V.$$

Necessarily, I is not an ideal of V and so $I = a\varphi^{-1}(W)$, for some $0 \neq a \in I$ and k-subspace W with $k \subsetneqq W \subsetneqq K$. Let

$$I = \sum_{i=1}^{n} a_i T$$

where $\{a_i\}_{1\leq i\leq n}$ is a minimal generating set of I in T. Next, let J be an arbitrary reduction of I in R. By [29, Lemma 2.11], $JT = I$.

Claim 1. *J is not an ideal of V and contains a finitely generated reduction B of I.*

If J is an ideal of V, then

$$V = (J : J) \subseteq (JT : JT) = (I : I) = T$$

which is absurd, proving the first statement. Next, for each $i = 1, \ldots, n$, let

$$a_i = \sum_{j=1}^{n_i} b_{ij} t_{ij}$$

for some $b_{ij} \in J$ and $t_{ij} \in T$. Let B be the ideal of R generated by all the b_{ij}'s. Clearly, $B \subseteq J$ and $BT = I$; that is, B is a finitely generated reduction of I, proving the claim.

Claim 2. *$J = a\varphi^{-1}(H)$, for some k-subspace H of W, and if J is finitely generated, then* $\dim_k(H) < \infty$.

By Claim 1, J is not an ideal of V. So, $J = b\varphi^{-1}(F)$, for some $0 \neq b \in J$ and k-subspace F with $k \subseteqq F \subsetneqq K$. Hence the inclusion $J \subseteq I$ forces $\varphi(ba^{-1})F \subseteq W$. Therefore, by considering the k-subspace H of W given by

$$H := \varphi(ba^{-1})F,$$

it is easy to check that $J = a\varphi^{-1}(H)$. Now, suppose that $J = \sum_{j=1}^{r} b_j R$. Then, the fact $J = a\varphi^{-1}(H)$ implies that $H = \sum_{j=1}^{r} \varphi(b_i a^{-1})k$. Thus, H is a finitely generated k-subspace of W; that is, $\dim_k(H) \leq r$, proving the claim.

Next, consider the finitely generated reduction J_0 of I given by

$$J_0 = \sum_{i=1}^{n} a_i R = a\varphi^{-1}(H_0)$$

for some finite-dimensional k-subspace H_0 of W. If J_0 is not minimal, then it contains a proper reduction J_1 of I. By Claims 1 and 2, we may assume that J_1 is finitely generated with

$$J_1 = a\varphi^{-1}(H_1) \subsetneqq a\varphi^{-1}(H_0) = J_0$$

for some finite dimensional k-subspace H_1 such that $H_1 \subsetneq H_0$. If J_1 is not minimal, then it contains a proper reduction J_2 of I. Suppose, for a contradiction, that the process can be reiterated indefinitely. Then, the fact "$\dim_k(H_0) < \infty$" eventually forces this process to terminate, producing a decreasing chain of reductions of I

$$J_s \subsetneq J_{s-1} \subsetneq \cdots \subsetneq J_1 \subsetneq J_0$$

giving rise to a decreasing chain of k-vector spaces

$$H_s \subsetneq H_{s-1} \subsetneq \cdots \subsetneq H_1 \subsetneq H_0$$

with $\dim_k(H_s) = 1$. Therefore, $H_s = \varphi(b)k \subseteq W$, for some $b \in T$, and so

$$\begin{array}{rcl} J_s & = & a\varphi^{-1}(H_s) \\ & = & a\varphi^{-1}(\varphi(b)k) \\ & = & ab\varphi^{-1}(k) \\ & = & abR. \end{array}$$

But, $J_sT = I$ forces $abT = I$ and so I is strongly stable, the desired contradiction. Consequently, I has a minimal reduction $J \subseteq J_0$.

Finally, the implication (1) $\Longrightarrow$ (2) is trivial, completing the proof of the theorem. □

The next result investigates the existence of minimal reductions for the class of ideals which are incomparable to M in generic pullbacks.

Theorem 2.7. *Let R be a generic pullback issued from (T, M, D). Let $J \subseteq I$ be ideals of R such that $I + M = R$. Then, the following assertions are equivalent:*

(1) *J is a minimal reduction of I;*
(2) *JT is a minimal reduction of IT with $J + M = R$;*
(3) *$J = H \cap R$, for some minimal reduction H of IT.*

Proof. Recall first that since I is coprime to M then, by [31, Theorem 2.1], we have

$$\begin{array}{rcl} \mathrm{Red}_R(I) & = & \left\{ J \subseteq I \mid JT \in \mathrm{Red}_T(IT) \text{ with } J + M = R \right\} \\ & = & \left\{ H \cap R \mid H \in \mathrm{Red}_T(IT) \right\} \end{array}$$

where $\mathrm{Red}_R(I)$ denotes the set of all reductions of I in R.

(1) $\Rightarrow$ (2) Suppose that J is a minimal reduction of I. So, JT is a reduction of IT with $J + M = R$. Hence $JT + M = T$ and so

$$\begin{array}{rcl} JT \cap R & = & JT \cap (J + M) \\ & = & J + (JT \cap M) \\ & = & J + JM \\ & = & J. \end{array}$$

Now, let H be a reduction of IT in T such that $H \subseteq JT$. Then, $H \cap R$ is a reduction of I with $H \cap R \subseteq JT \cap R = J$. By minimality, $H \cap R = J$. It follows that $J \subseteq H \subseteq JT$ and hence $H = JT$. Thus, JT is a minimal reduction of IT, as desired.

(2) $\Rightarrow$ (3) Straightforward since $J + M = R$ forces $JT \cap R = J$.

(3) $\Rightarrow$ (1) Assume that $J = H \cap R$, for some minimal reduction H of IT. Then, J is a reduction of I. Let B be a reduction of I in R such that $B \subseteq J$. Hence, BT is a reduction of IT and whence $BT = H$. Further, as $I^{n+1} \subseteq B$ for some positive integer

n, any maximal ideal containing B contains also I; that is, $B+M=R$. Similar arguments as above, yield

$$B = BT \cap R = H \cap R = J.$$

Thus, J is a minimal reduction of I, completing the proof of the theorem. □

3. Core

This section investigates the core of ideals in pullback constructions. Recall, for convenience, that the core of an ideal I of R, denoted $\mathrm{core}_R(I)$, is the intersection of all (minimal) reductions of I in R.

As a direct application of Theorem 2.7, the next result features the core and the basic property for the class of ideals of R which are incomparable to M in generic pullbacks.

Corollary 3.1. *Let R be a generic pullback issued from (T,M,D) and let I be an ideal of R such that $I+M=R$. Then:*

(1) $\mathrm{core}_R(I) = \mathrm{core}_T(IT) \cap R$.
(2) *I is basic in R if and only if IT is basic in T.*

Proof. (1) Follows easily from Theorem 2.7.
(2) Combine (1) with the fact $IT \cap R = I$. □

Note that Corollary 3.1 is illustrated by Example 4.2, and however Example 4.2 shows that this result is not true, in general, if I is comparable to M in R.

The next result of this section investigates the core for the class of ideals of both R and T in generic pullbacks.

Theorem 3.2. *Let R be a generic pullback issued from (T,M,D) and let I be a nonzero ideal of both R and T. Then:*

(1) $\mathrm{core}_R(I)$ *is an ideal of T satisfying*

$$M\,\mathrm{core}_T(I) \subseteq \mathrm{core}_R(I) \subseteq \mathrm{core}_T(I).$$

Moreover, If T_M is a valuation domain, then

$$\mathrm{core}_R(I) = \mathrm{core}_T(I) \ \ or \ \ M\,\mathrm{core}_T(I).$$

(2) *If $k \subsetneq K$, then*

$$M\,\mathrm{core}_T(I) \subseteq \mathrm{core}_R(I) \subseteq MI \cap \mathrm{core}_T(I).$$

Moreover, if I is basic in T, then

$$\mathrm{core}_R(I) = MI.$$

Proof. (1) Set $A := \mathrm{core}_R(I)$ and let $u \in T \setminus M$. Then, $u^2T + M = T$; that is

$$u^2 t = 1 - m \in R$$

for some $t \in T$ and $m \in M$. Let J be an arbitrary reduction of I in R. So, $JI^n = I^{n+1}$, for some positive integer n. Now, let

$$F_u := utR + M.$$

Then, we have

$$JF_u \subseteq JT \subseteq I.$$

Further, the fact $F_uT = T$ yields
$$JF_uI^n = JI^n = I^{n+1}.$$
That is, JF_u is a reduction of I in R. Therefore, we obtain
$$\begin{array}{rcl} uA & \subseteq & uJF_u \\ & = & u^2tJ + uJM \\ & = & (1-m)J + uJM \\ & \subseteq & J + JM \\ & \subseteq & J. \end{array}$$
Consequently, $uA \subseteq A$ and thus A is an ideal of T. Next, set $B := \text{core}_T(I)$. Clearly, $A \subseteq B$. Now, let J be any reduction of I in R. Then JT is a reduction of I in T and so $BM \subseteq JM \subseteq J$. Therefore, $BM \subseteq A$, as desired.

For the proof of the last statement, assume that T_M is a valuation domain. For every maximal ideal $Q \neq M$ of T, we have
$$\begin{array}{rcl} BT_Q & = & BMT_Q \\ & \subseteq & AT_Q \\ & \subseteq & BT_Q \end{array}$$
and so
$$BMT_Q = AT_Q = BT_Q.$$
Now, if $AT_M = BT_M$, then $A = B$. Next, assume that $AT_M \subsetneqq BT_M$ and let $x \in B$ such that $x/1 \notin AT_M$. Then, $AT_M \subsetneqq xT_M$ and hence $x^{-1}AT_M \subsetneqq T_M$. Whence $x^{-1}AT_M \subseteq MT_M$, which yields
$$\begin{array}{rcl} AT_M & \subseteq & xMT_M \\ & \subseteq & BMT_M \\ & \subseteq & AT_M. \end{array}$$
Therefore, $AT_M = BMT_M$ and so $A = BM$, completing the proof of the theorem.

(2) In view of (1), we only need to prove that $A := \text{core}_R(I) \subseteq MI$. Suppose, by way of contradiction, that $A \nsubseteq MI$ and let $\left\{\overline{e_i}\right\}_{i\in\Delta}$ be a basis for the nonzero vector space I/MI over the field T/M, where the elements e_i's are in the ideal I. Necessarily, we have
$$I = \sum_{i\in\Delta} Te_i + MI.$$
Next, consider the following ideal of R
$$J := \sum_{i\in\Delta} Re_i + MI.$$
Clearly, $JT = I$ and so $JI = I^2$; i.e., J is a reduction of I in R. Next, let $a \in A \setminus MI$. Then $a \in J$. Write
$$a = \sum_{i\in\Delta_1} a_ie_i + b$$
where Δ_1 is a finite subset of Δ, $a_i \in R$ and $b \in MI$. Without loss of generality, we may assume that, for each $i \in \Delta_1$, $a_i \notin M$. Let $\lambda \in K \setminus k$ and let $u \in T \setminus R$ such that $\lambda = \varphi(u)$. Then, there is $t \in T$ and $m \in M$ such that $1 = ut + m$. Set
$$F := Ru + Rm.$$

Since $JF \subseteq JT = I$ and $FT = T$, then $JFT = JT = I$ and so JF is a reduction of I in R. Therefore, $a \in JF$. Hence, $a = yu + zm$ for some $y, z \in J$. Clearly, $zm \in JM = MI$ and, without loss of generality, we may write

$$a = \sum_{j \in \Delta_2} ub_j e_j + c$$

where Δ_2 is a finite subset of Δ, $b_j \in R \setminus M$ and $c \in MI$. Consequently, modulo MI, we obtain

$$\sum_{i \in \Delta_1} \overline{a_i}\,\overline{e_i} = \sum_{j \in \Delta_2} \overline{ub_j}\overline{e_j}.$$

As a_i and $ub_j \in T \setminus M$, for all i and j, necessarily $\Delta_1 = \Delta_2$ and then, for each i, we have $\overline{a_i} = \overline{ub_i}$; that is,

$$\varphi(a_i) = \varphi(ub_i) = \lambda\varphi(b_i).$$

It follows that $\lambda \in k$, the desired contradiction. Thus, $A \subseteq MI$.

Moreover, if I is basic in T, then the double inclusion yields $\mathrm{core}_R(I) = MI$. □

Recall at this point that, in a generic pullback (T, M, D), we have:

"*M is basic in $R \Longleftrightarrow M = M^2$*" [31, Theorem 2.1(1)].

Obviously, if any one of the above two conditions holds, then M is basic in T. However, the converse is not true in general (e.g., when T is a discrete valuation domain). As a direct application of Theorem 3.2, we have the following extension of this result.

Corollary 3.3. *Let R be a generic pullback issued from (T, M, D) such that $k \subsetneqq K$ and let I be a nonzero ideal of both R and T. Then:*

I is basic in $R \Longleftrightarrow I$ is basic in T and $MI = I$.

The next result investigates the core of ideals in the special case of classical pullbacks for all types of ideals; namely, $I = M$, $M \subsetneqq I$, $0 \neq I \subseteq M$ with I an ideal of the valuation domain V, and $I \subsetneqq M$ with I not an ideal of V. To this purpose, recall that a domain R is said to have the trace property if for each nonzero ideal I of R, either I is invertible in R or $I(R : I)$ is a prime ideal of R [10]. Valuation domains [2] and pseudo-valuation domains [19] have the trace property.

Theorem 3.4. *Let R be a classical pullback issued from (V, M, D) and I an ideal of R.*

(1) *If $M \subsetneqq I$; i.e., $I = \varphi^{-1}(A)$, for some nonzero ideal A of D, then:*

$$\mathrm{core}_R(I) = \varphi^{-1}\Big(\mathrm{core}_R(A)\Big).$$

Moreover, I is basic in R if and only if A is basic in D.

(2) *If $0 \neq I \subseteq M$ and I is an ideal of V, then:*

$$\mathrm{core}_R(I) = I^2 I^{-1}.$$

Moreover, I is basic in R if and only if I is not strongly stable in R.

(3) *If $I \subsetneqq M$ and I is not an ideal of V; i.e., $I = a\varphi^{-1}(W)$, for some $0 \neq a \in M$ and D-submodule W with $D \subseteq W \subsetneqq K$, then:*

$$\mathrm{core}_R(I) = a\varphi^{-1}\Big(\bigcap_{H \in \mathcal{H}} H\Big)$$

where $\mathcal{H} := \left\{\text{all } D\text{-submodules } H \text{ of } W \text{ with } HW^n = W^{n+1} \text{ for some } n \geq 0\right\}$. *Moreover, I is basic in R if and only if* $W = \bigcap_{H \in \mathcal{H}} H$.

Proof. The proofs of Assertions (1) and (3) follow easily from Proposition 2.1 and [31, Theorem 2.2], respectively, and so we leave the details to the reader.

(2) Suppose $0 \neq I \subseteq M$ and I is an ideal of V. Without loss of generality, we suppose that I is not invertible in R. If I is strongly stable, then $\text{core}_R(I) = I^2 I^{-1}$ [29, Theorem 2.12]. Next, assume that I is not strongly stable. Then, I is not invertible in V and so $I(V : I) = P$ is a prime ideal of V. Hence

$$(V : I) = (P : I) \subseteq (R : I) = I^{-1}$$

and so $I^{-1} = (V : I)$. Whence $II^{-1} = I(V : I) = P$ and, by [2, Theorem 2.8], $(I : I) = V_P$. We claim that

$$IP = I.$$

Otherwise, suppose that $IP \subsetneq I$. Let $x \in I \setminus IP$. Then, $IP \subsetneq xV$ and so

$$x^{-1}IP \subsetneq V \subseteq V_P.$$

Observe that $P = PV_P$ is the maximal ideal of V_P. Hence $x^{-1}IP$ is an ideal of V_P contained in P. It follows that

$$\begin{array}{rcl} I & \subseteq & x(P : P) \\ & = & xV_P \\ & = & x(I : I) \\ & \subseteq & I \end{array}$$

and so $I = x(I : I)$; i.e., I is a strongly stable ideal of R, the desired contradiction.

Now, if $P = M$, then $I = IP = IM =$ and so

$$MI^2 I^{-1} = I^2 I^{-1}.$$

Next, suppose $P \subsetneq M$. Since, we have

$$PM = (PV_P)M = PMV_P = PV_P = P,$$

then

$$MI^2 I^{-1} = MPI = PI = I^2 I^{-1}.$$

Finally, combine Theorem 3.2(1) with [29, Theorem 2.3] to get $\text{core}_R(I) = I^2 I^{-1}$, proving the first statement.

Next, suppose that I is basic. Then, I is not strongly stable; otherwise, $I = a(I : I)$ for some $a \in I$ forces aR to be a proper reduction of I in R. Conversely, suppose that I is not strongly stable in R. Then, the same arguments as above yield

$$II^{-1} = P \text{ and } PI = I$$

where $I(V : I) = P$ is a prime ideal of V. By the first statement, we obtain

$$\text{core}_R(I) = I^2 I^{-1} = PI = I.$$

Consequently, I is basic, completing the proof of the theorem. □

The special case of the conductor ideal M is given below.

Corollary 3.5. *Let R be a classical pullback issued from (V, M, D). Then, for any positive integer $n \geq 1$, we have*

$$\text{core}_R(M^n) = M^{n+1}.$$

Proof. Recall that M is either principal in V or idempotent. By Theorem 3.4(2), $\mathrm{core}_R(M^n) = M^{2n}(M^{-n})$, where M^{-n} denotes $(M^n)^{-1}$. If M is idempotent, then $M^{2n}(M^{-n}) = MM^{-1} = M = M^{n+1}$. If $M = aV$, for some $0 \neq a \in V$, then $M^{-n} = a^{-n}M$ and so $M^{2n}(M^{-n}) = a^nM = M^{n+1}$. □

Next, we recover two known results on PVDs.

Corollary 3.6 ([29, Proposition 2.10]). *Let R be a non-trivial PVD and P a prime ideal of R. Then, for any positive integer $n \geq 1$, we have*

$$\mathrm{core}_R(P^n) = P^{n+1}.$$

Proof. Recall that a PVD and its parent valuation overring share the same spectrum. The case when P is maximal is handled by the above corollary. Next, assume that P is not maximal. Then, $P^{-1} = (P : P)$ and so $PP^{-1} = P$. Further, recall that R satisfies the trace property and hence, by [19, Remark 2.13], we get $P^nP^{-n} = PP^{-1}$. So, by Theorem 3.4(2), $\mathrm{core}(P^n) = P^{2n}P^{-n} = P^nP = P^{n+1}$, as desired. □

Corollary 3.7 ([29, Proposition 3.5(a)]). *Let R be a PVD issued from (V,M,k) and I a nonzero ideal of both R and V. Then,* $\mathrm{core}(I)$ *is a reduction of I if and only if I is basic.*

Proof. Sufficiency is trivial. For necessity, suppose, for a contradiction, that $\mathrm{core}_R(I)$ is a proper reduction. Then, by Theorem 3.4, $\mathrm{core}_R(I) = I^2I^{-1}$ and I is strongly stable. Therefore, $I = a(I : I) = aV_P$, for some nonzero $a \in M$ and prime ideal P of V. Hence $\mathrm{core}_R(I) = IP = aP$, whence $aPI^n = I^{n+1}$, for some $n \geq 1$. Thus, $a^{n+1}P = a^{n+1}V_P$, which is absurd. So, I is basic. □

4. Examples

This section features illustrative examples and counterexamples, where we compute the core, explicitly, for various classes of ideals in pullbacks.

Example 4.1. This is an illustrative example for Corollary 3.1. Let $\mathbb{Q}$ denote the field of rational numbers and X, Y two indeterminates over $\mathbb{Q}$. Let $T_1 = \mathbb{Q}(\sqrt{2})((Y))[[X]]$ and $T_2 = \mathbb{Q}(\sqrt{2})[[X]] + Y\mathbb{Q}(\sqrt{2})((X))[[Y]]$. Then, T_1 is a one-dimensional valuation domain with spectrum $\mathrm{Spec}(T_1) = \left\{0 \subsetneq M_1\right\}$ and T_2 is a two-dimensional valuation domain with $\mathrm{Spec}(T_2) = \left\{0 \subsetneq P_2 \subsetneq M_2\right\}$. Now, let $T := T_1 \cap T_2$. Then, T is a two-dimensional Prüfer domain with prime spectrum $\mathrm{Spec}(T) = \left\{0 \subsetneq P \subsetneq M, 0 \subsetneq N\right\}$. Here N is a one-height maximal ideal of T and T/N contains $\mathbb{Q}$. Finally, consider the pullback R issued from $(T, N, \mathbb{Q})$ and let $p := P \cap R$. Clearly, p is a prime ideal of R with $p + N = R$. Then, by [29, Theorem 2.6], we obtain $\mathrm{core}_T(P) = P^2$. Moreover, $pT_N = PT_N$ and $pT_M = PT_M$ so that $pT = P$. It follows, via Corollary 3.1, that $\mathrm{core}_R(p) = \mathrm{core}_T(pT) \cap R = \mathrm{core}_T(P) \cap R = P^2 \cap R = p^2$. □

Example 4.2. This example shows that Corollary 3.1(1) is not true if I is comparable to M in R; and also Lemma 2.2 is not true if I is not supposed to be an ideal of B. Let $\mathbb{Z}$ and $\mathbb{Q}$ denote the ring of integers and field of rational numbers, respectively, and let X, Y be two indeterminates over $\mathbb{Q}$. Let $T := \mathbb{Q}[X, Y] := \mathbb{Q} + M$, where $M := (X, Y)T$, and $R := \mathbb{Z} + M$.

Let $I := (X, Y)R$. Then, $I \subsetneq M$ with $IT = M$. So, since M is a two-generated height-two ideal of the Noetherian domain T [5, Theorem 4], it is of the principal class and hence a basic ideal of T [16, Theorem 2.3]. It follows that $\mathrm{core}_R(I) \subsetneq \mathrm{core}_T(IT) \cap R$.

Next, let $I := (X-1, Y)R$. Then, $I+M = R$ and same arguments as above show that $IT = (X-1, Y)T$ is a basic ideal of T. By Corollary 3.1, I is a basic ideal of R. Therefore, I is a minimal reduction of itself (in R), although $(R : T) = M$ is not a maximal ideal of R. □

Example 4.3. This is an illustrative example for Theorem 3.2, which also shows that the assumption $k \subsetneq K$ is not necessary. Let $T := \mathbb{Q}(X)[Y] = \mathbb{Q}(X)+M$, where $M := YT$, $S := \mathbb{Q}[X]+M$, and $R := \mathbb{Z}+M$. Consider the ideal I of T, S and R given by $I := Y(Y-1)T$. Obviously, $\operatorname{core}_T(I) = I$. Moreover, since $Y(Y-1)S$ is a reduction of I in S, we have $\operatorname{core}_S(I) \subsetneq I = \operatorname{core}_T(I)$. Further, T_M is a valuation domain. Therefore, by Theorem 3.2(1)&(2), applied respectively to the pullbacks $S \subset T$ and $R \subset T$, we obtain $\operatorname{core}_S(I) = M\operatorname{core}_T(I) = MI = \operatorname{core}_R(I) \subsetneq I \subsetneq M$. □

Example 4.4. These two examples show that the two inequalities of Theorem 3.2 can be strict. Let K be a field containing (strictly) a domain D and X an indeterminate over K. For the first inequality, let $T := K[[X^2, X^3]] = K+M$, where $M := (X^2, X^3)T$, and $R := D+M$. Then, we have

$$M^{-1} = (R : M) = (M : M) = (T : M) = K[[X]].$$

Observe that M is strongly stable, since $M = X^2K[[X]]$, and $(M : M)$ is local. So, by [29, Theorem 2.12] applied to R and T, we get

$$\operatorname{core}_T(M) = M^2(T : M) = M^2 = M^2M^{-1} = \operatorname{core}_R(M).$$

Therefore, $M\operatorname{core}_T(M) = X^6K[[X]] \subsetneq \operatorname{core}_R(M) = X^4K[[X]]$.

For the second inequality, let $T := K[[X]] = K+M$, where $M := XT$, and $R = D+M$. Then, by Corollary 3.5, $\operatorname{core}_R(M) = M^2 \subsetneq \operatorname{core}_T(M) = M$. □

Example 4.5. These are three illustrative examples for Assertions (1), (2), and (3) of Theorem 3.4. Let K be a field and let X, Y be two indeterminates over K.

(1) Let $V := K((X))[[Y]] = K((X))+M$, where $M := YV$, $R := K[[X^2, X^3]]+M$, and $I := (X^2, X^3)K[[X^2, X^3]]+M$. Then, by Theorem 3.4(1), we get

$$\operatorname{core}_R(I) = \operatorname{core}_{K[[X^2,X^3]]}(X^2, X^3)+M.$$

Moreover, in Example 4.4, we saw that $\operatorname{core}_{K[[X^2,X^3]]}(X^2, X^3) = X^4K[[X]]$. Hence

$$\operatorname{core}_R(I) = X^4K[[X]]+YK((X))[[Y]].$$

(2) Let $V := K[[X]]+YK((X))[[Y]] = K+M$, where $M := XK[[X]]+YK((X))[[Y]]$, $R := D+M$, where D is any domain strictly contained in K. Let $I := Y^2K((X))[[Y]] = N^2$, where $N := YK((X))[[Y]]$. Clearly, $I \subsetneq M$ is an ideal of both R and V, and we have

$$I^{-1} = (R : I) = (R : N^2) = ((R : N) : N) = ((N : N) : N) = (V_1 : N) = Y^{-1}V_1$$

where $V_1 := K((X))[[Y]]$. Thus, by Theorem 3.4(2), we obtain

$$\operatorname{core}_R(I) = I^2I^{-1} = N^4Y^{-1}V_1 = N^3 = Y^3K((X))[[Y]].$$

(3) Let $V := K(X)[[Y]] = K(X)+M$, where $M := YV$, $R := K+M$, $W := K+KX$, and $I := Y(W+M)$. By Theorem 3.4(3), we have

$$\operatorname{core}_R(I) = Y\left(\left(\bigcap_{H\in\mathcal{H}} H\right)+M\right)$$

where $\mathcal{H} := \Big\{$all K-subspaces H of W with $HW^n = W^{n+1}$ for some $n \geq 0\Big\}$. Next, let $H \in \mathcal{H}$. If $\dim_K(H) = 2$, then $H = W$. Suppose $\dim_K(H) = 1$. Then, $H = (a + bX)K$, for some $a, b \in K$ such that $a + bX \neq 0$. If $a = 0$, then the assumption $HW^n = W^{n+1}$, for some $n \geq 1$, yields $KX + KX^2 + \cdots + KX^{n+1} = K + KX + KX^2 + \cdots + KX^{n+1}$, which is absurd. So, $a \neq 0$. Further, $b \neq 0$ since $K \notin \mathcal{H}$. It follows that

$$\bigcap_{H \in \mathcal{H}} H = \bigcap_{a,b \in K \setminus \{0\}} \Big(a + bX\Big)K \subseteq \Big(1 + X\Big)K \cap \Big(1 - X\Big)K.$$

Now, let $f \in \bigcap_{H \in \mathcal{H}} H$. Then, $f \in K[X]$ with degree equal to 1 and $f(1) = f(-1) = 0$. Consequently, for $K := \mathbb{Z}/2\mathbb{Z}$, we have

$$\text{core}_R(I) = Y\Big((1 + X)K + M\Big) = Y(1 + X)K + M^2 = Y(1 + X)K + Y^2K(X)[[Y]]$$

and, for any $K \neq \mathbb{Z}/2\mathbb{Z}$, we have $\bigcap_{H \in \mathcal{H}} H = 0$ yielding

$$\text{core}_R(I) = YM = M^2 = Y^2K(X)[[Y]].$$

Notice that I is not an ideal of V and not stable in R, and yet $\text{core}_R(I) = I^2I^{-1}$. □

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